Strong Bruhat order

(Björner–Brenti, Chapter 2)

It's an important poset on a Coxeter group W

partially ordered set

= binary relation $x \le y$

with $\begin{cases} x \le x & \text{(reflexive)} \\ x \le y, y \le x \Rightarrow x = y & \text{(antisymmetric)} \\ x \le y, y \le z \Rightarrow x \le z & \text{(transitive)} \end{cases}$

defined like this...

DEF'N: Given a Coxeter system (W,S) with reflections $T := \bigcup_{\substack{w \in W \\ s \in S}} wsw^{-1}$ as usual,

the Bruhat graph on W is a directed graph with arcs $u \longrightarrow w$ if $w = tu$ with $\ell(u) < \ell(w)$ for some $t \in T$ (and we'll sometimes write $u \xrightarrow{t} w$ here).

The (strong) Bruhat order is the transitive, reflexive closure of $u \to w$, meaning $u \le w$ if

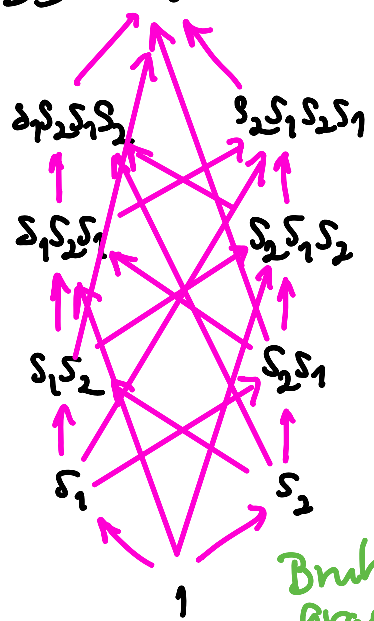
$\exists$ a path $u = u_0 \to u_1 \to \cdots \to u_{k-1} \to u_k = w$ in the Bruhat graph

EXAMPLES

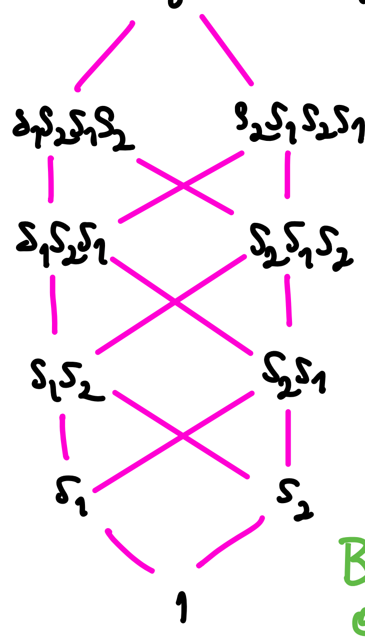
(1) $I_2(m) = W\left(\underset{s_1}{\circ}\overset{m}{\text{—}}\underset{s_2}{\circ}\right)$

Hasse diagram of poset:
edges $u - w$ are
cover relations $u \lessdot w$
meaning $u < w$ and $\nexists v$ with
$u < v < w$

$m=5$ $w_0 = s_1s_2s_1s_2s_1 = s_2s_1s_2s_1s_2$ $w_0 = s_1s_2s_1s_2s_1 = s_2s_1s_2s_1s_2$

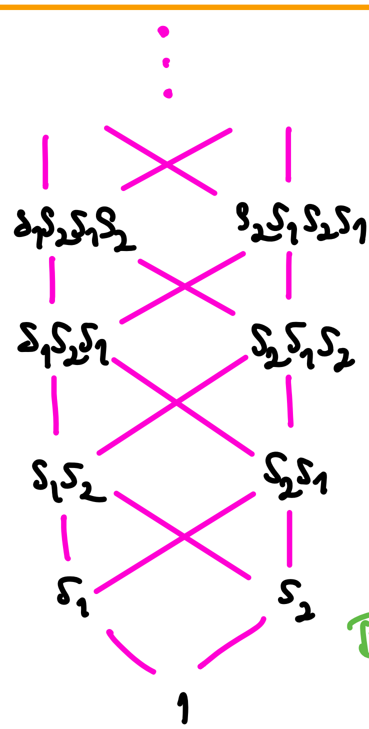


Bruhat graph



Bruhat order

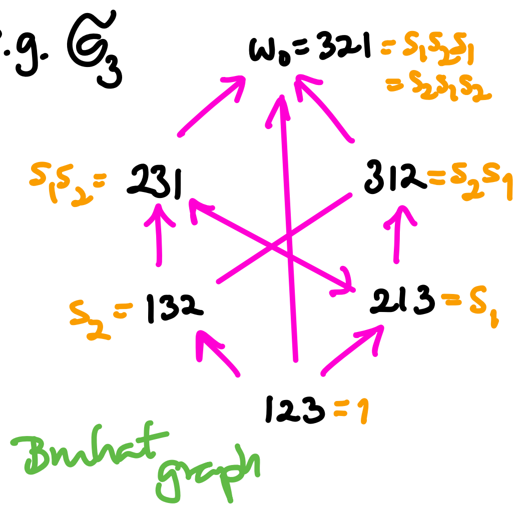
$m=\infty$

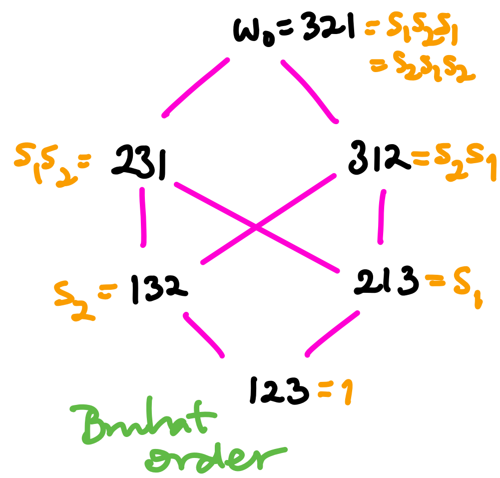


Bruhat order

(2) $\mathfrak{S}_n = W(\circ\!-\!\circ \cdots - \circ)$

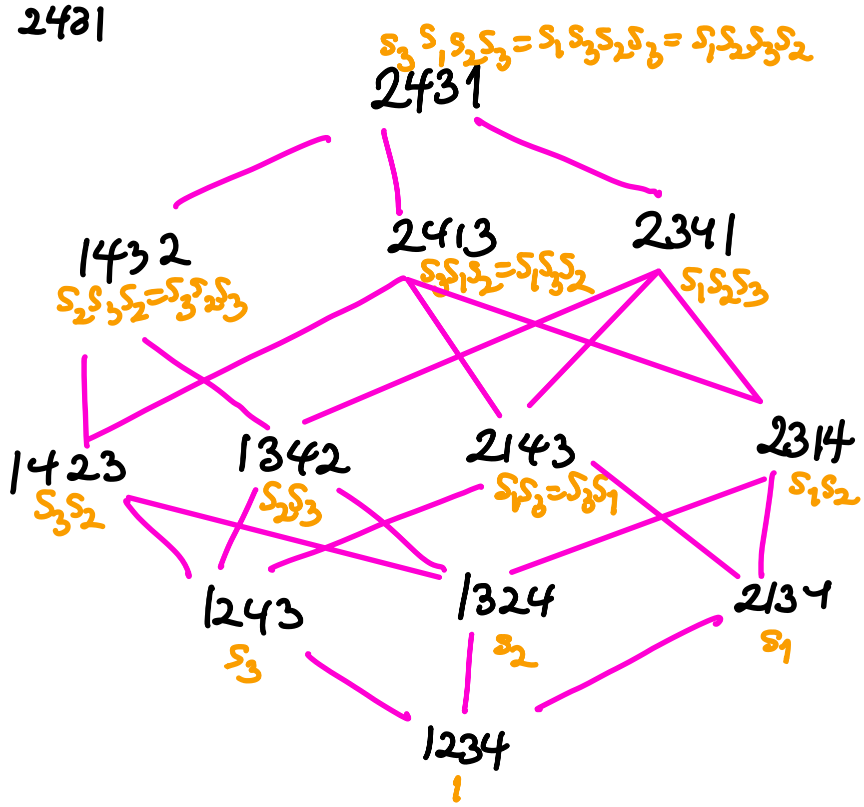
e.g. $\mathfrak{S}_3$





In $W = \mathfrak{S}_4$, here is the (lower) interval

$[u, w] \overset{\text{DEF}}{:=} \{v \in W : u \le v \le w\}$, $u = 1234$, $w = 2431$



PROPOSITION: $W = \mathfrak{S}_n$ has Bruhat covering relations:

$$u \lessdot w \iff w = u\cdot(a,b) \text{ with } a<b \text{ and } u_a < u_b$$
$$\text{but } \nexists c \text{ with } a<c<b \text{ and } u_a < u_c < u_b$$
$$\iff w = u\cdot(a,b) \text{ with } \mathrm{inv}(w) = \mathrm{inv}(u)+1$$

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e.g. $u = 2\,7\,\underline{1}\,4\,6\,\underline{3}\,\underline{5} \quad < \quad 2\,7\,\underline{5}\,4\,6\,\underline{3}\,\underline{1} = w = u\cdot(1,5)$

but $u \not\lessdot w$, since $v = 2\,7\,\underline{3}\,4\,6\,\underline{1}\,\underline{5}$ (and $u \lessdot v$)

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proof: If $w = u\cdot(a,b)$ with $u_a < u_b$, then $\ell(u) < \ell(w)$ so $u < w$ in Bruhat. And if furthermore $\nexists c$ with $a<c<b$ and $u_a < u_c < u_b$, can check $\ell(w) = \mathrm{inv}(w) = \mathrm{inv}(u)+1 = \ell(u)+1$, so $u \lessdot w$. But if such an index c does exist, then $u < u\cdot(u_a, u_c) < w$ $\blacksquare$

REMARK: Later we'll prove a faster algorithm for checking $u < w$ in $\mathfrak{S}_n$, called the Tableaux Criterion (THM. 2.6.3 in Björner-Brenti).

DIGRESSION: Where does Bruhat order on W come from?

For W a Weyl group, one has a semisimple complex Lie group G

e.g. $G = SL_n(\mathbb{C})$ $\quad n=3: \quad A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \quad \det = 1$

with a choice of Borel subgroup $B < G$

e.g. $B = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \text{ upper triangular} \right\} \subset SL_n(\mathbb{C})$ $\quad n=3: \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$

and choice of a maximal torus $T < B$

e.g. $T = \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \text{ diagonal} \right\} \subset SL_n(\mathbb{C})$ $\quad n=3: \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$

with $W = N_G(T)/T$. normalizer of T in G

This lets one define $G/B :=$ generalized flag manifold

e.g. $SL_n/B \cong \{ \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n : \dim_{\mathbb{C}} V_i = i \}$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} B \longmapsto \left(\{0\} \subset \mathbb{C}v_1 \subset \mathbb{C}v_1 + \mathbb{C}v_2 \subset \cdots \subset \mathbb{C}v_1 + \cdots + \mathbb{C}v_{n-1} \subset \mathbb{C}^n \right)$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} B \longmapsto \left(\{0\} \subset \mathbb{C}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \subset \mathbb{C}\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \mathbb{C}\begin{bmatrix} d \\ e \\ f \end{bmatrix} \subset \mathbb{C}^3 \right)$$

G/B is not only a smooth manifold of dimension $\ell(w_0) = |T| = |\Phi^+|$,

but has an embedding $G/B \overset{P}{\hookrightarrow} \mathbb{P}^{N-1} := (\mathbb{C}^N - \{0\})/\mathbb{C}^\times$ projective space

making it a projective variety.

e.g. $SL_n/B \underset{\text{Plücker embedding}}{\overset{P}{\hookrightarrow}} \mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \ldots \times \mathbb{P}^{\binom{n}{n-1}-1} \underset{\text{Segre embedding}}{\hookrightarrow} \mathbb{P}^{N-1}$ where $N := \binom{n}{1}\binom{n}{2}\cdots\binom{n}{n-1}$

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} B \overset{P}{\longmapsto} \left(\left[\begin{matrix}\text{left-justified} \\ 1\times 1 \\ \text{subdeterminants}\end{matrix} \right], \left[\begin{matrix}\text{left-justified} \\ 2\times 2 \\ \text{subdeterminants}\end{matrix} \right], \ldots \right)$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \longmapsto \left(\underbrace{[a:b:c]}, \left[\underbrace{\begin{vmatrix} a & d \\ b & e \end{vmatrix} : \begin{vmatrix} a & d \\ c & f \end{vmatrix} : \begin{vmatrix} b & e \\ c & f \end{vmatrix}} \right] \right) \in \mathbb{P}^{3-1} \times \mathbb{P}^{3-1}$$

homogeneous coordinates

defined only up to simultaneous scaling

And G has a double coset decomposition (Bruhat decomposition)

$$G = \bigsqcup_{w \in W} BwB$$

that turns into a cell decomposition of the flag manifold

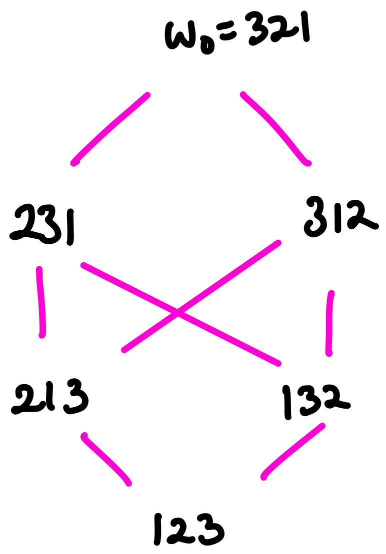
$$G/B = \bigsqcup_{w \in W} \underbrace{BwB/B}_{X_w^{\text{open}}}$$

$X_w^{\text{open}} :=$ (open) Bruhat cell for w

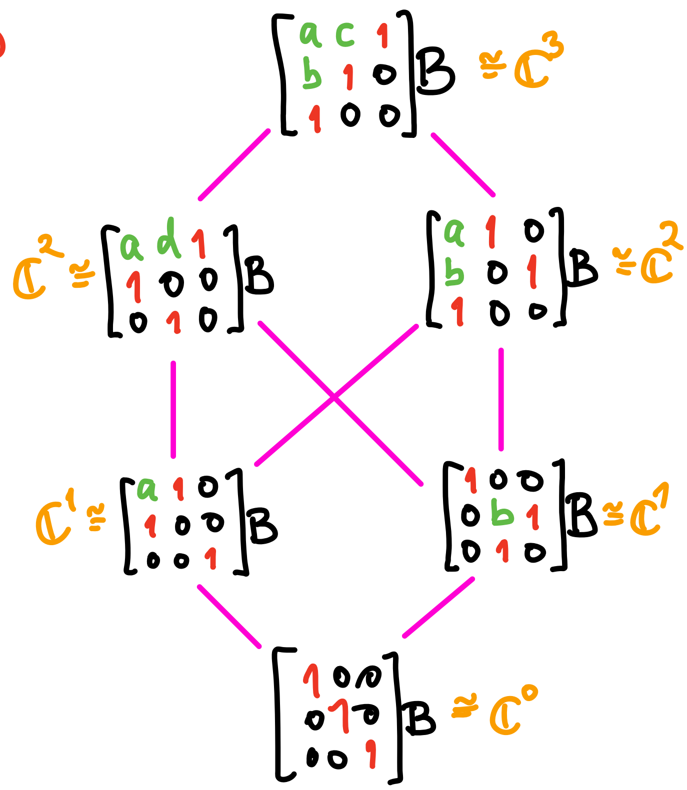
$\cong \mathbb{C}^{\ell(w)}$ an affine space

THEOREM (Bruhat 1954): The closures $X_w := \overline{X_w^{\text{open}}}$, called Schubert varieties, have $X_u \subset X_w \iff u \leq w$ in Bruhat order.

$W = \mathfrak{S}_3$, $G/B = SL_3(\mathbb{C})/B$



Bruhat order



$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}B}_{X_{123}^{\text{open}}} \overset{p}{\longmapsto} \big([1:0:0],[1:0:0]\big) \in \mathbb{P}^2\times\mathbb{P}^2 \qquad X_{123}$$

$$\Big\uparrow \lim_{b\to\infty} \qquad\qquad \cap$$

if $b\neq 0$

$$= \big([1:0:0],[1:\tfrac{1}{b}:0]\big)$$

$$\underbrace{\begin{bmatrix}1&0&0\\0&b&1\\0&1&0\end{bmatrix}B}_{X_{213}^{\text{open}}} \overset{p}{\longmapsto} \big([1:0:0],[b:1:0]\big) \qquad X_{132}$$

$$\Big\uparrow \lim_{a\to\infty} \qquad\qquad \cap$$

$$= \big([1:\tfrac{1}{a}:0],[b:1:\tfrac{1}{a}]\big)$$

if $a\neq 0$ and we set $b = -d/a$

$$= \big([a:1:0],[ab:a:1]\big)$$

$$\underbrace{\begin{bmatrix}a&d&1\\1&0&0\\0&1&0\end{bmatrix}B}_{X_{231}^{\text{open}}} \overset{p}{\longmapsto} \big([a:1:0],[-d:a:1]\big) \qquad X_{231}$$

Basic properties of Bruhat order (B-B §2.2)

Most come from this...

LEMMA: Let $w = s_1 s_2 \cdots s_q$ reduced and assume $u \in W$ has a reduced subexpression

(*) $u = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots\cdots \hat{s}_{i_k} \cdots\cdots s_q$ with $1 \le i_1 < \cdots < i_k \le q$.

Then $\exists v \in W$ with
$$\left\{\begin{array}{l} \text{(a) } \ell(v) = \ell(u)+1 \\ \text{(b) } u < v \\ \text{(c) } v \text{ also has a reduced subexpression of } s_1 s_2 \cdots s_q \end{array}\right.$$

proof: Choose the expression (*) for u with i_k minimal (leftmost).

Let $t := s_q s_{q-1} \cdots s_{i_k} \cdots s_{q-1} s_q$

and $v := ut = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_{k-1}} \cdots\cdots s_q$.

Hence $\ell(v) \le \ell(u)+1$. We CLAIM: $\ell(v) > \ell(u)$ and hence all 3 of (a),(b),(c) hold.

To prove the CLAIM, assume not, i.e. $\ell(v) < \ell(u)$.

Strong Exchange implies either....

- $t = s_q s_{q-1} \cdots s_p \cdots s_{q-1} s_q$ for some $p > i_k$, leading to the contradiction $w = t^2 = s_1 s_2 \cdots \hat{s}_{i_k} \cdots \hat{s}_{i_p} \cdots s_q$ of length $< q = \ell(w)$

or • $t = s_q s_{q-1} \cdots \hat{s}_{i_k} \cdots \hat{s}_{i_d} \cdots s_r \cdots \hat{s}_{i_d} \cdots \hat{s}_{i_k} \cdots s_{q-1} s_q$ for some $r < i_k$, $r \neq i_j$

leading to

$$u = ut^2 = (s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_k} \cdots s_q) \cdot (s_q \cdots \hat{s}_{i_k} \cdots s_r \cdots \hat{s}_{i_k} \cdots s_q)(s_q \cdots s_{i_k} \cdots s_q)$$

$$= s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_r \cdots s_{i_k} \cdots s_q$$

contradicting i_k being minimal ▨

COROLLARY (Subword characterization of Bruhat)
(B-B Thm 2.2.2, Cor 2.2.3)

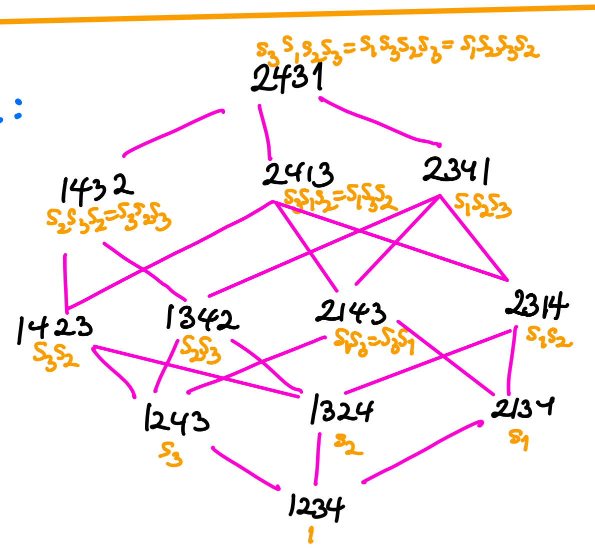
For any Cox. sys. (W,S) and $u, w \in W$, TFAE:

(i) $u \leq w$ in Bruhat order

(ii) Every reduced word for w contains a reduced subexpression for u.

(iii) Some reduced word for w contains a reduced subexpression for u.

EXAMPLE:



$= [1234, 2431]$ in $\mathfrak{S}_4$

Proof:

(ii) $\Rightarrow$ (iii): Clear.

(iii) $\Rightarrow$ (i): If $w = s_1 \cdots s_q$ reduced contains a reduced subexpression $u = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_k} \cdots s_q$, then induct on $k = \ell(w) - \ell(u)$ to conclude $u \le w$, using LEMMA above to find v with $u \lessdot v$, $\ell(v) = \ell(u) + 1$ and v also has such a reduced subexpression of $s_1 \cdots s_q$.

(i) $\Rightarrow$ (ii): Given $u \le w$, so there exists a path $u = u_0 \to u_1 \to \cdots \to u_{k-1} \to u_k = w$ in the Bruhat graph, assume we are given any reduced word $w = s_1 s_2 \cdots s_q$.

Since $u_{k-1} \xrightarrow{t} w$ for some $t \in T$, Strong Exchange shows $u_{k-1} = wt = s_1 s_2 \cdots \hat{s}_i \cdots s_q$ for some i.

Repeating this k times, one concludes u has some expression (possibly not reduced) that is a subexpression of $s_1 s_2 \cdots s_q$. But then Deletion Condition lets one conclude u also has a reduced such subexpression. $\blacksquare$

A few immediate consequences...

COROLLARY

(i) Bruhat order is ranked with $\mathrm{rank}(w) = \ell(w)$, meaning if $u \leq w$ then $\exists$ a chain
$$u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_{\ell(w)-1} \lessdot u_{\ell(w)} = w$$

(ii) Bruhat intervals $[u,w]$ are always finite, with $\#[u,w] \leq 2^{\ell(w)}$.

(iii) $u \leq w \iff u^{-1} \leq w^{-1}$.

proof: (i): comes from Subword Characterization and the Lemma at the beginning.

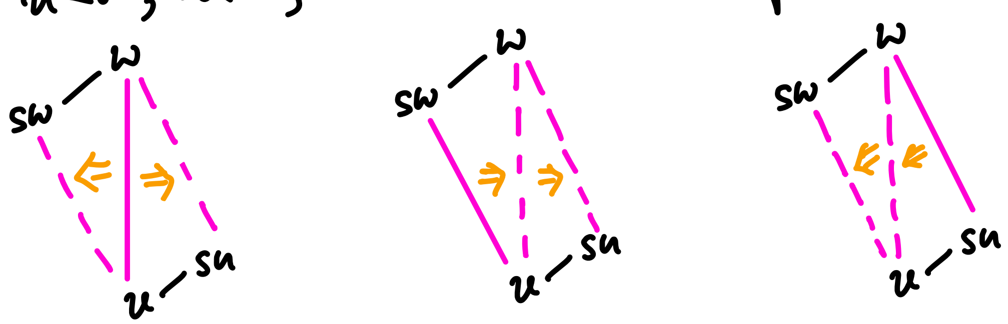
(ii): $w = s_1 s_2 \cdots s_{\ell(w)}$ has at most $2^{\ell(w)}$ subexpressions.

(iii): comes from Subword Characterization ▨

Here's a subtle property of Bruhat order...

PROPOSITION (The Lifting/Zig-zag/N/Z Property)

Suppose u, w have $s \in S$ with $s \in D_L(w) \setminus D_L(u)$.
Then any of the three Bruhat order relations $u < w$, $u < sw$, $su < w$ shown here implies the other two:



proof: Since $sw<w$ and $u<su$, by transitivity it suffices to show only the leftmost diagram implications. So assume $u<w$ (can't have $u=w$ if $s\in D_L(w)\setminus D_L(u)$). Pick a reduced expression $sw = s_1 s_2 \cdots s_q$.

$$\Downarrow$$

$w = s s_1 s_2 \cdots s_q$ is also reduced since $s \in D_L(w)$.

By Subword Characterization, u has a reduced subexpression $s_{i_1} s_{i_2} \cdots s_{i_k}$ of $w = \underbrace{s}_{\text{call this } s_0} s_1 s_2 \cdots s_q$.

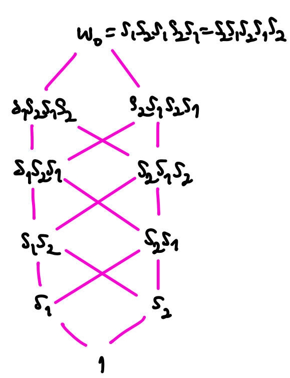
Then $s_{i_1} \neq s_0 = s$ since $su>u$, so $u < s_1 s_2 \cdots s_q = sw$. Also $su = s s_{i_1} s_{i_2} \cdots s_{i_k}$ is reduced since $s \notin D_L(u)$, hence $su < w$. ■

Here's an application of Lifting Property:

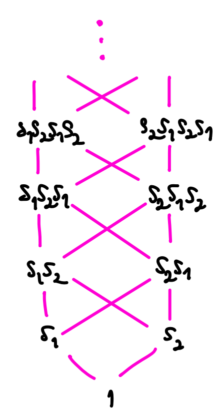
COROLLARY: Bruhat order is always a **directed poset**, meaning $\forall u,v\in W\ \exists w\in W$ with $w \geq u,v$. In particular, if W is finite, the longest element $w_0 \geq w\ \forall w \in W$.

EXAMPLE: $I_2(m)$

m= 5:



m= ∞:



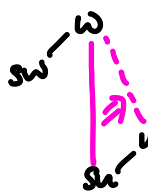
proof: Prove $w \geq u,v$ exists by induction on $\ell(u)+\ell(v)$.

BASE CASE where $\ell(u)+\ell(v)=0$, so $u=v=1$ is trivial.

INDUCTIVE STEP:

W.L.O.G. $\ell(u)\geq 1$, so $\exists\, s\in S$ with $su<u$.

By induction, $\exists\, w\in W$ with $w\geq su, v$

Either $sw<w$ and use Lifting  to get $w>u$ (so $w>u,v$)

or $sw>w$ and use Lifting  to get $sw>u$ (and $sw>w>v$)

When W is finite, since w_0 is the unique element of the longest length $\ell(w_0)=\#T$, for any $w\in W$, it must be the common upper bound of w, w_0. So $w_0\geq w$ ▨

Not only does w_0 give a top element in Bruhat order for finite W, it also gives a poset anti-automorphism:

PROPOSITION: When W is finite,

(i) $\ell(ww_0)=\ell(w_0)-\ell(w)$ $\quad(=\ell(w_0w))$

(ii) $u\leq w \iff uw_0\geq ww_0$ $\quad(\iff w_0u\geq w_0w)$

(iii) $T_L(ww_0)=T\setminus T_L(w)$ $\quad$ (and $T_R(w_0w)=T\setminus T_R(w)$)

proof: (ii),(iii) follow from (i), since (i) implies for any $t \in T$ that one has

$$t \in T_L(u) \iff \ell(tu) < \ell(u) \iff tu \xrightarrow{t} u \text{ in Bruhat graph}$$

$$\Updownarrow \text{(i)}$$

$$t \in T \setminus T_L(uw_0) \iff \underset{\ell(w_0)-\ell(tu)}{\underset{\|}{\ell(tuw_0)}} > \underset{\ell(w_0)-\ell(u)}{\underset{\|}{\ell(uw_0)}} \iff uw_0 \xrightarrow{t} tuw_0 \text{ in Bruhat graph}$$

For (i), the inequality $\ell(ww_0) \geq \ell(w_0) - \ell(w)$ comes from

$$\ell(w_0) = \ell(w^{-1} \cdot ww_0) \leq \ell(w^{-1}) + \ell(ww_0) = \ell(w) + \ell(ww_0).$$

For the reverse inequality $\ell(ww_0) \leq \ell(w_0) - \ell(w)$, one proceeds by induction on $\ell(w_0) - \ell(w)$.

BASE CASE: $w = w_0$, so $\ell(w_0w_0) = \ell(1) = 0 = \ell(w_0) - \ell(w_0)$ ✓

INDUCTIVE STEP: If $\ell(w_0) - \ell(w) \geq 1$, so $w \neq w_0$, then we know $D_L(w) \neq S$, so $\exists\, s \in S$ with $sw > w$.

Then
$$\begin{aligned}\ell(ww_0) &\leq \ell(sww_0) + 1 \\ &\leq \ell(w_0) - \ell(sw) + 1 \quad \text{by induction} \\ &= \ell(w_0) - (\ell(w)+1) + 1 = \ell(w_0) - \ell(w) \quad \blacksquare\end{aligned}$$

COROLLARY When W is finite,

(a) $\left.\begin{array}{l} w \longmapsto ww_0 \\ w \longmapsto w_0 w \end{array}\right\}$ are poset anti-automorphisms of Bruhat

(b) $\left.\begin{array}{l} w \longmapsto w^{-1} \\ w \longmapsto w_0 w w_0 \end{array}\right\}$ are poset automorphisms of Bruhat

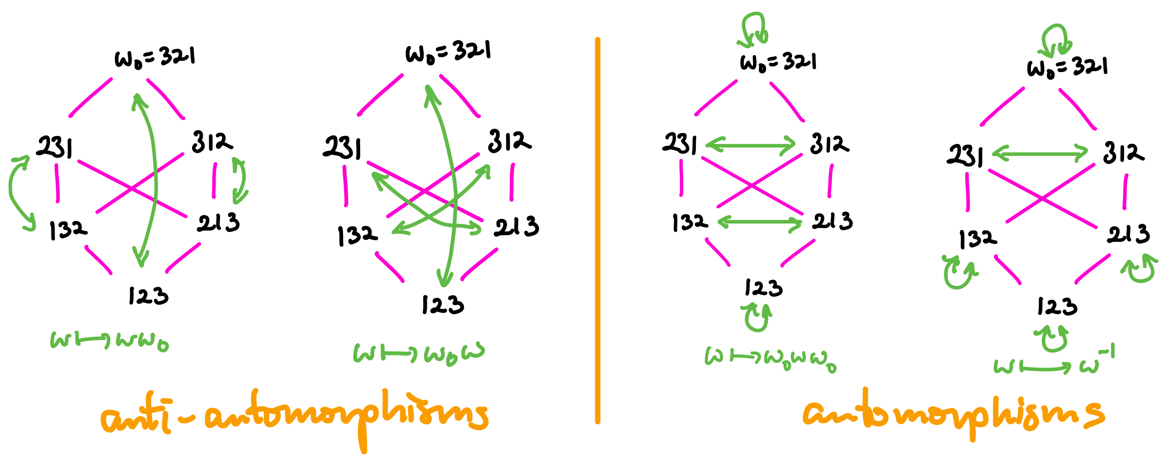
(c) $s_i \overset{\sigma}{\longmapsto} s_j := w_0 s_i w_0$ is a (Coxeter) diagram automorphism,

meaning σ permutes S, and $m_{\sigma(i),\sigma(j)} = m_{ij}$

EXAMPLES (1) For $\mathfrak{S}_n = W(\underset{s_1}{\circ}\!-\!\underset{s_2}{\circ} \cdots -\underset{s_{n-1}}{\circ})$, $w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$

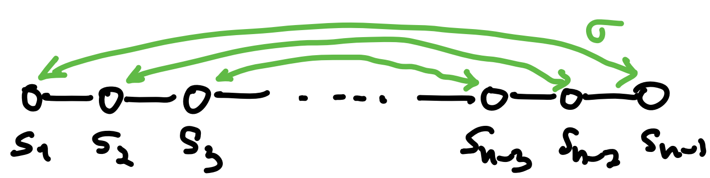
has $ww_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_n & w_{n-1} & \cdots & 1 \end{pmatrix}$ $\quad w_0 w = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1-w_1 & n+1-w_2 & & n+1-w_n \end{pmatrix}$

$$w_0 w w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1-w_n & n+1-w_{n-1} & & n+1-w_1 \end{pmatrix}$$



$$s_i \overset{\sigma}{\longmapsto} w_0 s_i w_0 = s_{n-i}$$

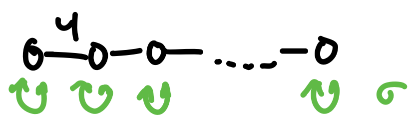
$$\underset{(i,i+1)}{\shortparallel} \qquad\qquad \underset{(n+1-i,\,n-i)}{\shortparallel}$$



(2) For $W = W(B_n) = W(C_n)$ = signed permutations

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ -1 & -2 & \cdots & -n \end{pmatrix} = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix}$$

so $w_0 w = w w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ -w_1 & -w_2 & & -w_n \end{pmatrix}$ and $w_0 w w_0 = w$



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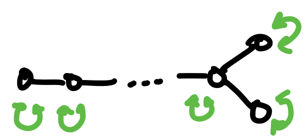
(3) For $W(D_n)$ = signed permutations with evenly many negative signs,

$$w_0 = \begin{cases} \begin{pmatrix} 1 & 2 & \cdots & n \\ -1 & -2 & \cdots & -n \end{pmatrix} = \begin{bmatrix} -1 & & 0 \\ & -1 \ddots & \\ 0 & & -1 \end{bmatrix} & \text{if } n \text{ even} \\[2ex] \left(\begin{array}{cccc|c} 1 & 2 & \cdots & n-1 & n \\ -1 & -2 & \cdots & -(n-1) & n \end{array}\right) = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots -1 & \\ 0 & & & +1 \end{bmatrix} & \text{if } n \text{ odd} \end{cases}$$

and $\sigma(s_i) = w_0 s_i w_0$ does this:



n even

n odd

EXERCISE When W is finite, show T.F.A.E.

(a) $w_0 = -1_V$ in the geom. rep'n $W \xrightarrow{\sigma} GL(V)$

(b) $w \mapsto w_0 w w_0$ is the trivial diagram automorphism of (W,S)

(c) The center $Z(W) = \langle w_0 \rangle = \{1, w_0\}$.