Math 5707 Graph theory
Spring 2023, Vic Reiner
Final exam- Due Wednesday, May 3 by midnight at the class Canvas site

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1.(20 points total) **True** or **False**. To receive full credit, assertions which are true must be proven, and those that are false must have a counterexample exhibited.

(a) (5 points) If a graph G has degree sequences $(2, 2, 4, 4, 4, 4, 4)$ then it is eulerian, that is, it contains a (closed) Euler tour.

(b) (5 points) If a tree has $n \geq 2$ vertices in total, in which ℓ of them are leaves and all other vertices have degree 3, then $\ell = \frac{n+2}{2}$.

(c) (5 points) Given a graph G with connected components $G_1, G_2, \ldots, G_t$, its chromatic polynomial $p_G(k)$ always satisfies

$$p_G(k) = p_{G_1}(k) \cdot p_{G_2}(k) \cdot \cdots \cdot p_{G_t}(k).$$

(d) (5 points) Recall (from lecture or Bondy & Murty's Section 3.2) that for every graph G, one can uniquely define 2-vertex-connected subgraphs $B_1, B_2, \ldots, B_s$ called its *blocks* or *2-connected components*, whose union is G. Assuming G is connected, then its chromatic polynomial $p_G(k)$ always satisfies

$$p_G(k) = \frac{p_{B_1}(k) \cdot p_{B_2}(k) \cdot \cdots \cdot p_{B_s}(k)}{k^{s-1}}.$$

2.(20 points) Define for each $n = 1, 2, 3, \ldots$ a simple bipartite graph $G_n = (X \sqcup Y, E)$ with vertex set $V = X \sqcup Y$ partitioned into two sets of size n labeled

$$X = \{x_1, \ldots, x_n\},$$
$$Y = \{y_1, \ldots, y_n\},$$

and edge set E of size $n^2 - n$ defined as follows:

$$E := \{\{x_i, y_j\} : 1 \leq i, j \leq n \text{ but } i \neq j\}$$

(a) (10 points) For exactly which values of $n = 1, 2, 3, \ldots$ is G_n eulerian, that is, containing a (closed) Euler tour?

(b) (10 points) For exactly which values of $n = 1, 2, 3, \ldots$ is G_n hamiltonian, that is, containing a (closed) Hamilton tour?

3. (20 points) Let $G = (V, E)$ be a simple graph, with $\Delta(G)$ the maximum of its vertex degrees, and $\alpha(G)$ the maximum size of an *independent* or *stable* subset of vertices $V' \subset V$, that is, a subset V' having no edges of the form $e = \{x, y\} \subset V'$ in E. Prove that

$$\alpha(G) \geq \frac{|V|}{1 + \Delta(G)}.$$

4. (20 points) When a soccer-ball-like sphere is made up only of pentagonal and hexagonal faces, sewn together so that its seams form a graph which is 3-regular (at every vertex, there are exactly 3 edges incident), how many pentagonal faces will there be in total?
(Note: it turns out that the number of hexagons can vary, but you will end up proving that the number of pentagonal faces is a fixed number!)

5. (20 points total)
(a)(10 points). Show that in any directed graph $D = (V, A)$, one has

$$\sum_{v \in V} \mathrm{outdeg}_D(v) = \sum_{v \in V} \mathrm{indeg}_D(v).$$

where $\mathrm{outdeg}_D(v), \mathrm{indeg}_D(v)$ are the *outdegree* and *indegree* at a vertex v.

(b)(10 points). Let $G = (V, E)$ be an undirected graph which is *connected* and has $|E|$ *even.* Show that there exists at least one choice of a directed graph $D = (V, A)$ obtained from G by orienting each edge $e = \{x, y\}$ in E either as the arc $x \to y$ or $y \to x$ in A, such that the outdegree $\mathrm{outdeg}_D(v)$ is *even* for all v in V.