Bruhat order on quotients W^J and tableau criterion

Recall the unique factorizations for $J \subseteq S$

$$\begin{array}{ccc} W & \longleftrightarrow & W^J \times W_J \\ w & \longmapsto & (v, u) \end{array}$$

where $w = v \cdot u$

and $\ell(w) = \ell(v) + \ell(u)$

PROPOSITION:

The surjections $W \xrightarrow{p^J} \twoheadrightarrow W^J$, $w \longmapsto v$

are all order-preserving for Bruhat order,

i.e. $w \leq w' \Rightarrow p^J(w) \leq p^J(w')$

proof: Induct on $\ell(w')$.

BASE CASE: $w' \in W^J$, so $p^J(w') = w'$.

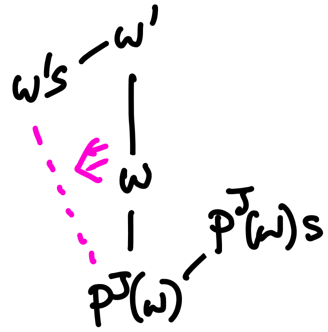
Then $p^J(w) \leq w \leq w' = p^J(w')$, so done.

because $w = u \cdot v$, $u = p^J(w)$ reduced

$= s_{i_1} \cdots s_{i_{\ell(u)}} \cdot s_{j_1} \cdots s_{j_{\ell(v)}}$

INDUCTIVE STEP: $w' \notin W^J$, so $\exists\, s \in J$ with $w's < w'$.

Apply Lifting Property here



to conclude $p^J(w) \le w's$.

So by induction, $p^J(p^J(w)) \le p^J(w's)$

$$\begin{array}{ccc} p^J(p^J(w)) & \le & p^J(w's) \\ \| & & \| \\ p^J(w) & & p^J(w') \end{array} \qquad \blacksquare$$

Pleasantly, on certain quotients W^J, Bruhat order is much simpler to check, and quite familiar...

PROPOSITION: For $W = \mathfrak{S}_n = W(\circ\!-\!\circ \cdots -\!\circ)$ and $J = S \setminus \{s_k\}$ for $k = 1, 2, \ldots, n-1$,

$$W^J \longleftrightarrow \binom{\{1,2,\ldots,n\}}{k} \longleftrightarrow \left\{ \begin{array}{l} \text{Ferrers diagrams } \lambda \\ \text{in } k \times (n-k) \end{array} \right\}$$

$$w = \left(\begin{array}{cccc|ccc} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ w_1 < & w_2 < & \cdots < & w_k & w_{k+1} < & \cdots < & w_n \end{array} \right) \longmapsto \{w_1, w_2, \ldots, w_k\} \longmapsto \lambda$$

whose right boundary vertical steps are $\{w_1, w_2, \ldots, w_k\}$

and

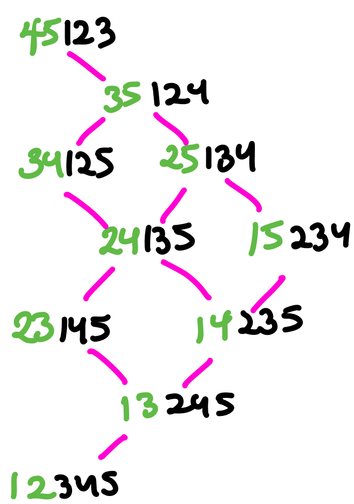
$$u \le w \iff \begin{array}{c} u_1 \le w_1 \\ \vdots \\ u_k \le w_k \end{array} \iff \lambda(u) \le \lambda(w)$$

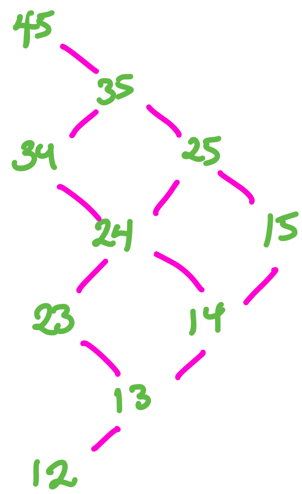
(Bruhat on W^J) (Gale order) (Young's lattice)

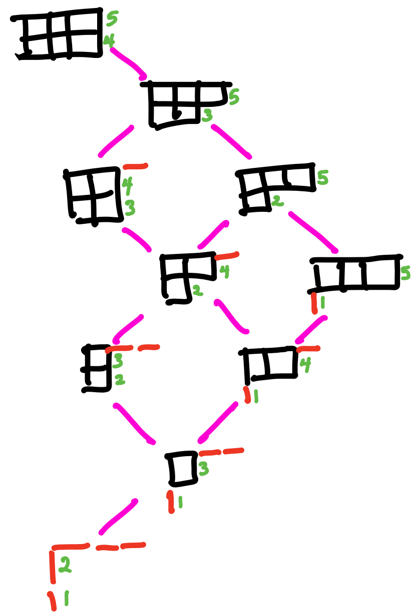
EXAMPLE $W = \mathfrak{S}_5 = W(s_1 — s_2 — s_3 — s_4)$

$J = S \setminus \{s_2\}$, $W_J = (s_1 \;\; s_3 — s_4) = \mathfrak{S}_2 \times \mathfrak{S}_3$

W^J







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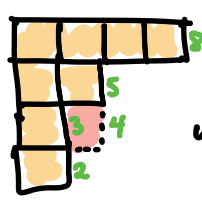
proof sketch:

Checking the maps are bijections is straightforward.

Not hard to check that if $u \xrightarrow{t} w$ in Bruhat graph then $\{u_1, \dots, u_k\} \leq_{Gale} \{w_1, \dots, w_k\}$, so $\lambda(u) \leq \lambda(w)$.

Conversely, if $\lambda < \mu$ in Young's Lattice
$\lambda = \lambda(u)$, $\mu = \lambda(w)$



$w = (3,4) \cdot u$ and $\ell(w) > \ell(u)$

$\{u_1, \dots, u_k\} = \{2,3,5,8\}$ 2358 1467

$\{w_1, \dots, w_k\} = \{2,4,5,8\}$ 2458 1367

Can exhibit $w = s_i u$ and $\ell(w) > \ell(u)$, showing $u \leq w$ ∎

This will have a nice consequence for Bruhat on $\mathfrak{S}_n$:

THEOREM (Tableau Criterion, (B-B Thm 2.6.3))

In $\mathfrak{S}_n$, $u \le w$ in Bruhat order

$$\Updownarrow$$

$\{u_1, \ldots, u_k\} \le_{Gale} \{w_1, \ldots, w_k\}$ for all $k \in \mathrm{Des}(u) = D_R(u)$

EXAMPLE In $\mathfrak{S}_9$, to check

$u = 368\underbrace{\cdot 47 \cdot 59 \cdot}_{\mathrm{Des}(u)}12 \overset{?}{<} w = 694287531$

compare entrywise ...

$$\begin{matrix} 368 \\ \boxed{3}4678 \\ \boxed{3}456789 \end{matrix} \quad \overset{?}{\le} \quad \begin{matrix} 469 \\ \boxed{2}4689 \\ \boxed{2}456789 \end{matrix}$$

failure

$\Longrightarrow u \not\le w$ in Bruhat.

The Tableau Criterion is a special case of ...

THEOREM (B-B Thm. 2.6.1) Given subsets $\{J_i\}$ of S with $I := \bigcap_i J_i$ one has for any $u \in W^I$, $w \in W$

$$u \leq w \iff p^{J_i}(u) \leq p^{J_i}(w) \quad \forall i$$

EXAMPLES:

(1) If $W = \mathfrak{S}_n$, $\{J_i\} = \{S \setminus \{s_k\}\}$ $\forall k \in Des(u)$,

so $I = \bigcap_i J_i = S \setminus Des(u)$, $u \in W^I$,

and this is exactly Tableau Criterion

(2) If $I = \bigcap_i J_i = \emptyset$, so $W^I = W^\emptyset = W$,

then $u \leq w \iff p^{J_i}(u) \leq p^{J_i}(w)$ $\forall i$

sketch proof of THEOREM (see B-B pp 45-46):

Forward implication we know:

$$u \leq w \implies p^{J_i}(u) \leq p^{J_i}(w) \quad \forall J_i$$

(with no assumption of $u \in W^I$ for $I = \bigcap_i J_i$ needed)

For the reverse implication

$$p^{J_i}(u) \le p^{J_i}(w) \quad \Rightarrow \quad u \le w,$$

induct on $\ell(w)$.

BASE CASE $\ell(w)=0 \Rightarrow w=e$

$$\Rightarrow p^{J_i}(u) \le p^{J_i}(w) = e \quad \forall i$$

$$\Rightarrow p^{J_i}(u) = e \quad \forall i$$

$$\Rightarrow u \in W_{J_i} \ \forall i \Rightarrow u \in W_I$$

$$\Rightarrow u \in W_I \cap W^I$$

$$\Rightarrow u = e \quad (=w).$$

INDUCTIVE STEP $\ell(w) \ge 1$, so pick $s \in S$ with $sw < w$.

Now rely on a general...

CLAIM: $\forall J \subset S$

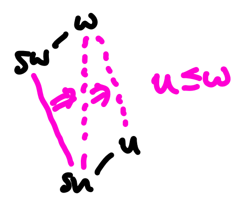
$$p^J(u) \le p^J(w) \Rightarrow \begin{cases} p^J(su) \le p^J(sw) & \text{if } su < u \\ p^J(u) \le p^J(sw) & \text{if } su > u \end{cases}$$

allowing one to finish in 2 cases:

- $su < u \xRightarrow{\text{CLAIM}} p^{J_i}(su) \le p^{J_i}(sw) \ \forall i$

  $\xRightarrow[\text{induction,}]{} su \le sw \xRightarrow[\text{Lifting}]{}$

  Since $su \in W^I$ again, $\ell(sw) < \ell(w)$



- $su > u \xRightarrow{\text{CLAIM}} p^{J_i}(u) \le p^{J_i}(sw) \ \forall i$

  $\xRightarrow[\text{induction}]{} u \le sw < w$

But then proving the CLAIM is a slightly painful case-by-case check, based on $su \underset{>u}{<u}$ and $s\,p^J(w) \underset{>}{\lessgtr} p^J(w)$? $\blacksquare$