Length generating function revisited

(Humphreys §3.14, 3.15)

Let's return ~~better~~-equipped to a result alluded to in our overview.

THEOREM (Solomon 1966) For (W,S) with W finite (= a finite real refln group) having $\mathbb{C}[\underline{x}]^W = \mathbb{C}[f_1, \ldots, f_n]$ and degrees $d_1, \ldots, d_n$,

one has $W(q) := \sum_{w \in W} q^{\ell(w)} = [d_1]_q [d_2]_q \cdots [d_n]_q$

EXAMPLE For $W = \mathfrak{S}_n = W(\underset{s_1}{\circ}\!-\!\underset{s_2}{\circ} \cdots -\!\underset{s_{n-1}}{\circ})$,

$$W(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [1]_q [2]_q \cdots [n]_q = [n]!_q$$

How does the invariant theory play any role?

Let's massage the statement a bit.

Want to show $W(q) = [d_1]_q \cdots [d_n]_q$

$$\Updownarrow$$

$$\frac{1}{W(q)} = \frac{1}{[d_1]_q \cdots [d_n]_q}$$

$$= (1-q)^n \cdot \frac{1}{(1-q^{d_1}) \cdots (1-q^{d_n})}$$

$$\frac{1}{W(q)} = (1-q)^n \mathrm{Hilb}(\mathbb{C}[x]^W, q)$$

We've also seen $W(q)$ is characterized by a recurrence on $\#S$ in (W,S):

$$\sum_{J:\, J \subseteq S} (-1)^{\#J} \frac{1}{W_J(q)} = q^{\ell(w_0)} \frac{1}{W(q)}$$

So it would suffice to show

$$\sum_{J:\, J \subseteq S} (-1)^{\#J} \cancel{(1-q)^n} \mathrm{Hilb}(\mathbb{C}[x]^{W_J}, q) = q^{\ell(w_0)} \cancel{(1-q)^n} \mathrm{Hilb}(\mathbb{C}[x]^W, q)$$

or equivalently

$$\sum_{J:\, J \subseteq S} (-1)^{\#J} \mathrm{Hilb}(\mathbb{C}[x]^{W_J}, q) = q^{\ell(w_0)} \mathrm{Hilb}(\mathbb{C}[x]^W, q)$$

Now note $\ell(w_0) = \#\Phi^+ = \#$ refl'n hyperplanes for W
$= \#$ refl'ns in W (since W is real)
$= \deg(J)$,

so $q^{\ell(w_0)} \mathrm{Hilb}(\mathbb{C}[\underline{x}]^W, q)$
$$= \mathrm{Hilb}(J \cdot \mathbb{C}[\underline{x}]^W, q)$$
$$= \mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W, \det}, q).$$

same as
sgn: $W \to \{\pm 1\}$
$w \mapsto (-1)^{\ell(w)}$

Hence the THEOREM follows if we can show
$$\sum_{J : J \subseteq S} (-1)^{\#J} \mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W_J}, q) \overset{(*)}{=} \mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W, \mathrm{sgn}}, q)$$

PROPOSITION: The equality $(*)$ would follow from an equality of characters of W-rep'ns
$$\sum_{J : J \subseteq S} (-1)^{\#J} \chi_{\underbrace{\mathbb{C}[W/W_J]}} = \chi_{\mathrm{sgn}}$$
W-permutation rep on cosets W/W_J
$u(wW_J) = uwW_J$

proof: Assuming this equality of characters, for each $d=0,1,2,\ldots$, take inner product with the character $\chi_{\mathbb{C}[x]_d}$ for W acting on $\mathbb{C}[x]_d$, multiply by q^d, then sum on d:

$$\sum_{d=0}^{\infty} q^d \sum_{J: J\subseteq S} (-1)^{\#J} \underbrace{\left\langle \chi_{\mathbb{C}[W/W_J]}, \chi_{\mathbb{C}[x]_d} \right\rangle_W}$$

$$= \sum_{d=0}^{\infty} q^d \underbrace{\left\langle \chi_{sgn}, \chi_{\mathbb{C}[x]_d} \right\rangle_W}_{=\dim_{\mathbb{C}}\left(\mathbb{C}[x]_d^{W,sgn}\right)}$$

equals $\dim_{\mathbb{C}}\left(\mathbb{C}[x]_d^{W_J}\right)$:

for any subgroup $H<G$ and any G-rep U,

$$\left\langle \chi_{\mathbb{C}[G/H]}, \chi_U \right\rangle_G$$

$$= \left\langle \mathrm{Ind}_H^G \chi_{triv}^H, \chi_U \right\rangle_G$$

$$= \left\langle \chi_{triv}^H, \mathrm{Res}_H^G \chi_U \right\rangle_H$$

Frobenius reciprocity

$$= \dim_{\mathbb{C}}\left(U^H\right)$$

$$\Updownarrow$$

$$\sum_{J: J\subseteq S} (-1)^{\#J} \mathrm{Hilb}\left(\mathbb{C}[x]^{W_J}, q\right) = \mathrm{Hilb}\left(\mathbb{C}[x]^{W,sgn}, q\right)$$

$\blacksquare$

So why should $\sum_{J : J \subseteq S} (-1)^{\#J} \chi_{\mathbb{C}[W/W_J]} = \chi_{sgn}$ hold?

One proof is through topology and this important tool:

THEOREM:
(Hopf Trace Formula)

Let a finite group G act on a chain complex of $\mathbb{C}$-vector spaces

$$0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0,$$

commuting with the boundary maps (i.e. $g\partial_i = \partial_i g$)

(e.g. if G are symmetries of a simplicial complex Δ, and $C_i = C_i(\Delta, \mathbb{C})$)

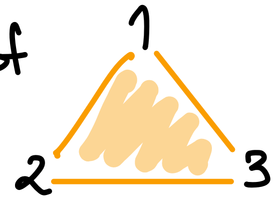
Then $$\sum_{i=0}^{m} (-1)^i \underbrace{\chi_{C_i}}_{\text{character of } G \text{ on } C_i} = \sum_{i=0}^{m} (-1)^i \underbrace{\chi_{H_i}}_{\text{character of } G \text{ on homology}}$$

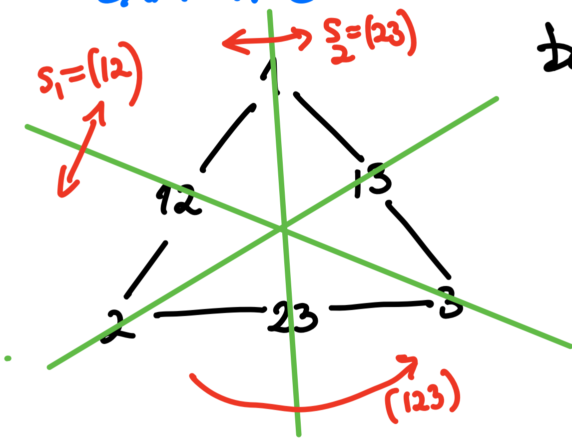
$H_i = Z_i / B_i$
$= \ker \partial_i / \operatorname{im} \partial_{i+1}$

evaluate characters on $1 \in G$

Euler-Poincaré relation: $$\sum_{i=0}^{m} (-1)^i \dim_{\mathbb{C}} C_i = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{C}} H_i$$

EXAMPLE $W = \mathfrak{S}_3$ acting on Δ = barycentric subdivision of the boundary of





$$0 \to \tilde C_1 \xrightarrow{\partial_1} \tilde C_0 \xrightarrow{\partial_0} \tilde C_{-1} \to 0$$

$\mathbb{C}$-bases:

$\mathbb{C}[W] = \mathbb{C}[W/W_\emptyset] \rightsquigarrow \left\{ \begin{array}{l} [1,12] \\ [1,13] \\ [2,12] \\ [2,23] \\ [3,12] \\ [3,13] \end{array} \right.$

$\left. \begin{array}{l} [1] \\ [2] \\ [3] \end{array} \right\} \mathbb{C}[W/W_{\{s_2\}}]$

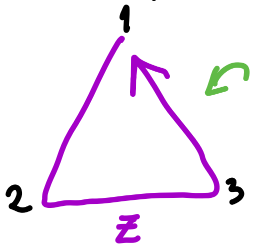
$\left. \begin{array}{l} [12] \\ [13] \\ [23] \end{array} \right\} \mathbb{C}[W/W_{\{s_1\}}]$

$[\,] \; \} \; \mathbb{C}[W/W_{\{s_1,s_2\}}]$

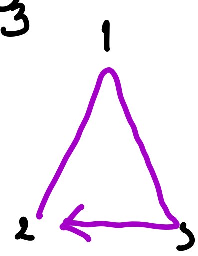
$\dim_{\mathbb{C}} =$	6	6	1

Homology $\tilde H_i(\Delta) = 0$ except for $i = 1$

where $\tilde H_1(\Delta) \cong$ sgn repn of $W = \mathfrak{S}_3$



orientation cycle, $\mathbb{C}$-basis for $\tilde H_1(\Delta)$



$s_1(z) = -z$

Here Hopf Trace Formula says

$$-\chi_{\tilde{C}_{-1}} + \chi_{\tilde{C}_0} - \chi_{\tilde{C}_1} = -\chi_{\tilde{H}_1}$$

	$(1)(2)(3)$	(12) (13) (23)	(123) (132)	
$\chi_{triv} = \chi_{\tilde{C}_{-1}}$	1	1	1	A
$\chi_{\mathbb{C}[W/W_{\{s_i\}}]}$	3	1	0	
$2\chi_{\mathbb{C}[W/W_{\{s_i\}}]} = \chi_{\tilde{C}_0}$	6	2	0	B
$\chi_{\mathbb{C}[W]} = \chi_{\tilde{C}_1}$	6	0	0	C
sgn	1	-1	1	← This row is row A − row B + row C

Hopf Trace formula comes from a simple fact:

PROPOSITION: When one has a short exact sequence of vector spaces $0 \to A \to B \to B/A \to 0$ and a linear map $g: B \to B$ that also preserves A, i.e. $g(A) \subseteq A$, then g induces a linear map $g: B/A \to B/A$, with $\mathrm{Trace}(g|_B) = \mathrm{Trace}(g|_A) + \mathrm{Trace}(g|_{B/A})$

proof: Pick bases so g looks like this:

$$g = \begin{array}{c} \\ A \\ B/A \end{array} \begin{array}{cc} A & B/A \\ \left[\begin{array}{c|c} g|_A & * \\ \hline 0 & g|_{B/A} \end{array}\right] \end{array} \qquad \blacksquare$$

proof of Hopf Trace Formula: The chain complex $0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \dots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$ and group elements $g \in G$ commuting with ∂_i give rise to two kinds of short exact sequences as in the PROPOPOSITION ...

$$0 \to \underset{\ker \partial_i}{\underset{\parallel}{Z_i}} \to C_i \to \underset{\operatorname{im} \partial_i}{\underset{\parallel}{B_{i-1}}} \to 0 \quad \Longrightarrow \quad \chi_{C_i} \overset{(a)}{=} \chi_{Z_i} + \chi_{B_{i-1}}$$

take traces of $g \in G$

$$0 \to B_i \to Z_i \to \underset{Z_i/B_i}{\underset{\parallel}{H_i}} \to 0 \quad \Longrightarrow \quad \chi_{Z_i} \overset{(b)}{=} \chi_{B_i} + \chi_{H_i}$$

Hence

$$\chi_{C_0} - \chi_{C_1} + \chi_{C_2} - \chi_{C_3} + \dots$$

via (a)

$$= \left(\chi_{Z_0} + \chi_{B_{-1}}\right) - \left(\chi_{Z_1} + \chi_{B_0}\right) + \left(\chi_{Z_2} + \chi_{B_1}\right) - \left(\chi_{Z_3} + \chi_{B_2}\right) + \dots$$

via (b)

$$= \left(\chi_{B_0} + \chi_{H_0} + \chi_{B_{-1}}\right) - \left(\chi_{B_1} + \chi_{H_1} + \chi_{B_0}\right) + \left(\chi_{B_2} + \chi_{H_2} + \chi_{B_1}\right) - \left(\chi_{B_3} + \chi_{H_3} + \chi_{B_2}\right) + \dots$$

cancel cancel cancel

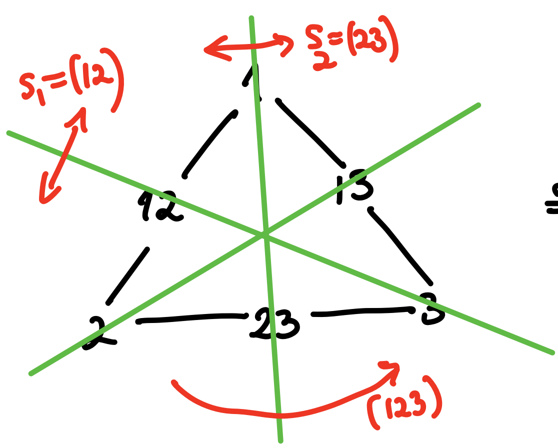
$$= \cancel{\chi_{B_{-1}}}^{\,0} + \chi_{H_0} - \chi_{H_1} + \chi_{H_2} - \chi_{H_3} + \dots \qquad \blacksquare$$

Now let's apply this to the simplicial complex Δ, called the Coxeter Complex for (W,S), that comes from intersecting the Tits cone $\mathcal{U} = V^* \cong V$ with the unit sphere $\mathbb{S}^{n-1}$ in $V = \mathbb{R}^n$.

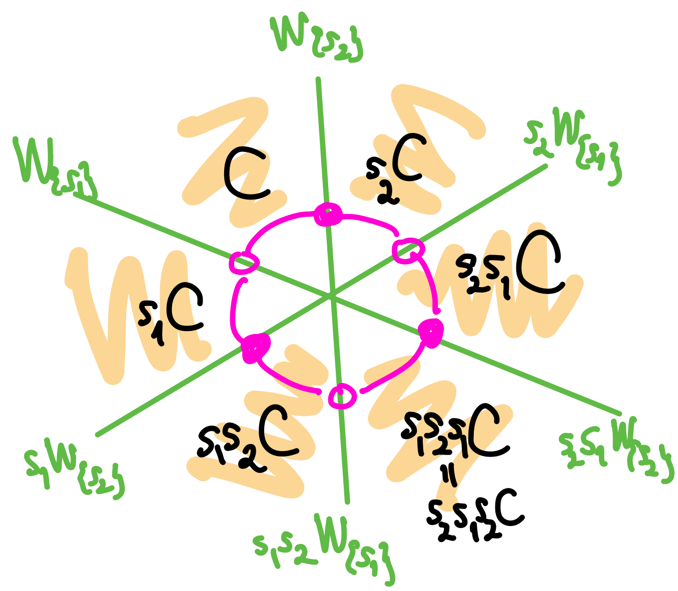
One finds that the oriented simplicial chain group

$$\tilde{C}_i \cong \bigoplus_{\#S \setminus J = i+1} \mathbb{C}[W/W_J]$$

isomorphism as W-rep'ns



$\cong$



$$\begin{array}{ccccccccc}
0 & \to & \tilde{C}_1 & \to & \tilde{C}_0 & \to & \tilde{C}_{-1} & \to & 0 \\
 & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\
 & & \mathbb{C}[W] & & \mathbb{C}[W/W_{\{s_1\}}] & & \mathbb{C}[W/W_{\{s_1,s_2\}}] & & \\
 & & & & \oplus & & & & \\
 & & & & \mathbb{C}[W/W_{\{s_2\}}] & & & &
\end{array}$$

Meanwhile, $\tilde{H}_i(\Delta) = \tilde{H}_i(\mathbb{S}^{n-1}) = 0$ unless $i = n-1$

with $\tilde{H}_{n-1}(\Delta) \cong$ sgn = det

as W-rep'ns

since W acts orthogonally on $V = \mathbb{R}^n$,
and elements $g \in O_n(\mathbb{R})$ act on the orientation cycle z of $\tilde{H}_{n-1}(\mathbb{S}^{n-1})$ via the $\det(g) \in \pm 1$, either preserving $\det(g)=+1$ or reversing $\det(g)=-1$ orientation.

Hence Hopf Trace Formula says

$$(-1)^{n-1} \chi_{sgn} = \sum_{i \geq -1} (-1)^i \chi_{\tilde{C}_i}$$

$$= \sum_{i \geq -1} (-1)^i \sum_{\substack{J: \\ \#S \setminus J = i+1}} \chi_{\mathbb{C}[W/W_J]}$$

$$\Rightarrow \quad \chi_{sgn} = \sum_{J : J \subseteq S} (-1)^{\#J} \chi_{\mathbb{C}[W/W_J]} \qquad \blacksquare$$