**Math 8669 Introductory Grad Combinatorics, 2nd semester**
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**Group representations homework**

1. Given two finite groups $G, G'$ and complex representations

$$\rho : G \to GL(V)$$
$$\rho' : G' \to GL(V')$$

define a new representation

$$\rho \otimes \rho' : G \times G' \to GL(V \otimes V')$$

by

$$(\rho \otimes \rho')(g, g')(v \otimes v') = \rho(g)v \otimes \rho'(g')v'.$$

(a) Show $\chi_{\rho\otimes\rho'}(g, g') = \chi_\rho(g) \cdot \chi_{\rho'}(g')$.

(b) Show that $\rho \otimes \rho'$ is irreducible for $G \times G'$ if and only if both $\rho, \rho'$ are irreducibles for $G, G'$.

(c) If $\{\rho_i\}_{i \in I}, \{\rho'_{i'}\}_{i' \in I'}$ are complete sets of representatives of the (equivalence classes of) irreducible representations of $G, G'$, respectively, show that $\{\rho_i \otimes \rho_{i'}\}_{(i,i') \in I \times I'}$ gives a complete set of representatives for the irreducibles of $G \times G'$.

2. If $G$ is a finite group acting on $[n]$, say that the action is

- *transitive* if there is only one $G$-orbit on $[n]$,
- *doubly transitive* if it is transitive on *ordered pairs*, that is, for every pair $i \neq j$ and $i' \neq j'$ in $[n]$ there exists $g \in G$ with $g(i) = i', g(j) = j'$.

Let $\chi$ be the permutation representation/character associated with the $G$-action.

(a) Show that the action is transitive if and only if $\langle \chi, \chi_{trivial} \rangle = 1$.
(b) Show that a transitive action is doubly transitive if and only if $\chi - \chi_{trivial}$ is irreducible.

3. Construct all the irreducible representations/characters for the symmetric group $\mathfrak{S}_4$ according to the following plan (and using a labelling convention, to be explained later, by partitions $\lambda$ of the number 4).

There are two obvious (irreducible) 1-dimensional representations, namely

- the *trivial represention*, which we will denote $\chi_{(4)}$, and
- the *sign representation* $\chi_{(1,1,1,1)}$.

(a) Show that the defining permutation representation $\chi_{def}$ of $\mathfrak{S}_4$, in which it permutes the coordinates in $\mathbb{C}^4$, decomposes

$$\chi_{def} = \chi_{(4)} \bigoplus \chi_{(3,1)}$$

where $\chi_{(3,1)}$ is an *irreducible* representation of degree 3.
(b) Show that the permutation representation $\chi_{pairs}$ of $\mathfrak{S}_4$, in which it permutes all *unordered pairs* $\{i, j\} \in \binom{[4]}{2}$, decomposes

$$\chi_{pairs} = \chi_{(4)} \bigoplus \chi_{(3,1)} \bigoplus \chi_{(2,2)}$$

where $\chi_{(2,2)}$ is an *irreducible* representation of degree 2.
(d) Define $\chi_{(2,1,1)} := \chi_{(1,1,1,1)} \otimes \chi_{(3,1)}$. Check that $\chi_{(2,1,1)}$ is irreducible, and that $\chi_\lambda$ for $\lambda = (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$ give the complete list of irreducible representations of $\mathfrak{S}_4$.
(e) Write down the conjugacy classes and character table for $\mathfrak{S}_4$.

4. Let $D_{2n}$ be the dihedral group of order $2n$, with presentation

$$D_{2n} = \langle s, r : s^2 = r^n = 1, srs = r^{-1} \rangle$$

and defining representation as the symmetries of a regular convex $n$-gon, in which $s$ is any fixed reflection symmetry, and $r$ is rotation through $\frac{2\pi}{n}$ counterclockwise.

Consider the cyclic (normal) subgroup $C_n = \langle r \rangle$ inside $D_{2n}$, and its irreducible (degree 1) representations $\chi_k$ for $k \in \mathbb{Z}/n\mathbb{Z}$:

$$\begin{array}{rcl} C_n & \overset{\chi_k}{\to} & \mathbb{C}^\times \\ r & \mapsto & \omega^k \end{array}$$

where $\omega = e^{\frac{2\pi i}{n}}$

(a) Compute explicitly the characters $\mathrm{Ind}_{C_n}^{D_{2n}} \chi_k$ as functions $D_{2n} \to \mathbb{C}$. Under what conditions on $k, k' \in \mathbb{Z}/n\mathbb{Z}$ are they the same? For which values of $k$ is it equivalent to the defining representation?
(b) Find all the degree 1 characters of $D_{2n}$.
(Hint: the answer depends upon $n$ mod 2 and was discussed somewhat in lecture, but please give a complete discussion with proof).
(c) Find all the irreducible characters of $D_{2n}$, all its conjugacy classes, and write down its character table.

5. Let $G$ be a finite group, and $H \subset G$ a subgroup *of index* 2.

(a) Recall (and explain) why $H$ is a *normal* subgroup, and hence a union of conjugacy classes from $G$.
(b) Show that a conjugacy class in $G$ which intersects $H$ will either form a single conjugacy class in $H$, or split into two conjugacy classes

in $H$. Furthermore, show that a conjugacy class $C$ in $G$ does *not* split in $H$ if and only if there exists some $c \in C$ which commutes with some $g \notin H$.
(c) Let $\chi$ be an irreducible character/representation for $G$. Show that $\mathrm{Res}^G_H \chi$ is either irreducible for $H$, or is the sum of two inequivalent irreducibles for $H$. Furthermore, show that $\mathrm{Res}^G_H \chi$ is irreducible for $H$ if and only if $\chi(g) \neq 0$ for some $g \notin H$.

6. Use problems 4 and 6 to find the conjugacy classes and irreducible characters for the alternating subgroup $A_4 \subset \mathfrak{S}_4$, and write down its character table.

7. Write down explicitly the entire character table for the symmetric group $\mathfrak{S}_5$ using the Murnaghan-Nakayama rule.

8. Show that the irreducible character $\chi^\lambda$ of the symmetric group $\mathfrak{S}_n$ has $\chi^\lambda(w) = 0$ whenever the side-length of $\lambda$'s Durfee square (the largest square contained inside the Ferrers diagram of $\lambda$) is larger than the number of cycles of $w$.

9. Prove that the character table for $\mathfrak{S}_n$ has determinant

$$\pm \prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i.$$

(Hint: Depending upon your approach, Exercise 10 could be useful.)
The absolute value of this determinant gives the index of the sublattice of virtual characters $\chi_\rho - \chi_{\rho'}$ of $\mathfrak{S}_n$ inside the lattice of all $\mathbb{Z}$-valued class functions $f : \mathfrak{S}_n \to \mathbb{Z}$.

10. Given a partition $\lambda$ and $j \geq 1$, let $m_j(\lambda)$ denote the multiplicity of the part $j$ in $\lambda$, so that $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \cdots)$.

(a) Prove these two generating function identities (*) and (**):
For each fixed positive integeer $j = 1, 2, 3, \ldots$, one has

$$\sum_\lambda q^{|\lambda|} \cdot m_j(\lambda) \quad \overset{(*)}{=} \quad \frac{q^j}{1-q^j} \prod_{i=1}^{\infty} \frac{1}{1-q^i} \quad \overset{(**)}{=} \quad \sum_\lambda q^{|\lambda|} \cdot \#\{k : m_k \geq j\}.$$

(b) From (a), deduce that with $j$ fixed as before, for all $n \geq 0$ one has

$$\sum_{\lambda \vdash n} m_j(\lambda) = \sum_{\lambda \vdash n} \#\{k : m_k \geq j\}.$$

(c) From (b), deduce that for all $n \geq 0$ one has

$$\prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i = \prod_{\lambda \vdash n} \prod_{j=1}^{\infty} m_j(\lambda)!.$$

(d) From (c), deduce that

$$\prod_{\lambda \vdash n} z_\lambda = \left( \prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} \lambda_i \right)^2$$

where recall that

$$z_\lambda := 1^{m_1(\lambda)} \cdot m_1(\lambda)! \cdot 2^{m_2(\lambda)} \cdot m_2(\lambda)! \cdot 3^{m_2(\lambda)} \cdot m_3(\lambda)! \cdots$$