# Solomon's Theorem & consequences (Kane §VI.22)

Shephard & Todd (1955) observed the following stronger generating function result, but could only prove it case-by-case via their classification; L. Solomon (1963) gave an insightful proof.

**THEOREM** For a $\mathbb{C}$-refl'n group $G \subset GL_n(\mathbb{C})$ with $\mathbb{C}[x]^G = \mathbb{C}[f_1, \ldots, f_n]$, one has of degrees: $d_1, \ldots, d_n$

$$\sum_{g \in G} u^{\dim(V^g)} = \prod_{i=1}^{n} (u + (d_i - 1)).$$

set $u=1$

$$|G| = d_1 d_2 \cdots d_n$$

extract coeff of $u^{n-1}$

$$\#\text{refl'ns in } G = \sum_{i=1}^{n} (d_i - 1)$$

$W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$

Since $\mathbb{C}[x]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$
degrees: $1, 2, \ldots, n$

$$\sum_{w \in \mathfrak{S}_n} u^{\#\text{cycles}(w)} = \prod_{i=1}^{n} (u + i - 1)$$

$$\sum_{k=1}^{n} \underbrace{c(n,k)}_{\substack{\text{signless Stirling numbers}\\ \text{of 1st kind}}} u^k = u(u+1)(u+2)\cdots(u+n-1)$$

Solomon's proof considered $G$ acting on these...

DEF'N: $\bigwedge(V^*) = \bigwedge V^* = \bigwedge\langle y_1,\ldots,y_n\rangle =: \bigwedge\langle \underline{y}\rangle$

= exterior algebra on a $\mathbb{C}$-basis $y_1,\ldots,y_n$ for $V^*$

So $y_i \wedge y_j = -y_j \wedge y_i$ anti-commutative

and $\bigwedge V^* = \bigoplus_{k=0}^{n} \bigwedge^k V^*$

with $\mathbb{C}$-basis: $\{y_{i_1}\wedge y_{i_2}\wedge\cdots\wedge y_{i_k} : 1\le i_1<\cdots<i_k\le n\}$

e.g. $n=3$ $\bigwedge\langle y_1,y_2,y_3\rangle = \mathrm{span}_{\mathbb{C}}\{1, \; y_1, y_2, y_3, \; y_1\wedge y_2, y_1\wedge y_3, y_2\wedge y_3, \; y_1\wedge y_2\wedge y_3\}$

| $\bigwedge^0 V^*$ | $\bigwedge^1 V^*$ | $\bigwedge^2 V^*$ | $\bigwedge^3 V^*$ |
|---|---|---|---|
| $1$ | $y_1, y_2, y_3$ | $y_1\wedge y_2, y_1\wedge y_3, y_2\wedge y_3$ | $y_1\wedge y_2\wedge y_3$ |

Then one also has the superalgebra on $V^*$ or polynomial tensor exterior algebra (or differential forms on $V$ with polynomial coeffs)

$S(V^*)\otimes_{\mathbb{C}} \bigwedge V^* = \mathbb{C}[\underline{x}]\otimes_{\mathbb{C}} \bigwedge\langle\underline{y}\rangle := \mathbb{C}[x_1,\ldots,x_n]\otimes_{\mathbb{C}}\bigwedge\langle y_1,\ldots,y_n\rangle$

= free $\mathbb{C}[\underline{x}]$-module on $\mathbb{C}[\underline{x}]$-basis $\{y_{i_1}\wedge\cdots\wedge y_{i_k} : 1\le i_1<\cdots<i_k\le n\}$

Typical element of $\mathbb{C}[\underline{x}]\otimes\bigwedge^k\langle\underline{y}\rangle$ is $\sum f_{i_1\ldots i_k}(\underline{x})\,\underbrace{y_{i_1}\wedge\cdots\wedge y_{i_k}}_{\text{call this } \underline{y}_I \text{ for } I=\{i_1<\cdots<i_k\}}$ (omitting $\otimes$ symbol)

**THEOREM** (Solomon 1963) When a $\mathbb{C}$ refln group $G$ with $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1, \ldots, f_n]$ acts on $\mathbb{C}[\underline{x}] \otimes \wedge\langle \underline{y} \rangle$ via linear substitutions, the $G$-invariant subalgebra

$(\mathbb{C}[\underline{x}] \otimes \wedge^k \langle \underline{y} \rangle)^G$ is a free $\mathbb{C}[\underline{x}]^G$-module on basis $\{ df_{i_1} \wedge \ldots \wedge df_{i_k} : 1 \le i_1 < \ldots < i_k \le n \}$

where
$$df \underset{\text{DEF'N}}{:=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes y_i = \frac{\partial f}{\partial x_1} \otimes y_1 + \ldots + \frac{\partial f}{\partial x_n} \otimes y_n.$$

In other words,
$$(\mathbb{C}[\underline{x}] \otimes \wedge\langle \underline{y} \rangle)^G \cong \underbrace{\wedge_{\mathbb{C}[\underline{x}]^G} \langle df_1, \ldots, df_n \rangle}_{\substack{\text{exterior algebra over } \mathbb{C}[\underline{x}]^G \\ \text{on } n \text{ generators}}}$$

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**EXAMPLE** $G = I_2(m) \cong G(m,m,2)$
$$= \left\{ \begin{bmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{bmatrix}, \begin{bmatrix} 0 & \zeta^k \\ \zeta^{-k} & 0 \end{bmatrix} : k = 0, 1, \ldots, m-1 \right\}$$

has $\mathbb{C}[\underline{x}]^G = \mathbb{C}[\underbrace{x_1^m + x_2^m}_{f_1}, \underbrace{x_1 x_2}_{f_2}]$

Then $\mathbb{C}[\underline{x}] \otimes \wedge\langle \underline{y} \rangle$ = free $\mathbb{C}[x_1, x_2]$-module on basis
$$\{1, \; y_1, \; y_2, \; y_1 \wedge y_2\}$$

and Solomon's Theorem says

$$\left(\mathbb{C}[\underline{x}] \otimes \wedge\langle \underline{y} \rangle\right)^G = \text{free } \mathbb{C}[x_1^m + x_2^m,\ x_1x_2]\text{-module on basis}$$

$\{\ 1$

$df_1 =$

$$\frac{\partial f_1}{\partial x_1} \otimes y_1 + \frac{\partial f_1}{\partial x_2} \otimes y_2 = m\left(x_1^{m-1} y_1 + x_2^{m-1} y_2\right)$$

$df_2 =$

$$\frac{\partial f_2}{\partial x_1} \otimes y_1 + \frac{\partial f_2}{\partial x_2} \otimes y_2 = x_2 y_1 + x_1 y_2$$

$df_1 \wedge df_2 =$

$$m\left(x_1^{m-1} y_1 + x_2^{m-1} y_2\right) \wedge \left(x_2 y_1 + x_1 y_2\right)$$

$$= m\left(x_1^m y_1 \wedge y_2 + x_2^m y_2 \wedge y_1\right)$$

$$= m\left(x_1^m - x_2^m\right) y_1 \wedge y_2 \ \}$$

$\mathbb{C}[\underline{x}]^G$-basis for $\left(\mathbb{C}[\underline{x}] \otimes \wedge^0 \langle \underline{y} \rangle\right)^G$

$\mathbb{C}[\underline{x}]^G$-basis for $\left(\mathbb{C}[\underline{x}] \otimes \wedge^1 \langle \underline{y} \rangle\right)^G$

$\mathbb{C}[\underline{x}]^G$-basis for $\left(\mathbb{C}[\underline{x}] \otimes \wedge^2 \langle \underline{y} \rangle\right)^G$

It's useful to track Hilbert series for these objects with separate gradings in $\mathbb{C}[\underline{x}]$ using a variable $q$
$\Lambda\langle\underline{y}\rangle$ using a variable $t$

So $\mathbb{C}[\underline{x}]\otimes\Lambda\langle\underline{y}\rangle = \bigoplus_{d,k=0}^{\infty} \mathbb{C}[\underline{x}]_d \otimes \Lambda^k\langle\underline{y}\rangle$

has $\mathrm{Hilb}(\mathbb{C}[\underline{x}]\otimes\Lambda\langle\underline{y}\rangle; q,t) := \sum_{d,k=0}^{\infty} q^d t^k \dim_{\mathbb{C}} \mathbb{C}[\underline{x}]_d \otimes \Lambda^k\langle\underline{y}\rangle$

$$= \underbrace{(1+q+q^2+\dots)}_{x_1}\cdots\underbrace{(1+q+q^2+\dots)}_{x_n}\underbrace{(1+t)}_{y_1}\cdots\underbrace{(1+t)}_{y_n} = (1+q^1+q^2+\dots)^n (1+t^1)^n = \frac{(1+t)^n}{(1-q)^n}$$

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COROLLARY (to Solomon's Thm.) $G$ a $\mathbb{C}$-refl'n group has

$\mathrm{Hilb}\left((\mathbb{C}[\underline{x}]\otimes\Lambda\langle\underline{y}\rangle)^G; q,t\right) =$

$$\mathrm{Hilb}(\underbrace{\mathbb{C}[\underline{x}]^G}_{\mathbb{C}[f_1,\dots,f_n]}, q)\cdot \sum_{k=0}^{n} \sum_{1\le i_1<\dots<i_k\le n} \underbrace{tq^{d_{i_1}-1}\cdot tq^{d_{i_2}-1}\cdots tq^{d_{i_k}-1}}_{\substack{\text{the } (q,t)\text{-monomial} \\ \text{tracking } \mathbb{C}[\underline{x}]^G\text{-basis} \\ \text{element} \\ df_{i_1}\wedge\dots\wedge df_{i_k}}}$$

$$= \frac{(1+tq^{d_1-1})(1+tq^{d_2-1})\cdots(1+tq^{d_n-1})}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})} = \prod_{i=1}^{n} \frac{1+tq^{d_i-1}}{1-q^{d_i}}$$

**EXAMPLE** For $G = I_2(m) \cong G(m,m,2)$

with $(d_1, d_2) = (m, 2)$

$$m = \deg(x_1^m + x_2^m), \qquad 2 = \deg(x_1 x_2)$$

$$\mathrm{Hilb}\left(\left(\mathbb{C}[\underline{x}] \otimes \wedge\langle \underline{y} \rangle\right)^G; q, t\right) = \frac{(1 + t\overbrace{q^1}^{x_2 y_1 + x_1 y_2})(1 + t\overbrace{q^{m-1}}^{x_1^{m-1} y_1 + x_2^{m-1} y_2})}{(1 - \underbrace{q^2}_{x_1 x_2})(1 - \underbrace{q^m}_{x_1^m + x_2^m})}$$

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Solomon's Thm makes magic happen when combined with ...

**Super-Molien's THEOREM:** For any finite group $G \subset GL_n(\mathbb{C})$,

$$\mathrm{Hilb}\left(\left(\mathbb{C}[\underline{x}] \otimes \wedge\langle \underline{y} \rangle\right)^G; q, t\right) = \frac{1}{|G|} \sum_{g \in G} \frac{\det(I_n + t \cdot g)}{\det(I_n - q \cdot g)}$$

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**proof:** Same proof as before: if we change basis to have $g$ act triangularly on $V^*$, so

$$g = \begin{array}{c|c} & x_1 \;\text{—}\; x_n \\ \begin{array}{c} x_1 \\ | \\ x_n \end{array} & \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \end{array} \quad \text{and same on} \quad \begin{array}{c|c} & y_1 \;\text{—}\; y_n \\ \begin{array}{c} y_1 \\ | \\ y_n \end{array} & \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \end{array}$$

then it also acts triangularly on each $\mathbb{C}[\underline{x}]_d \otimes \wedge^k\langle \underline{y} \rangle$

with eigenvalues $\left\{ \lambda_1^{a_1} \cdots \lambda_n^{a_n} \cdot \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} : \begin{array}{l} a_1 + \cdots + a_n = d \\ 1 \le i_1 < \cdots < i_k \le n \end{array} \right\}$

So can check that the bigraded trace of $g$

$$\sum_{d,k} q^d t^k \, \mathrm{Trace}\left( \mathbb{C}[\underline{x}]_d \otimes \wedge^k \langle \underline{y} \rangle \xrightarrow{g} \mathbb{C}[\underline{x}]_d \otimes \wedge^k \langle \underline{y} \rangle \right)$$

$$= \frac{(1+t\lambda_1)\cdots(1+t\lambda_n)}{(1-q\lambda_1)\cdots(1-q\lambda_n)} = \frac{\det(I_n + t\cdot g^{-1})}{\det(I_n - q\cdot g^{-1})}$$

and then

$$\mathrm{Hilb}(\mathbb{C}[\underline{x}] \otimes \wedge \langle \underline{y} \rangle; q,t) = \frac{1}{|G|} \sum_{g\in G} \frac{\det(I_n + t\cdot g^{-1})}{\det(I_n - q\cdot g^{-1})}$$

$$= \frac{1}{|G|} \sum_{g\in G} \frac{\det(I_n + t\cdot g)}{\det(I_n - q\cdot g)} \quad \blacksquare$$

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**COROLLARY:** $\displaystyle\sum_{g\in G} u^{\dim(V^g)} = \prod_{i=1}^{n} \left(u + (d_i - 1)\right)$

for a $\mathbb{C}$-refn group $G$ with $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1, \ldots, f_n]$ of degs $d_1, \ldots, d_n$ :

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**proof:** Solomon + super-Molien $\Longrightarrow$

$$\frac{1}{|G|} \sum_{g\in G} \frac{\det(I_n + t\cdot g)}{\det(I_n - q\cdot g)} = \prod_{i=1}^{n} \frac{1 + t q^{d_i - 1}}{1 - q^{d_i}}$$

$$\Vert$$

$$\frac{1}{|G|} \sum_{g\in G} \prod_{i=1}^{n} \frac{1 + t\lambda_i(g)}{1 - q\lambda_i(g)}$$ where $g$ has eigenvalues $\lambda_1(g), \lambda_2(g), \ldots, \lambda_n(g)$

Substitute $t = u(1-q)-1$

$$\frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{n}\frac{1+(u(1-q)-1)\lambda_i(g)}{1-q\lambda_i(g)} = \prod_{i=1}^{n}\frac{1+(u(1-q)-1)q^{d_i-1}}{1-q^{d_i}}$$

$\Vert$ ← since $\dim(V^{\omega}) = \#\{\lambda_i(g)=1\}$

$$\frac{1}{|G|}\sum_{g\in G}\left(\prod_{\lambda_i(g)\neq 1}\frac{1+(u(1-q)-1)\lambda_i(g)}{1-q\lambda_i(g)}\right)\cdot\left(\frac{1+(u(1-q)-1)}{1-q}\right)^{\dim(V^{\omega})}$$

$$\Vert$$

$$\prod_{i=1}^{n}\frac{1-q^{d_i-1}+u(1-q)q^{d_i}}{1-q^{d_i}}$$

$\Vert$ cancel some $(1-q)$s

$$\prod_{i=1}^{n}\frac{[d_i-1]_q+uq^{d_i}}{[d_i]_q}$$

take $\lim_{q\to 1}$

take $\lim_{q\to 1}$

$$\frac{1}{|G|}\sum_{g\in G}u^{\dim(V^{\omega})} = \prod_{i=1}^{n}\frac{d_i-1+u}{d_i} \quad ▨$$

So how to prove...

**THEOREM (Solomon)** A refln group $G$ with $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1, \ldots, f_n]$ has $\left(\mathbb{C}[\underline{x}] \otimes \Lambda^k\langle \underline{y}\rangle\right)^G$ a free $\mathbb{C}[\underline{x}]^G$-module on basis $\{df_{i_1} \wedge \ldots \wedge df_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}$.

Let's start by noting that each $df_i$ and hence $df_{i_1} \wedge \ldots \wedge df_{i_k}$ are actually $G$-invariant, due to ...

**PROPOSITION:** Every $g \in GL_n(\mathbb{C})$ commutes with $d$ when acting on $\mathbb{C}[\underline{x}] \otimes \Lambda\langle \underline{y}\rangle$.

*proof:* It reduces to checking $gd = dg$ when acting on any homog. $h(\underline{x}) \otimes 1 \in \mathbb{C}[\underline{x}] \otimes \Lambda^0\langle \underline{y}\rangle$, since $d(h(\underline{x}) \otimes \underline{y}_I) = d(h(\underline{x}) \otimes 1) \cdot (1 \otimes \underline{y}_I)$.

Checking it for $h(\underline{x}) \otimes 1$ can be done using induction on $\deg(h)$, since one has Leibniz rule:

$$d(h_1 \cdot h_2 \otimes 1) = d(h_1) \cdot h_2 \otimes 1 + h_1 \, d(h_2) \otimes 1$$

And then it's easy to check when $\deg(h) = 1$ by noting $d(x_{i_0} \otimes 1) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_{i_0} \otimes y_i = 1 \otimes y_{i_0}$ ▨

The $G$-invariance of

$$df_1 \wedge \dots \wedge df_n = \left(\sum_i \frac{\partial f_1}{\partial x_i} y_i\right) \wedge \dots \wedge \left(\sum_i \frac{\partial f_n}{\partial x_i} y_i\right)$$

$$= \underbrace{\det\left(\left[\frac{\partial f_j}{\partial x_i}\right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}\right)} \cdot \; y_1 \wedge \dots \wedge y_n$$

call this $J$ $:=$ Jacobian determinant of $f_1, \dots, f_n$

DEF'N

together with the fact that

$$g(y_1 \wedge \dots \wedge y_n) = (g^{-1} y_1) \wedge \dots \wedge (g^{-1} y_n) = \det(g)^{-1} \; y_1 \wedge \dots \wedge y_n$$

("$y_1 \wedge \dots \wedge y_n$ is a $\det^{-1}$-relative invariant")

implies $g(J) = \det(g) \cdot J$

("$J$ is a $\det$-relative invariant")

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Something much stronger about $J$ is true, and gets used in several proofs ...

**PROPOSITION:** Given a $\mathbb{C}$ refl'n group $G$, the det-relative invariants inside $\mathbb{C}[\underline{x}]$

$$\mathbb{C}[\underline{x}]^{G,\det} \underset{\text{DEF'N}}{:=} \{ h(\underline{x}) \in \mathbb{C}[\underline{x}] : g(h) = \det(g)\cdot h \}$$

$$= J \cdot \mathbb{C}[\underline{x}]^G,$$

that is, they form a free $\mathbb{C}[\underline{x}]^G$-submodule of rank 1, with $\mathbb{C}[\underline{x}]^G$-basis $\{J\}$.

Furthermore, $J = c \cdot \displaystyle\prod_{\substack{\text{refl'n hyperplanes} \\ H \text{ for } G}} \ell_H(\underline{x})^{d_H - 1}$ for some $c \in \mathbb{C}^\times$

and $d_H := |G_H|$ where $G_H := \{ g \in G : gH = H \}$
$= \{1\} \cup \underbrace{\{ \text{refl'ns } s \text{ in } G \text{ with hyperplane } H \}}_{\text{size } d_H - 1}$

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**EXAMPLES**

(1) A familiar case: if $G = \mathfrak{S}_n \circlearrowright \mathbb{C}[x_1, \ldots, x_n]$

then $\mathbb{C}[\underline{x}]^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[\overset{\text{power sums}}{p_1, p_2, \ldots, p_n}]$

$p_1 = x_1 + x_2 + \ldots + x_n$, $p_n = x_1^n + x_2^n + \ldots + x_n^n$

so $J = \det\left(\dfrac{\partial p_j}{\partial x_i}\right) = \det\left[ j \cdot x_i^{j} \right] = c \displaystyle\prod_{1 \le i < j \le n} (x_i - x_j) = c \prod_H \ell_H^{d_H - 1}$

Vandermonde determinant

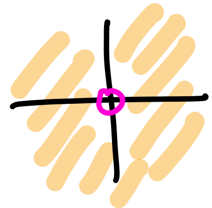
(all $d_H = 2$)

and $\mathbb{C}[\underline{x}]^{\mathfrak{S}_n, \det}$ = alternating polynomials $= \displaystyle\prod_{1 \le i < j \le n} (x_i - x_j) \cdot \mathbb{C}[\underline{x}]^{\mathfrak{S}_n}$

(2) The essence of the PROP is the simplest case:

$$G = \langle \overbrace{[\zeta]}^{g:=} \rangle \subset GL_1(\mathbb{C}) = \mathbb{C}^\times, \quad \zeta = e^{2\pi i/d}$$
$$= G(d,1,1)$$

$V = \mathbb{C}^1$



$H = \{0\}$

$\ell_H = x$

$d_H = d$ since $G = G_H$

$$\mathbb{C}[\underline{x}] = \mathbb{C}[x] \quad \text{via } g(x) = \zeta^{-1} x$$
$$\cup$$
$$\mathbb{C}[\underline{x}]^G = \mathbb{C}[\underbrace{x^d}_{= f_1}]$$

while we can directly check

$$\mathbb{C}[x]^{G,\det} = \{ h(x) \in \mathbb{C}[x] : h(\zeta^{-1}x) = \zeta \cdot h(x) \}$$
$$= \mathrm{span}_{\mathbb{C}} \{ x^j : (\zeta^{-1}x)^j = \zeta \cdot x^j \}$$

i.e. $-j \equiv 1 \mod d$

$j \equiv d-1 \mod d$

$j = d-1+dk$ for $k \geq 0$

$$= \mathrm{span}_{\mathbb{C}} \{ x^{d-1}(x^d)^k : k \geq 0 \}$$
$$= x^{d-1} \cdot \mathbb{C}[x^d]$$
$$= J \cdot \mathbb{C}[x^d] \quad \text{since } J = \det\left[\frac{\partial}{\partial x}(x^d)\right] = d \cdot x^{d-1}$$

proof of PROPOSITION: 1st prove

CLAIM: Any $h(\underline{x}) \in \mathbb{C}[\underline{x}]^{G, \det}$ is divisible by $\prod_H \ell_H^{d_H - 1}$.

This is equivalent by unique factorization, to showing it is divisible by each $\ell_H^{d_H-1}$. Changing bases $x_1, \ldots, x_n$ in $V^*$, assume $\ell_H(\underline{x}) = \ker(x_1)$

and $s = \begin{array}{c|c} & e_1 \;\; e_2 \;\; \cdots \;\; e_n \\ \hline \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{matrix} & \begin{bmatrix} \zeta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \end{array}$ with $\zeta = e^{2\pi i/d}$ where $d = d_H$ generates $G_H = \{1, s, s^2, \ldots, s^{d-1}\}$

and on $V^*$, $s = \begin{array}{c|c} & x_1 \;\; x_2 \;\; \cdots \;\; x_n \\ \hline \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} \zeta^{-1} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \end{array}$

Then can check as in above EXAMPLE that

$h(\underline{x}) \in \mathbb{C}[\underline{x}]$ has $s(h) = \det(s) \cdot h = \zeta \cdot h$

$\Longleftrightarrow$ every monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in $h(\underline{x})$ has $a_1 \equiv d-1 \bmod d$

$\Longrightarrow$ $x_1^{d-1}$ divides $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$

$\Longrightarrow$ $\underbrace{x_1^{d-1}}_{= \ell_H^{d_H-1}}$ divides $h(\underline{x})$, proving CLAIM.

Now since $J = \det\left(\frac{\partial f_i}{\partial x_j}\right) \in \mathbb{C}[\underline{x}]^{G,\det}$,
the CLAIM shows $\prod_H \ell_H^{d_H - 1}$ divides $J$.

But they have same degree:

$$\deg J = \sum_{i=1}^{n} \underbrace{(d_i - 1)}_{\substack{\| \\ \deg\left(\frac{\partial f_i}{\partial x_j}\right)}} = \#\text{refl'ns in } G = \sum_H \#\begin{pmatrix}\text{refl'ns in}\\ G \text{ fixing } H\end{pmatrix}$$

$$= \sum_H (d_H - 1) = \deg \prod_H \ell_H^{d_H - 1}.$$

Also $J \neq 0$ since $f_1, \ldots, f_n$ are alg. indep.
(not so obvious; see Humphreys §3.10 for quick proof)

Hence $J = c \cdot \prod_H \ell_H^{d_H - 1}$ for some $c \in \mathbb{C}^\times$.

And CLAIM shows $\mathbb{C}[\underline{x}]^{G,\det} = \prod_H \ell_H^{d_H - 1} \cdot \mathbb{C}[\underline{x}]^G$

$$= J \cdot \mathbb{C}[\underline{x}]^G \quad \blacksquare$$

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Now Solomon shows $(\mathbb{C}[\underline{x}] \otimes \Lambda^k \langle \underline{y} \rangle)^G$ has
free $\mathbb{C}[\underline{x}]^G$-basis $\{ df_I := df_{i_1} \wedge \cdots \wedge df_{i_k} \}_{\substack{I = \{i_1 < \cdots < i_k\} \\ \subset \{1,2,\ldots,n\}}}$

by checking their $\begin{cases} \text{lin. independence} \\ \quad + \\ \text{spanning} \ldots \end{cases}$

$\mathbb{C}[\underline{x}]^G$-linear independence of $\{df_I\}$:

It's enough to show they're $\mathbb{C}(\underline{x})$-lin. indep. inside the $\mathbb{C}(\underline{x})$-vector space $\mathbb{C}(\underline{x}) \otimes \Lambda^k\langle \underline{y}\rangle$.

If $\displaystyle\sum_I h_I(\underline{x})\, df_I = 0$ for some $h(\underline{x}) \in \mathbb{C}(\underline{x})$

then for each $k$-subset $I_0 \subset \{1,2,\ldots,n\}$, let $I_0^c := \{1,2,\ldots,n\} \setminus I_0$, and mult. by $df_{I_0^c}$:

$$\sum_I h_I(\underline{x})\, \underbrace{df_I \wedge df_{I_0^c}}_{\substack{\text{vanishes unless } I = I_0 \text{, due} \\ \text{to a repeat } df_i \wedge df_i = 0}} = 0$$

$$\Downarrow$$

$$h_{I_0}(\underline{x})\, df_{I_0} \wedge df_{I_0^c} = 0$$

$$= \pm h_{I_0}(\underline{x}) \cdot J\, \underbrace{dy_1 \wedge \cdots \wedge dy_n}_{\substack{= \text{the unique } \mathbb{C}(\underline{x})\text{-basis} \\ \text{element of } \mathbb{C}(\underline{x}) \otimes \Lambda^n\langle \underline{y}\rangle}}$$

$$\Downarrow$$

$$h_{I_0}(\underline{x}) \cdot J = 0$$

$$\Downarrow \text{ since } J \neq 0$$

$$h_{I_0}(\underline{x}) = 0 \qquad \text{for all } I_0.$$

The $\{df_I\}$ $\mathbb{C}[\underline{x}]^G$-span $(\mathbb{C}[\underline{x}] \otimes \Lambda^k\langle \underline{y} \rangle)^G$:

Note the above lin. independence, together $\dim_{\mathbb{C}(\underline{x})}$-counting shows that

$\{df_I\}$ are a $\mathbb{C}(\underline{x})$-basis for $\mathbb{C}(\underline{x}) \otimes \Lambda^k\langle \underline{y} \rangle$.

Hence given any $\omega \in (\mathbb{C}[\underline{x}] \otimes \Lambda^k\langle \underline{y} \rangle)^G$
$(\subseteq \mathbb{C}(\underline{x}) \otimes \Lambda^k\langle \underline{y} \rangle)$,

one can write it as

$$\omega = \sum_I h_I(\underline{x})\, df_I \quad \text{with } h_I(\underline{x}) \in \mathbb{C}(\underline{x})$$

and then by applying $\pi_G$,

$$\omega = \pi_G(\omega) = \sum_I \pi_G(h_I(x)) \cdot df_I$$

we can assume $\pi_G(h_I(x)) = \dfrac{p_I(\underline{x})}{q_I(\underline{x})} \in \mathbb{C}(\underline{x})^G$.

As before, fix a $k$-subset $I_0 \subset \{1, 2, \ldots, n\}$

and mult. by $df_{I_0^c}$:

$$\omega \wedge df_{I_0^c} = \sum_I \frac{p_I(\underline{x})}{q_I(\underline{x})} \cdot df_I \wedge df_{I_0^c}$$

$\parallel$

$r_{I_0}(\underline{x})\, dy_1 \wedge \cdots \wedge dy_n$

lying in $\left(\mathbb{C}[\underline{x}] \otimes \wedge^n \langle \underline{y} \rangle\right)^G$,

since $\omega$, $df_{I_0^c}$

were $G$-invariant

$$= \frac{p_{I_0}(\underline{x})}{q_{I_0}(\underline{x})}\, df_{I_0} \wedge df_{I_0^c}$$

$$= \frac{p_{I_0}(\underline{x})}{q_{I_0}(\underline{x})} \cdot J \cdot dy_1 \wedge \cdots \wedge dy_n$$

Hence $\dfrac{p_{I_0}(\underline{x})}{q_{I_0}(\underline{x})} \cdot J = r_{I_0}(\underline{x})$,

and $r_{I_0}(\underline{x}) \in \mathbb{C}(\underline{x})^{G,\det} \cap \mathbb{C}[\underline{x}]$

$= \mathbb{C}[\underline{x}]^{G,\det}$

$\overset{\text{PROP}}{=} J \cdot \mathbb{C}[\underline{x}]^G$

Thus $J$ divides $r_{I_0}(\underline{x})$ in $\mathbb{C}[\underline{x}]$,

so $\dfrac{p_{I_0}(\underline{x})}{q_{I_0}(\underline{x})} = \dfrac{r_{I_0}(\underline{x})}{J} \in \mathbb{C}[\underline{x}] \cap \mathbb{C}(\underline{x})^G$

$= \mathbb{C}[\underline{x}]^G \quad \blacksquare$