Math 8202 Graduate abstract algebra
Spring 2011, Vic Reiner
Midterm exam 2- Due Friday April 1, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points total) Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

(a) (5 points) Express the $\mathbb{Z}$-module $\mathrm{coker}(\mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4) := \mathbb{Z}^4/\,\mathrm{im}(A)$ as a direct sum of cyclic $\mathbb{Z}$-modules.
(b) (5 points) Write down the unique representative for the similarity class of A in $\mathbb{C}^{4\times 4}$ in Jordan canonical form over $\mathbb{C}$.
(c) (5 points) Write down the unique representative for the similarity class of A in $\mathbb{C}^{4\times 4}$ in rational canonical form over $\mathbb{C}$.

2.(15 points total) Compute the ranks (with explanation) for the following $\mathbb{Z}$-modules:
(a) (5 points) $\mathrm{coker}(\mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4)$ where A is the matrix in Problem 1.
(b) (5 points) $\mathbb{Q}$
(c) (5 points) $\mathbb{Q}/\mathbb{Z}$

3. (15 points) Prove, or disprove by counterexample: for any domain R, *not* necessarily a PID, and any finitely generated R-module M, there exists an R-submodule $F \subset M$ with F free and $M \cong F \oplus \mathrm{Tor}\, M$. (Here $\mathrm{Tor}\, M := \{m \in M : \exists r \in R \setminus \{0\} \text{ with } rm = 0\}$.)

4. (15 points) Let $\mathbb{F} \subset \mathbb{K}$ be an extension of fields with $[\mathbb{K} : \mathbb{F}] = n$, and let $f(x) \in \mathbb{F}[x]$ be an irreducible polynomial with degree d. If $\gcd(d, n) = 1$, show that $f(x)$ remains irreducible when considered as an element of $\mathbb{K}[x]$.

5. (15 points) Dummit and Foote, §13.2, Problem 17, on page 530.

6. (25 points total) For a (not necessarily commutative) ring R, and a sequence of (left) R-modules M_i and R-module homomorphisms

$$\cdots \to M_{i+1} \stackrel{f_{i+1}}{\to} M_i \stackrel{f_i}{\to} M_{i-1} \to \cdots$$

say that the sequence is

- a *complex* if $\operatorname{im}(f_{i+1}) \subset \ker(f_i)$ for each i, i.e. $f_i \circ f_{i+1} = 0$,
- *exact at* M_i if $\operatorname{im}(f_{i+1}) = \ker(f_i)$, and
- an *exact sequence* if it is exact at M_i for every i.

(a) (3 points) Explain why a sequence of the form

- $0 \to A \stackrel{\alpha}{\to} B$ is exact at A if and only if α is injective,
- $B \stackrel{\beta}{\to} C \to 0$ is exact at C if and only if β is surjective,
- $0 \to A \stackrel{\alpha}{\to} B \to 0$ is exact if and only if α is an isomorphism,
- $0 \to B \to 0$ is exact at B if and only if $B = 0$.
- $0 \to A \to B \to C \to 0$ is exact if and only if B contains an R-submodule A' isomorphic to A for which B/A' is isomorphic to C. These are called *short exact sequences*.

(b) (2 points) Show that every homomorphism $\alpha : A \to B$ gives rise to a short exact sequence of the form $0 \to \ker(\alpha) \to A \to \operatorname{im}(\alpha) \to 0$, and also to an exact sequence $0 \to \ker(\alpha) \to A \stackrel{\alpha}{\to} B \to \operatorname{coker}(\alpha) \to 0$

(c) (5 points) Show that an exact sequence of R-modules

$$0 \to M_\ell \to \cdots \to M_1 \to M_0 \to 0$$

for a domain R with $\operatorname{rank}_R M_i$ finite implies $\sum_{i=0}^{\ell}(-1)^i \operatorname{rank}_R M_i = 0$. (Hint: the case where one has a short exact sequence, that is, $\ell = 2$, was mentioned in lecture and proven on homework, so it may be assumed.)

(d) (5 points) An exact sequence $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ with each F_i a *free* R-module is called a *free resolution of* M. Show that every R-module has a free resolution, although the free modules F_i might not be of finite rank, and the resolution itself may be infinite.

(e) (5 points) Show that a finitely generated R-module over a Noetherian ring R has a free resolution with each F_i a free R-module of some *finite* rank, that is, $F_i \cong R^{\beta_i}$ for some positive integers β_i.

(f) (5 points) Show that if R is a principal ideal domain and M is a finitely generated R-module, one can choose a free resolution as in (e) in the form $0 \to F_1 \to F_0 \to M \to 0$.