(Exercise 1.24.) Computing the derivative of $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $|x|^2 x$. There are two ways of doing this:

- Recall that since $x \in \mathbb{R}^n$, $|x|^2 = x \cdot x = x_1^2 + x_2^2 + \cdots + x_n^2$; so

$$f(x) = |x|^2 x = \begin{bmatrix} (x_1^2 + x_2^2 + \cdots + x_n^2)x_1 \\ (x_1^2 + x_2^2 + \cdots + x_n^2)x_2 \\ \vdots \\ (x_1^2 + x_2^2 + \cdots + x_n^2)x_n \end{bmatrix}.$$

  Now there are two results we can cite: Theorem 1.9.8 or Theorem 1.8.1 (3). We choose the former (and using the notation of the theorem, $U = \mathbb{R}^n$). To see why Theorem 1.9.8 applies, observe that for each component of $f(x)$ satisfies

$$D_i\left((x_1^2 + x_2^2 + \cdots + x_n^2)x_j\right) = 2x_i x_j$$

  when $i \neq j$, and

$$D_i\left((x_1^2 + x_2^2 + \cdots + x_n^2)x_j\right) = 2x_i x_i + (x_1^2 + x_2^2 + \cdots + x_n^2) = 2x_i^2 + x \cdot x$$

  when $i = j$, so all partials derivatives of f indeed exist and are continuous on $\mathbb{R}^n$. (They are polynomials.) By Theorem 1.9.8, the derivative of f is given by its Jacobian:

$$[Df(x)] = \begin{bmatrix} 2x_1^2 + x \cdot x & 2x_1 x_2 & \cdots & 2x_1 x_n \\ \vdots & \vdots & \vdots & \vdots \\ 2x_n x_1 & 2x_n x_2 & \cdots & 2x_n^2 + x \cdot x \end{bmatrix}.$$

- Work with the limit definition of the derivative. (Not recommended in this case, but many of you opted for this route.)

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x) - L(h)}{|h|} &= \lim_{h\to 0} \frac{|x+h|^2(x+h) - |x|^2 x - L(h)}{|h|} \\ &= \lim_{h\to 0} \frac{\left((x+h)\cdot(x+h)\right)(x+h) - |x|^2 x - L(h)}{|h|} \\ &= \lim_{h\to 0} \frac{(x\cdot x + 2h\cdot x + h\cdot h)(x+h) - |x|^2 x - L(h)}{|h|} \\ &= \lim_{h\to 0} \frac{(2h\cdot x + h\cdot h)x + (x\cdot x + 2h\cdot x + h\cdot h)h - L(h)}{|h|}. \end{aligned}$$

  Since $h \cdot h = |h|^2$, pick L so that it cancels out those terms in the numerator that have only one h —in other words, define L by $L(h) = 2(h\cdot x)x + (x\cdot x)h$. Note that L is linear: For any $h_1, h_2 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, $L(\alpha h_1 + \beta h_2) = \alpha L(h_1) + \beta L(h_2)$. (Write out the details if this is not immediate.) Now we want to show that for our choice of L, the above limit indeed goes to 0. Then, picking up from where we left off in the above string of equalities and taking magnitudes,

$$\lim_{h\to 0}\left|\frac{f(x+h)-f(x)-L(h)}{|h|}\right| = \lim_{h\to 0}\frac{|(h\cdot h)x+2(h\cdot x)h+(h\cdot h)h|}{|h|}$$
$$\leq \lim_{h\to 0}\frac{|h|^2x+2|h|^2|x|+|h|^3}{|h|}=0,$$

as desired. Hence, f is differentiable (for any $x \in \mathbb{R}^n$) and the matrix $[Jf(x)]$ representing L is the derivative of f.

Exercise: Check that if you find the matrix of the linear transformation L, you will end up with the same matrix as found by invoking Theorem 1.9.8. In fact, technically in order to answer the question in 1.24, you should do this computation.

Remark: The moral of the story should be: Use results you know so that you do not find yourself computing derivatives from the limit definition!