Math 8201 Graduate abstract algebra- Fall 2019, Vic Reiner
Final exam - Due Wednesday December 11, in class

Instructions: This is an open book, library, web, notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (20 points total; 5 points each part) Prove or disprove:

(a) If a group G contains three normal subgroups $N_1, N_2, N_3 \triangleleft G$ with $N_1N_2N_3 = G$ and $N_1 \cap N_2 \cap N_3 = \{e\}$, then $G \cong N_1 \times N_2 \times N_3$.

(b) One can find a linear operator $V \xrightarrow{\pi} V$ on a finite-dimensional $\mathbb{F}_2$-vector space V which satisfies $\pi^2 = \pi$ but is not diagonalizable.

(c) One can find a linear operator $V \xrightarrow{\pi} V$ on a finite-dimensional $\mathbb{F}_2$-vector space V which satisfies $\pi^3 = \pi$ but is not diagonalizable.

(d) A matrix A in $\mathbb{C}^{n\times n}$ has $\det(t \cdot I_n - A) = t^n$ if and only if $A^n = 0$.

2. (20 points total) Let σ be a permutation in the symmetric group S_n, and A_σ its associated $n \times n$ permutation matrix, that is,

$$(A_\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) If σ is an n-cycle, prove that $\det(t \cdot I_n - A_\sigma) = t^n - 1$.

(b) (5 points) If σ has cycle decomposition $\sigma = \sigma^{(1)}\sigma^{(2)} \cdots \sigma^{(k)}$ in which $\sigma^{(i)}$ is an n_i-cycle for each $i = 1, 2, \ldots, k$, prove that

$$\det(t \cdot I_n - A_\sigma) = \prod_{i=1}^{k} (t^{n_i} - 1).$$

(c) (10 points) If σ is a $(p-1)$-cycle for some prime number p, prove that when working over the finite field $\mathbb{F}_p$, the matrix A_σ in $\mathbb{F}_p^{(p-1)\times(p-1)}$ is diagonalizable.

3. (20 points total) Let V be an n-dimensional vector space over a field $\mathbb{F}$. Call a linear operator $V \xrightarrow{\varphi} V$ a *transvection* if there exists an $(n-1)$-dimensional subspace $W \subset V$ and a vector $v \in V \setminus W$, such that $\varphi(w) = w$ for all w in W, but $\varphi(v) = v + w_0$ for some $w_0 \in W \setminus \{\mathbf{0}\}$.

(a) (10 points) Prove all transvections are conjugate within $GL(V)(\cong GL_n(\mathbb{F}))$.

(b) (5 points) Prove transvections have $\det(\varphi) = 1$, so they lie in $SL(V)(\cong SL_n(\mathbb{F}))$.

(c) (5 points) Prove that these two matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

both representation transvections $\mathbb{R}^2 \to \mathbb{R}^2$, but are *not* conjugate within $SL_2(\mathbb{R})$.

4. (15 points) Given $\mathbb{F}$-vector spaces V, W with ordered bases $(v_1, \ldots, v_n)$ for V and $(w_1, \ldots, w_m)$ for W, recall that $V \otimes W$ has basis $\{v_i \otimes w_j\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$, that is, every tensor t in $V \otimes W$ can be written uniquely as

$$t = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} \; v_i \otimes w_j$$

for some matrix of coefficients $A = (a_{ij})$ in $\mathbb{F}^{n\times m}$.

Recall t is *decomposable* if $t = v \otimes w$ for some $v \in V, w \in W$. Show that tensor $t = \sum_{i,j} a_{i,j} v_i \otimes w_j$ is decomposable if and only if the matrix A has rank 0 or 1.

5. (25 points total, 5 points each part)

(a) Given abelian groups A, B, show that the set $\text{Hom}(A, B)$ of all group homomorphisms $A \stackrel{\varphi}{\to} B$ becomes an abelian group when endowed with pointwise addition: $(\varphi_1 + \varphi_2)(a) := \varphi_1(a) + \varphi_2(a)$.

(b) Given abelian groups A, B, C, and a homomorphism $B \stackrel{f}{\to} C$, show that one obtains a homomorphism $\text{Hom}(A, B) \stackrel{\tilde{f}}{\to} \text{Hom}(A, C)$ as follows: for φ in $\text{Hom}(A, B)$, define $\tilde{f}(\varphi) := f \circ \varphi$, that is, $\tilde{f}(\varphi)(a) := f(\varphi(a))$. Show furthermore that

(i) given homomorphisms $B \stackrel{f}{\to} C \stackrel{g}{\to} D$ one has

$$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f} : \text{Hom}(A, B) \to \text{Hom}(A, D)$$

(ii) and the identity map $B \stackrel{1_B}{\to} B$ has $\tilde{1}_B = 1_{\text{Hom}(A,B)}$.

(c) Given abelian groups A, B, C, D, and a short exact sequence

$$0 \to B \stackrel{f}{\to} C \stackrel{g}{\to} D \to 0$$

show that the induced maps $\tilde{f}, \tilde{g}$ give rise to a sequence

$$0 \to \text{Hom}(A, B) \stackrel{\tilde{f}}{\to} \text{Hom}(A, C) \stackrel{\tilde{g}}{\to} \text{Hom}(A, D)$$

which is exact at the positions $\text{Hom}(A, B)$ and $\text{Hom}(A, C)$.

(d) Assume in the set-up of part (c) that the sequence is *split short exact*, meaning one can relabel B, C, D so that $C = B \oplus D (= B \times D)$ with

- $f : B \to C$ the injection $f(b) = (b, 0)$, and
- $g : C \to D$ the surjection $g(b, d) = d$.

Show that this whole sequence is exact:

$$0 \to \text{Hom}(A, B) \stackrel{\tilde{f}}{\to} \text{Hom}(A, C) \stackrel{\tilde{g}}{\to} \text{Hom}(A, D) \to 0.$$

(e) Consider the following example of part (c),

$$\begin{array}{ccccccc} & B & & C & & D & \\ & \| & & \| & & \| & \\ 0 \to & \mathbb{Z}/2\mathbb{Z} & \stackrel{f}{\to} & \mathbb{Z}/4\mathbb{Z} & \stackrel{g}{\to} & \mathbb{Z}/2\mathbb{Z} & \to 0 \end{array}$$

where f is inclusion of subgroup $B = \{\overline{0}, \overline{2}\} \cong \mathbb{Z}/2\mathbb{Z}$ into $C = \mathbb{Z}/4\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, and g is the reduction map

$$\begin{array}{rcl} \mathbb{Z}/4\mathbb{Z} & \stackrel{g}{\longrightarrow} & \mathbb{Z}/2\mathbb{Z} \\ \overline{0}, \overline{2} & \longmapsto & \overline{0}(\text{mod} 2) \\ \overline{1}, \overline{3} & \longmapsto & \overline{1}(\text{mod} 2). \end{array}$$

Show that when one takes $A = \mathbb{Z}/2\mathbb{Z}$, then this sequence

$$0 \to \text{Hom}(A, \mathbb{Z}/2\mathbb{Z}) \stackrel{\tilde{f}}{\to} \text{Hom}(A, \mathbb{Z}/4\mathbb{Z}) \stackrel{\tilde{g}}{\to} \text{Hom}(A, \mathbb{Z}/2\mathbb{Z}) \to 0.$$

fails to be exact in one of its positions, namely the one corresponding to $\text{Hom}(A, D)$, that is, the rightmost occurrence of $\text{Hom}(A, \mathbb{Z}/2\mathbb{Z})$.