Math 8669 Introductory Grad Combinatorics
Spring 2010, Vic Reiner
Homework 3- Friday May 7

Hand in at least 6 of the 10 problems.

1. Construct all the irreducible representations/characters for the symmetric group $\mathfrak{S}_4$ according to the following plan (and using a labelling convention, to be explained later, by partitions λ of the number 4).

There are two obvious (irreducible) 1-dimensional representations, namely

- the *trivial represention*, which we will denote $\chi_{(4)}$, and
- the *sign representation* $\chi_{(1,1,1,1)}$.

(a) Show that the defining permutation representation χ_{def} of $\mathfrak{S}_4$, in which it permutes the coordinates in $\mathbb{C}^4$, decomposes

$$\chi_{def} = \chi_{(4)} \oplus \chi_{(3,1)}$$

where $\chi_{(3,1)}$ is an *irreducible* representation of degree 3.
(b) Show that the permutation representation χ_{pairs} of $\mathfrak{S}_4$, in which it permutes all *unordered pairs* $\{i, j\} \in \binom{[4]}{2}$, decomposes

$$\chi_{pairs} = \chi_{(4)} \oplus \chi_{(3,1)} \oplus \chi_{(2,2)}$$

where $\chi_{(2,2)}$ is an *irreducible* representation of degree 2.
(d) Define $\chi_{(2,1,1)} := \chi_{(1,1,1,1)} \otimes \chi_{(3,1)}$. Check that $\chi_{(2,1,1)}$ is irreducible, and that χ_λ for $\lambda = (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$ give the complete list of irreducible representations of $\mathfrak{S}_4$.
(e) Write down the conjugacy classes and character table for $\mathfrak{S}_4$.

2. Let D_{2n} be the dihedral group of order $2n$, with presentation

$$D_{2n} = \langle s, r : s^2 = r^n = 1, srs = r^{-1} \rangle$$

and defining representation as the symmetries of a regular convex n-gon, in which s is any fixed reflection symmetry, and r is rotation through $\frac{2\pi}{n}$ counterclockwise.

Consider the cyclic (normal) subgroup $C_n = \langle r \rangle$ inside D_{2n}, and its irreducible (degree 1) representations χ_k for $k \in \mathbb{Z}/n\mathbb{Z}$:

$$\begin{array}{rcl} C_n & \xrightarrow{\chi_k} & \mathbb{C}^\times \\ r & \mapsto & \omega^k \end{array}$$

where $\omega = e^{\frac{2\pi i}{n}}$

(a) Compute explicitly the characters $\mathrm{Ind}_{C_n}^{D_{2n}} \chi_k$ as functions $D_{2n} \to \mathbb{C}$. Under what conditions on $k, k' \in \mathbb{Z}/n\mathbb{Z}$ are they the same? For which values of k is it equivalent to the defining representation?
(b) Find all the degree 1 characters of D_{2n}.
(Hint: the answer depends upon n mod 2 and was discussed somewhat in lecture, but please give a complete discussion with proof).
(c) Find all the irreducible characters of D_{2n}, all its conjugacy classes, and write down its character table.

3. Let G be a finite group, and $H \subset G$ a subgroup *of index* 2.

(a) Recall (and explain) why H is a *normal* subgroup, and hence a union of conjugacy classes from G.
(b) Show that a conjugacy class in G which intersects H will either form a single conjugacy class in H, or split into two conjugacy classes in H. Furthermore, show that a conjugacy class C in G does *not* split in H if and only if there exists some $c \in C$ which commutes with some $g \notin H$.
(c) Let χ be an irreducible character/representation for G. Show that $\mathrm{Res}_H^G \chi$ is either irreducible for H, or is the sum of two inequivalent irreducibles for H. Furthermore, show that $\mathrm{Res}_H^G \chi$ is irreducible for H if and only if $\chi(g) \neq 0$ for some $g \notin H$.

4. Use problems 4 and 6 to find the conjugacy classes and irreducible characters for the alternating subgroup $A_4 \subset \mathfrak{S}_4$, and write down its character table.

5. Say that a sequence $a_0, a_1, \ldots, a_n$ of non-negative real numbers is *unimodal* if there exists an index k for which

$$a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_{n-1} \geq a_n.$$

Say that it is *log-concave* if for each $k \in \{2, 3, \ldots, n-1\}$ one has $a_k^2 \geq a_{k-1}a_{k+1}$.
(a) Assuming $a_k > 0$ for all k, show that log-concave implies unimodal, but not conversely.
(b) Show that whenever the a_k are non-negative, any real root of the polynomial $A(x) := \sum_{k=0}^n a_k x^k$ must be non-positive.
(c) Show that if $A(x)$ has *only real roots* and *positive* coefficients a_k, then this coefficient sequence is log-concave, and hence unimodal.
(Hint: there is more than one proof of this. One way factors $A(x)$ into linear factors according to its roots and proceeds combinatorially. Another way applies Rolle's Theorem to deduce that the quadratic

polynomial

$$\frac{d^{n-k-1}}{dx^{n-k-1}}\left(x^{n-k+1}\cdot\left[\frac{d^{k-1}}{dx^{k-1}}A(x)\right]_{x\mapsto x^{-1}}\right)$$

also has only real roots, so one can look at its discriminant, giving an even stronger inequality than log-concavity).

(d) Recall that the signless Stirling number of the 1st kind $c(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k cycles, and the Stirling number of the 2nd kind $S(n,k)$ is the number of partitions of the set $[n]$ into k blocks. Show that both sequences $(c(n,k))_{k=1}^n, (S(n,k))_{k=1}^n$ are log-concave, and hence unimodal.

(Hint: $\sum_k s(n,k)x^k$ has a simple factored expression that shows it has real roots. For $A_n(x)=\sum_k S(n,k)x^k$, show that $A_n(x)=xA_{n-1}(x)+xA'_{n-1}(x)$ and use this to give a proof of real-rootedness by induction on n involving the *interlacing* of the roots of $A_n(x), A_{n-1}(x)$ (that is, between every pair of roots of $A_{n-1}(x)$ there is one for $A_n(x)$, and then two more on the extreme right and extreme left).

6. Define two infinite upper-triangular matrices E,H by

$$E_{i,j}=(-1)^{j-i}e_{j-i}$$
$$H_{i,j}=h_{j-i}$$

where the elementary and complete symmetric functions e_r, h_r are both taken to be 1 for $r=0$, and 0 for $r<0$.

(a) Explain why the matrices E,H are inverse to each other.

(b) Specialize the $e_i(x_1,x_2,\ldots), h_i(x_1,x_2,\ldots)$ to $x_1=x_2=\cdots=x_n=1$ and $x_i=0$ for $i>n$. What do e_k, h_k specialize to, and what identity results from E,H being inverse to each other?

(c) Generalize part (b) by answering the same questions for the specialization $x_i=q^{i-1}$ for $i\le n$, $x_i=0$ for $i>n$.

(d) Answer the same questions for the specialization $x_i=i-1$ for $i\le n$, $x_i=0$ for $i>n$.

(Hint for part (d): Stirling numbers are relevant.)

7. Let $\mathbf{b}=(b_1,b_2,\ldots,b_{mn+1})$ be a sequence of length $mn+1$ in the alphabet $\{1,2,\ldots\}$. Show that $\mathbf{b}$ contains either a weakly increasing subsequence of length $m+1$, or a strictly decreasing subsequence of length $n+1$.

8. Write down explicitly the entire character table for the symmetric group $\mathfrak{S}_5$ using the Murnaghan-Nakayama rule.

9. Show that the irreducible character χ^λ of the symmetric group $\mathfrak{S}_n$ has $\chi^\lambda(w) = 0$ whenever the side-length of λ's Durfee square (the largest square contained inside the Ferrers diagram of λ) is larger than the number of cycles of w.

10. Prove that the character table for $\mathfrak{S}_n$ has determinant

$$\pm \prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i.$$