Math 8669 Introductory Grad Combinatorics
Spring 2010, Vic Reiner
Homework 2- Due April 9, 2010

Hand in at least 6 of the 11 problems.

1. Recall from lecture that a (combinatorial) projective geometry $(\mathcal{P}, \mathcal{L})$ was defined by 4 axioms PG1, PG2, PG3, PG4, and that its *dimension* was defined to be 1 less than the rank of its lattice of flats.

(a) Show that a projective *plane*, that is a projective geometry of dimension 2, has the following equivalent axiomatization:

PP1. Every two distinct points lie on a unique line.
PP2. Every two distinct lines have a unique point in common.
PP3. Every line contains at least 3 points.
PP4. There exist 3 non-collinear points.

(b) Show that in a finite projective plane, all lines have the same number of points, and call this number $q + 1$.

(c) Show that each point lies on $q+1$ lines, and $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$.

2. Given $G = (V, E)$ be a bipartite graph with bipartition $V = A \sqcup B$, let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension in which there exist elements $\{c_{a,b} : \{a, b\} \in E\}$ of $\mathbb{K}$ which are algebraically independent over $\mathbb{F}$, i.e. there are no polynomials in variables $x_{a,b}$ with coefficients in $\mathbb{F}$ which vanish when one plugs in $x_{a,b} = c_{a,b}$. Define vectors $\{v_a : a \in A\}$ in $\mathbb{K}^B$ by

$$v_a := \sum_{b \in B : \{a,b\} \in E} c_{a,b} \epsilon_b$$

where ϵ_b is a standard basis vector in $\mathbb{K}^B$.

Show that a subset $A' \subset A$ can be matched along edges in E to distinct elements of B if and only if the subset $\{v_a : a \in A'\}$ is $\mathbb{K}$-linearly independent. In other words, partial matchings of A into B form the independent sets of a matroid that is representable over $\mathbb{K}$. Such matroids are called *transversal* matroids.
(Hint: Consider the $|A'| \times |B|$ matrix having $\{v_a : a \in A'\}$ as its columns. Under what circumstances does the square submatrix with rows indexed by some subset $B' \subseteq B$ with $|B'| = |A'|$ have non-zero determinant? What does it mean for there to exist such a B'?)

3. Show that the following axiom systems are equivalent to the axiomizations of finite matroids given in lecture (by an exchange closure and/or independent sets):

(a) (Basis axioms) A family $\mathcal{B} \subseteq 2^E$ forms the set of *bases* of a matroid M on the finite set E if

- B1. All sets B in $\mathcal{B}$ have the same cardinality (called the *rank* of M).
- B2. Given $B, B' \in \mathcal{B}$, and $e \in B$, there exists some $e' \in B'$ with $B - \{e\} \cup \{e'\} \in \mathcal{B}$.

(Hint: The bases are supposed to model the maximal independent sets.)

(b) (Circuit axioms) A family $\mathcal{C} \subseteq 2^E$ forms the set of *circuits* of a matroid M on the finite set E if

- C1. The sets in $\mathcal{C}$ form an antichain under inclusion.
- C2. Given $C, C' \in \mathcal{C}$, with $C \neq C'$ and $e \in C \cap C'$, there exists some $C'' \in \mathcal{C}$ with $C'' \subseteq C \cup C' - \{e\}$.

(Hint: The circuits are supposed to model the minimal dependent sets.)

4. Given a graph $G = (V, E)$ with loops and multiple edges allowed, show that for any field $\mathbb{F}$, the matroid associated with the vector configuration in $\mathbb{F}^V$ defined by

$$\{v_e = \epsilon_i - \epsilon_j : e = \{i, j\} \in E(G)\}$$

satisfies the following.

(a) the *closure* $\bar{A}$ of a subset $A \subset E$ consists of all edges $e \in E$ for which there exists a path from the endpoints of e in G using only edges from A.

(b) its *independent sets* are the *subforests* of G, that is, the subsets of edges containing no cycles.

(c) its *bases* are the *spanning subforests* of G, that is, the subforests which consist of one spanning tree in each connected component of G (here *spanning* means connecting all vertices).

(d) its *circuits* are the *simple cycles* of G, that is sequences of edges $e_1, \ldots, e_k$ in E with the property that there are k *distinct* vertices $v_1, \ldots, v_k$ for which $e_i = \{v_i, v_{i+1}\}$ (and the subscripts on v_j's are taken modulo k).

5. Let M be a matroid on E, and choose a linear order $e_1, e_2, \ldots, e_n$ for the elements of E. Given a circuit C of M, with minimum element c in this order, call $C - \{c\}$ a *broken circuit*. Say that a subset $A \subseteq E$ is NBC if it contains no broken circuits $C - \{c\}$.

(a) Show that for any flat F in the geometric lattice of flats $L(M)$, one has

$$\mu_{L(M)}(\varnothing, F) = (-1)^{r(F)} |\{NBC \text{ sets } A \subseteq E : \bar{A} = F\}|.$$

(Hint: Show the right-hand side satisfies the proper identity that defines $\mu_{L(M)}(\varnothing, F)$, via a sign-reversing involution).

(b) The linear ordering on E gives an ordering on the join-irreducibles (=atoms) of the upper-semimodular lattice $L(M)$, and hence induces an R-labelling of $L(M)$ as explained in lecture. Show why the Möbius function calculation this R-labelling provides agrees with part (a), by exhibiting a bijection between NBC bases for M and maximal chains in $L(M)$ whose label set is decreasing.

6. (a) Explain why the partition lattice Π_n is the lattice of flats for the matroid associated with the complete graph K_n on n vertices.

(b) Indexing the atoms E of Π_n by pairs $\{i, j\}$ (i.e. edges of K_n), pick any linear ordering of E in which $\min\{i, j\} > \min\{i', j'\}$ implies that $\{i, j\}$ comes before $\{i', j'\}$. Show that for every triple $i < j < k$, the pair of edges $\{i, j\}, \{i, k\}$ forms a broken circuit. Show furthermore that every broken circuit contains at least one such pair.

(c) Use part (b) and Problem 8(a) to prove that

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!.$$

7. Let M be a matroid on ground set E, and $c : E \to \mathbb{R}$ any assignment of costs $c(e) \in \mathbb{R}$ to each e in E. Show that the following "greedy" algorithm for finding a basis B of M with *minimum total cost* $\sum_{e \in B} c(e)$ always works, that is, it will always terminate with a basis B for M, and B achieves the minimum:

> Start at stage 0 with $I_0 = \varnothing$, an independent set. At stage j, given the independent set I_{j-1}, choose an edge $e_j \in E$ with lowest cost among those such that $I_j := I_{j-1} \cup \{e_j\}$ remains independent. Repeat.

When $M = M(G)$ is a graphic matroid, this is called *Kruskal's algorithm* for finding a minimum cost spanning tree.

8. (a) Let G be a planar graph with a chosen planar embedding, and $G^\perp$ its planar dual with respect to this embedding. Show that $M(G)^\perp = M(G^\perp)$.

(b) For any orientation ω of the edges $E(G)$, let $\omega^\perp$ be the induced orientation of the dual edges $E(G^\perp)$ defined by the *right-hand rule*: if you place the origin at the crossing of some pair of dual edges $e, e^\perp$ in $E(G), E(G^\perp)$ respectively, then the pair of tangent vectors to those edges pointing in the directions of the edges should form a right-handed coordinate system in the plane (like the usual positive x-axis, positive y-axis). Show that ω is acyclic if and only if $\omega^\perp$ is totally cyclic.

9. Prove the following Tutte polynomial evaluation for graphic matroids: if G is a graph with $c(G)$ connected components, and p, q are positive integers, then

$$T_{M(G)}(1-p, 1-q) = (-p)^{-c(G)}(-1)^{|V(G)|} \sum_{(x,y)} (-1)^{|\mathrm{supp}(y)|}$$

where (x, y) runs over all pairs in which

- x is a vertex p-coloring,
- y is a $\mathbb{Z}/q\mathbb{Z}$-valued flow, and
- for every edge $e \in E(G)$, one has $y_e \neq 0$ if and only if x colors e improperly, i.e. $x_v = x_{v'}$ where $e = \{v, v'\}$.

Here $|\mathrm{supp}(y)|$ is the number of edges e with $y_e \neq 0$ or equivalently, the number of edges that are improperly colored by x.

10. (Character theory warm-up) Given two finite groups G, G' and complex representations

$$\rho : G \to GL(V)$$
$$\rho' : G' \to GL(V')$$

define a new representation

$$\rho \otimes \rho' : G \times G' \to GL(V \otimes V')$$

by

$$(\rho \otimes \rho')(g, g')(v \otimes v') = \rho(g)v \otimes \rho'(g')v'.$$

(a) Show $\chi_{\rho\otimes\rho'}(g, g') = \chi_\rho(g) \cdot \chi_{\rho'}(g')$.

(b) Show that $\rho \otimes \rho'$ is irreducible for $G \times G'$ if and only if both ρ, ρ' are irreducibles for G, G'.

(c) If $\{\rho_i\}_{i\in I}, \{\rho'_{i'}\}_{i'\in I'}$ are complete sets of representatives of the (equivalence classes of) irreducible representations of G, G', respectively, show that $\{\rho_i \otimes \rho_{i'}\}_{(i,i') \in I \times I'}$ gives a complete set of representatives for the irreducibles of $G \times G'$.

11. If G is a finite group acting on $[n]$, say that the action is
    - *transitive* if there is only one G-orbit on $[n]$,
    - *doubly transitive* if it is transitive on *ordered pairs*, that is, for every pair $i \neq j$ and $i' \neq j'$ in $[n]$ there exists $g \in G$ with $g(i) = i', g(j) = j'$.

Let χ be the permutation representation/character associated with the G-action.

(a) Show that the action is transitive if and only if $\langle \chi, \chi_{trivial} \rangle = 1$.
(b) Show that the action is doubly transitive if and only if $\chi - \chi_{trivial}$ is irreducible.