Math 8201 Graduate abstract algebra- Fall 2013, Vic Reiner
Midterm exam 2- Due Wednesday November 20, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (20 points total; 5 points each part) Prove or disprove:

(a) (5 points) If there exists an element of order n in a quotient group G/N of a finite group G, then there will also exist an element of order n in G.

(b) (5 points) A vector space V over a field can be isomorphic to one of its own proper subspaces $U \subsetneq V$.

(c) (5 points) A subset $U \subset V$ of an $\mathbb{R}$-vector space V is an $\mathbb{R}$-subspace if and only if $(U,+)$ is a subgroup of the additive group $V^+ := (V,+)$.

(d) (5 points) If $\mathbb{F}_p$ denotes the finite field $\mathbb{Z}/p\mathbb{Z}$ for a *prime* p, then a subset $U \subset V$ of an $\mathbb{F}_p$-vector space V is an $\mathbb{F}_p$-subspace if and only if $(U,+)$ is a subgroup of the additive group $V^+ := (V,+)$.

2. (15 points total; 5 points each part)

(a) (5 points) Given an exact sequence of finite-dimensional vector spaces over a field $\mathbb{F}$

$$0 \longrightarrow V_\ell \xrightarrow{f_\ell} V_{\ell-1} \xrightarrow{f_{\ell-1}} \cdots \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0$$

prove that

$$\dim_{\mathbb{F}} V_0 - \dim_{\mathbb{F}} V_1 + \dim_{\mathbb{F}} V_2 - \cdots + (-1)^\ell \dim_{\mathbb{F}} V_\ell = \sum_{i=0}^{\ell} (-1)^i \dim_{\mathbb{F}} V_i = 0$$

(b) (5 points) Given a short exact sequence of finite groups $1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$, prove $|B| = |A||C|$.

(c) (5 points) Given an exact sequence of finite groups

$$1 \longrightarrow G_\ell \xrightarrow{f_\ell} G_{\ell-1} \xrightarrow{f_{\ell-1}} \cdots \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0 \longrightarrow 1$$

prove that

$$|G_1||G_3||G_5| \cdots = |G_0||G_2||G_4| \cdots$$

3, (10 points; 5 points each part) For a field $\mathbb{F}$ and a linear operator $\varphi : V \to V$ on a finite-dimensional $\mathbb{F}$-vector space V, define the *trace* $\mathrm{Tr}_V(\varphi)$ as follows: make a choice of an ordered basis $(v_1, \ldots, v_n)$ for V in which to express φ by an $n \times n$ matrix $A = (a_{ij})_{i,j=1,2,\ldots,n}$, and then set

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}(A) := \sum_{i=1}^{n} a_{i,i} = a_{11} + a_{22} + \cdots + a_{nn}.$$

(a) Show $\mathrm{Tr}(\varphi)$ is well-defined, that is, independent of the choice of the ordered basis $(v_1, \ldots, v_n)$ for V.

(b) Given an $\mathbb{F}$-linear subspace $W \subseteq V$ with $\varphi(W) \subset W$, show that the restriction map φ_W on W and the induced map $\varphi_{V/W}$ on the quotient V/W satisfy

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi_W) + \mathrm{Tr}(\varphi_{V/W}).$$

4. (20 points total; 5 points each) Let $\mathbb{F}_q$ be a finite field with q elements, e.g., if q is prime then $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. Let $V = \mathbb{F}_q^n$ considered as an n-dimensional vector space over $\mathbb{F}_q$. Fixing k in the range $0 \leq k \leq n$, let $G(k, V) = G(k, \mathbb{F}_q^n)$ denote the set of all k-dimensional $\mathbb{F}_q$-linear subspaces of V.

(a) (5 points) Show that when then group $GL(V) = GL_n(\mathbb{F}_q)$ acts on V, it takes an $\mathbb{F}_q$-subspace to another $\mathbb{F}_q$-subspace, preserving dimension, so that it acts on the set $G(k, V)$.

(b) (5 points) Show that this action on $G(k, V)$ is transitive.

(c) (5 points) Let P_k be the subgroup P of $GL(V)$ which is the stabilizer of the particular k-dimensional subspace of $V = \mathbb{F}_q^n$ spanned by the first k standard basis vectors $\{e_1, \ldots, e_k\}$. Writing elements of $GL(V)$ as $n \times n$ matrices in block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{F}^{k \times k}, B \in \mathbb{F}^{k \times (n-k)}, C \in \mathbb{F}^{(n-k) \times k}, D \in \mathbb{F}^{(n-k) \times (n-k)}$, identify the elements of P_k by saying what are the conditions on A, B, C, D for this matrix to lie in P_k.

(d) (5 points) Find the cardinality of $G(k, V)$, as a function of k, n and q.

5. (15 points) For two *simple* groups G_1, G_2 and a normal subgroup $N \triangleleft G_1 \times G_2$, show that either

- $N = \{e\}$, or
- $N = G_1 \times G_2$, or
- N is isomorphic to one of G_1 or G_2.

6. (20 points) Let G be a finite group,

- with $|G| = pqr$ for primes $p < q < r$,
- with q *not* dividing $r - 1$, and
- containing a normal subgroup $N \triangleleft G$ having $|N| = p$.

Prove that G is cyclic.