Math 8201 Graduate abstract algebra- Fall 2010, Vic Reiner
Midterm exam 2- Due Friday November 19, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Each of the 7 problems is worth approximately the same number of points.

1. (a) How many abelian groups of order $28,000$ are there, up to isomorphism? Explain your answer– don't just write down a number.

(b) How many such groups as in (a) contain an element of order 100? Again, explain your answer.

2. Recall for commutative rings R, the general linear group over R is

$$\begin{aligned} GL_n(R) &:= \{A \in R^{n\times n} : A \text{ has an inverse in } R^{n\times n}\} \\ &= \{A \in R^{n\times n} : \det(A) \in R^{\times}\}. \end{aligned}$$

(a) Let p be a prime, and $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, a finite field with p elements. Write down the cardinality of $GL_n(\mathbb{F}_p)$ as a function of n and p.

(b) Write down the cardinality of $GL_n(\mathbb{Z}/105\mathbb{Z})$ as a function of n.

3. Let $\mathbb{F}_p$ be a finite field with p elements, and $V = \mathbb{F}_p^n$, an n-dimensional vector space over $\mathbb{F}_p$. Let $G(k, V) = G(k, \mathbb{F}_p^n)$ denote the set of all k-dimensional $\mathbb{F}_p$-linear subspaces of V.

(a) The group $GL(V) = GL_n(\mathbb{F}_p)$ acts on V, and takes $\mathbb{F}_p$-subspaces to $\mathbb{F}_p$-subspaces, preserving dimension, so that it acts on $G(k, V)$. Show that this action on $G(k, V)$ is transitive.

(b) Let P_k be the subgroup P of $GL(V)$ which is the stabilizer of the particular k-dimensional subspace of V spanned by the first k standard basis vectors. Writing elements of $GL(V)$ as $n \times n$ matrices in block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{F}^{k\times k}, B \in \mathbb{F}^{k\times(n-k)}, C \in \mathbb{F}^{(n-k)\times k}, D \in \mathbb{F}^{(n-k)\times(n-k)}$, identify the elements of P_k by saying what are the conditions on A, B, C, D for this matrix to lie in P_k.

(c) Find the cardinality of $G(k, V)$, as a function of k, n and p.

4. Let G_1, G_2 be simple groups, and $N \triangleleft G_1 \times G_2$. Show that either

- $N = \{e\}$, or
- $N = G_1 \times G_2$, or
- N is isomorphic to one of G_1 or G_2.

5. Given a linear operator $\varphi : V \to V$ on a finite-dimensional $\mathbb{F}$-vector space V, define its trace $\mathrm{Tr}_V(\varphi)$ by making a choice of an ordered basis $(v_1, \ldots, v_n)$ for V in which to express φ by an $n \times n$ matrix $A = (a_{ij})_{i,j=1,2,\ldots,n}$, and then setting

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}(A) := \sum_{i=1}^{n} a_{i,i}.$$

(a) Show that $\mathrm{Tr}(\varphi)$ is well-defined, that is, independent of the choice of the basis $(v_1, \ldots, v_n)$ for V.

(b) Given an $\mathbb{F}$-linear subspace $W \subseteq V$ with $\varphi(W) \subset W$, show that the restriction map φ_W on W and the induced map $\varphi_{V/W}$ on the quotient V/W satisfy

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi_W) + \mathrm{Tr}(\varphi_{V/W}).$$

6. Let G be a group of order pqr with primes $p < q < r$ and q not dividing $r - 1$. Show that if there exists a normal subgroup $N \triangleleft G$ having $|N| = p$, then G is cyclic.

7. Let G be a group, and $\mathrm{Aut}(G)$ its automorphism group. Show that $\mathrm{Aut}(G)$ cyclic implies G abelian.
(Hint: Consider the homomorphism $\varphi : G \to \mathrm{Aut}(G)$ that sends g to the automorphism $(x \mapsto gxg^{-1})$. What is $\ker(\varphi)$?)