5707 Spring 2017 Lecture 16

Network flows

We shall next learn the basics of network flow theory. There is a lot to be said, little of which we will actually say; for a more comprehensive survey, see Lex Schrijver's "A Course in Combinatorial Optimization" (http://homepages.cwi.nl/~lex/files/dict.pdf) Chapter 4.
We shall follow Jeremy Martin's notes instead.

Def. A *network* consists of

- a digraph (V, A);
- two vertices $s \in V$ and $t \in V$ called the *source* and the *sink* (although we don't require s to have indegree 0, or t to have outdegree 0);
- a function $c : A \to \mathbb{Q}_+$, called the *capacity function*.

~~(Recall (Recall that $0 \in \mathbb{N}$.)~~

(Here, $\mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\}$.)

We shall require $s \neq t$.

In this situation, we call (V, A) the underlying digraph of the network, and we say that the network is a network on (V, A).

We call $c(a)$ (for a given $a \in A$) the capacity of the arc a.

Def. Let N be a network with underlying digraph (V, A), source s, sink t, and capacity function c.

A flow (on N) means a function $f : A \to \mathbb{Q}_+$ with the following properties:

- $0 \leq f(a) \leq c(a) \quad \forall a \in A$.
  ("Capacity constraints")
- $\forall v \in V \setminus \{s, t\}$, we have
  $$f^-(v) = f^+(v),$$
  where
  $$f^-(v) = \sum_{\substack{a \text{ arc} \\ \text{with} \\ \text{target } v}} f(a) \qquad \text{and}$$
  $$f^+(v) = \sum_{\substack{a \text{ arc} \\ \text{with} \\ \text{source } v}} f(a).$$

("Conservation constraints".)

A flow is usually visualized as (e.g.) water flowing through the network (where the arcs $a \in A$ are pipes, and their capacities $c(a)$ are how much water they can maximally handle).
The conservation constraints then say that water does not appear out of thin air or dissipate other than at s and t.

~~Def. Let N be a network as before~~

The value of the flow f is defined to be

$$|f| := f^+(s) - f^-(s).$$

Prop. 1. Let N, V, A, s, t, c be as before. Let f be a flow. Then,

$$|f| = f^-(t) - f^+(t).$$

Proof. It is easy to see that

$$\sum_{v \in V} f^-(v) = \sum_{v \in V} f^+(v)$$

(indeed, both sums equal $\sum_{a \in A} f(a)$). Hence,

Hence, $\sum_{v \in V} (f^-(v) - f^+(v)) = 0.$

But all addends in this sum except for the ones at $v=s$ and at $v=t$ are 0. Hence,

$$(f^-(s) - f^+(s)) + (f^-(t) - f^+(t)) = 0,$$

so $f^-(t) - f^+(t) = f^+(s) - f^-(s) = |f|.$ $\square$

<u>Def.</u> Let N, V, A, s, t, c be as before. For any $S \subseteq V$, let $\bar{S} = V \setminus S$. For any two subsets S and T of V, we let $[S, T]$ be the set of all arcs whose source lies in S and whose target lies in T; furthermore, we set

$$c(S,T) = \sum_{a \in [S,T]} c(a).$$

If f is a flow, then, for any $S \subseteq V$, we set

$$f^-(S) = \sum_{a \in [\bar{S}, S]} f(a) \quad \text{and}$$

$$f^+(S) = \sum_{a \in [S, \bar{S}]} f(a).$$

(Note: $f^-(v) = f^-(\{v\})$ and
$f^+(v) = f^+(\{v\})$ $\forall v \in V$.)

<u>Prop. 2.</u> Let $S \subseteq V$. Let f be a flow.

(a) We have $f^+(S) - f^-(S) = \sum_{v \in S} (f^+(v) - f^-(v))$.

(b) We have $f^+(S) - f^-(S) = f^-(\bar{S}) - f^+(\bar{S})$.

(c) If $s \in S$ and $t \notin S$, then
$f^+(S) - f^-(S) = f^-(\bar{S}) - f^+(\bar{S}) = |f|$
and $|f| \leq c(S, \bar{S})$.

<u>Proof.</u> (a) We have

$$\sum_{v \in S} (f^+(v) - f^-(v))$$

$$= \underbrace{\sum_{v \in S} f^+(v)}_{} - \underbrace{\sum_{v \in S} f^-(v)}_{}$$

$$= \sum_{a \in [S, V]} f(a) \qquad\qquad = \sum_{a \in [V, S]} f(a)$$

$$= \sum_{a \in [S, S]} f(a) + \sum_{a \in [S, \bar{S}]} f(a) \qquad\qquad = \sum_{a \in [S, S]} f(a) + \sum_{a \in [\bar{S}, S]} f(a)$$

$$= \left(\sum_{a \in [S,S]} f(a) + \sum_{a \in [S,\bar{S}]} f(a) \right) - \left(\sum_{a \in [S,S]} f(a) + \sum_{a \in [\bar{S},S]} f(a) \right)$$

$$= \underbrace{\sum_{a \in [S,\bar{S}]} f(a)}_{= f^+(S)} - \underbrace{\sum_{a \in [\bar{S},S]} f(a)}_{= f^-(S)} = f^+(S) - f^-(S).$$

(b) Rewrite both $f^+(S) - f^-(S)$ and $f^+(\bar{S}) - f^-(\bar{S})$ as sums using part (a). These two sums add up to

$$\sum_{v \in V} \left(f^+(v) - f^-(v) \right) = 0$$ (as shown in the proof of Prop. 1).

(Alternatively: Observe that $f^+(S) = f^-(\bar{S})$ and $f^-(S) = f^+(\bar{S})$.)

(c) Assume that $s \in S$ and $t \notin S$. Then, (a) yields

$$f^+(S) - f^-(S) = \sum_{v \in S} \left(f^+(v) - f^-(v) \right)$$

$$= \underbrace{f^+(s) - f^-(s)}_{= |f|} + \sum_{v \in S;\ v \neq s} \underbrace{\left(f^+(v) - f^-(v) \right)}_{= 0}$$

(by conservation constraints (since $v \neq s$ and $v \neq t$))

$$= |f|.$$

Hence, $|f| = f^+(S) - \underbrace{f^-(S)}_{\geq 0}$

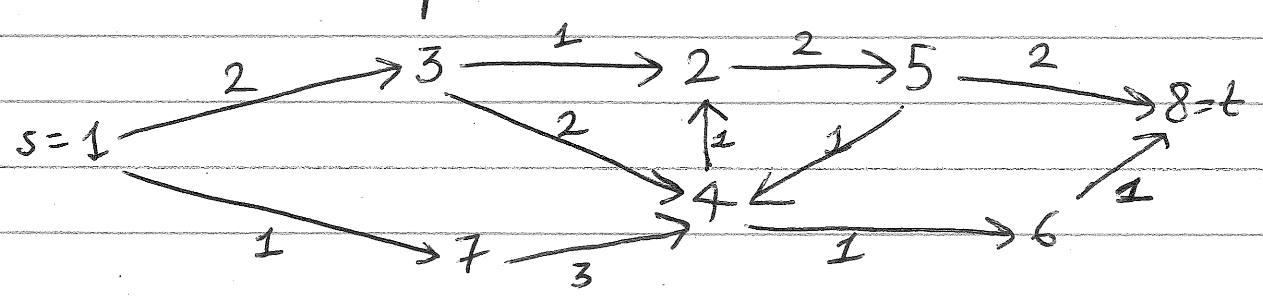
$$\leq f^+(S)$$

$$= \sum_{a \in [S, \bar{S}]} \underbrace{f(a)}_{\leq c(a)}$$

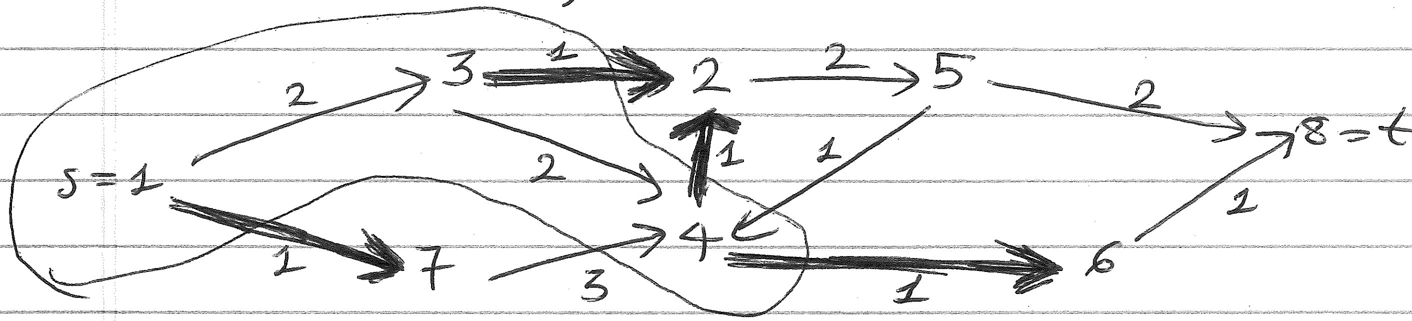
$$\leq \sum_{a \in [S, \bar{S}]} c(a) = c(S, \bar{S}). \quad \square$$

Def. Let N, V, A, s, t, c be as above. A *cut* of N means a subset of A having the form $[S, \bar{S}]$ for some $S \subseteq V$ satisfying $s \in S$ and $t \notin S$. The *capacity* of this cut is defined to be $c(S, \bar{S})$.

Example. Here is a network (with capacities drawn atop the arcs):



If $S = \{1, 3, 4\}$, then S gives rise to the cut $\{\overrightarrow{17}, \overrightarrow{32}, \overrightarrow{42}, \overrightarrow{46}\}$ (drawn in boldface below):



The capacity of this cut is
$$c(S, \bar{S}) \overset{\text{def}}{=} 1 + 1 + 1 + 1 = 4.$$

<u>Rmk.</u> The inequality in part (c) of Prop. 2 says that the value of a flow is always $\leq$ to the capacity of a cut. But can we achieve equality?

One of the most important results in combinatorics says "yes":

<u>Thm. 3.</u> Let N, V, A, s, t, c be as above.
(a) Then,

$$\max\{|f| \mid f \text{ is a flow}\}$$
$$= \min\{c(S, \bar{S}) \mid S \subseteq V;\ s \in S;\ t \notin S\}.$$

(In particular, the left-hand side is well-defined.)

In other words, the maximum value of a flow equals the minimum capacity of a cut.

(b) Assume that $c(a) \in \mathbb{N} \ \forall a \in A$ (where $\mathbb{N} = \{0, 1, 2, \ldots\} = \mathbb{Q}_+ \cap \mathbb{Z}$). Then, the claim of part (a) still holds if we restrict ourselves to flows $f : A \to \mathbb{N}$ (as opposed to flows $f : A \to \mathbb{Q}_+$).

(c) On the other hand, the claim of part (a) also holds if we replace $\mathbb{Q}_+$ by $\mathbb{R}_+$ throughout the definitions. (In other words, if we allow irrational capacities and flow values.)

We shall prove parts (a) and (b) only.

There are different proofs of Thm. 3. In particular, linear programming duality provides a brutal but efficient way to deal with parts (a) and (c). But our proof will be fully elementary.

First, we do some groundwork:

16/10

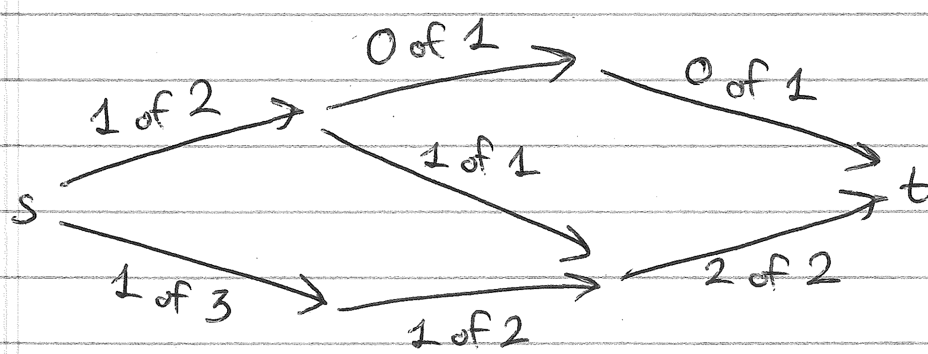
Def. Let N, V, A, s, t, c be as above.
~~Let f be a~~

(a) For any arc $a = (u, v) \in V \times V$, we let a^{-1} denote the arc (v, u). This arc may & may not lie in A.

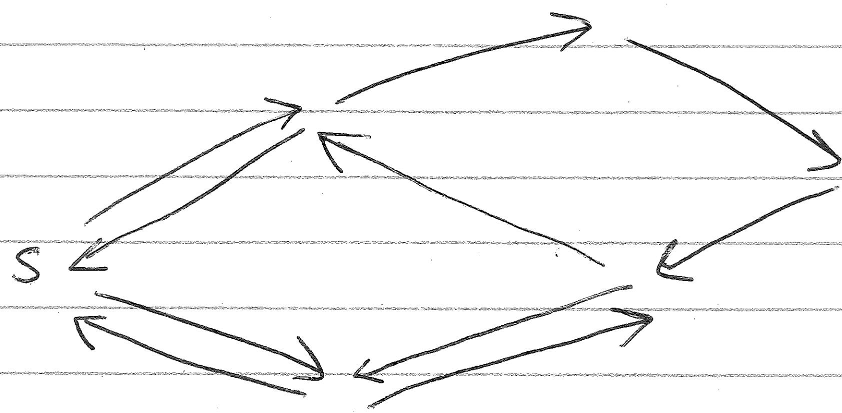
(b) Let f be a flow. We define the <u>residual digraph D_f</u> to be the digraph (V, A_f) where

$$A_f = \{ a \in A \mid f(a) < c(a) \} \cup \{ a^{-1} \mid a \in A \text{ and } f(a) > 0 \}.$$

<u>Example:</u> Consider the following flow:



where a "p of q" atop an arc a means that $c(a) = q$ and $f(a) = p$. Then, the residual digraph D_f is



Note that D_f may have cycles even if (V, A) has none!

(Intuition for D_f: It shows the wiggle room for modifying the flow without going below 0 or over capacity.)

<u>Lem. 4.</u> Let N, V, A, s, t, c be as above. Let f be a flow. Work over $\mathbb{N}$ or $\mathbb{Q}_+$ or $\mathbb{R}_+$.

(a) If the digraph D_f has a path from s to t, then $\exists$ flow f' with a larger value than f.

(b) If ~~no~~ the digraph D_f has no path from s to t, then $\exists$ subset $S \subseteq V$ satisfying $s \in S$ and $t \notin S$ and $c(S, \bar{S}) = |f|$.

Before we prove this, let's make a definition:

16/12

Def. Let N, V, A, s, t, c be as above. If $f : A \to \mathbb{Q}$ and $g : A \to \mathbb{Q}$ are two maps, then $f+g$ denotes a new map $A \to \mathbb{Q}$ defined by

$$(f+g)(a) = f(a) + g(a) \quad \forall a \in A.$$

Lem. 5. If f and g ~~are flows~~ satisfy the conservation constraints, then so does $f+g$.

~~If f and g are flows, then so is $f+g$.~~

Proof. Straightforward. $\square$

~~Proof of~~

Actually, let us take smaller steps and prove yet another lemma:

Lem. 6. Let N, V, A, s, t, c be as above. Let p be a path from s to t in the digraph $(V, A \cup A^{-1})$, where $A^{-1} := \{a^{-1} \mid a \in A\}$. Let P be the set of arcs of p.
~~Define~~ Let $\rho \in \mathbb{R}$.
~~Define a map~~
Assume each arc in P is colored either red

or blue, in such a way that

- if an arc $b \in \mathbf{p}$ is colored red, then $b \in A$;
- if an arc $b \in \mathbf{p}$ is colored blue, then $b \in A^{-1}$ (that is, $b^{-1} \in A$).

(Thus, only arcs belonging to both A and A^{-1} at the same time can choose their color freely — all other have only 1 choice.)

Define a map $\gamma : A \to \mathbb{R}$ by

$$\gamma(a) = \begin{cases} \rho & \text{if } a \in \mathbf{p} \text{ and } a \text{ is red;} \\ -\rho & \text{if } a^{-1} \in \mathbf{p} \text{ and } a^{-1} \text{ is blue;} \\ 0 & \text{otherwise} \end{cases} \qquad \forall a \in A.$$

(This is well-defined, since $a \in \mathbf{p}$ and $a^{-1} \in \mathbf{p}$ cannot happen at the same time (since $\mathbf{p}$ is a path).)

(a) This map γ satisfies the conservation constraints — i.e., we have

$$\gamma^-(v) = \gamma^+(v) \qquad \forall v \in V \setminus \{s, t\}.$$

(b) We have $|\gamma| = \rho$.

Proof. (a) Fix $v \in V \setminus \{s, t\}$.

If v does not lie on the path $\mathbf{p}$, then all is clear (since in this case, each arc a having source v or target v satisfies $\gamma(a) = 0$, and thus we have $\gamma^-(v) = 0$ and $\gamma^+(v) = 0$).

So, WLOG assume that it does.
Thus, $\exists$ exactly one arc $b \in P$ having source v, and $\exists$ exactly one arc $c \in P$ having target v. Consider these b and c.

We have $b \in P$. Thus, b is ~~either~~ colored red or blue. Hence, we are in one of the following two cases:

CASE 1: b is colored red.
CASE 2: b is colored blue.

Let us consider Case 2. Thus, b is colored blue. Hence, $b \in A^{-1}$.
Now, $c \in P$. Hence, c is ~~ei~~ colored red or blue. Hence, we are in one of the following two subcases:

SUBCASE 2.1: c is colored red.
SUBCASE 2.2: c is colored blue.

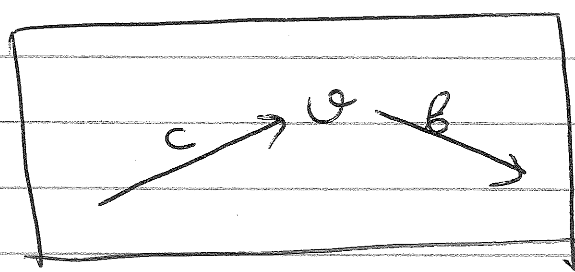
Let us consider Subcase 2.2. Thus, c is colored blue. Hence, $c \in A^{-1}$.

~~CASE 1: $b \in A$.~~
~~CASE 2: $b \in A^{-1}$~~
~~(The cases can overlap.)~~

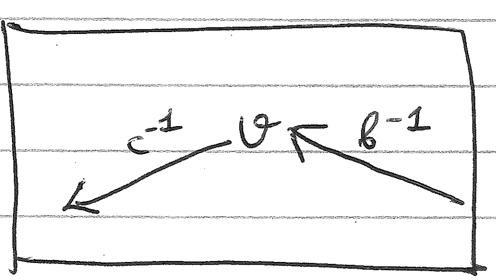
~~Let us consider Case 2 (by way of example). Thus, $b \in A^{-1}$. Now, $c \in P \subseteq A \cup A^{-1}$. Hence, we are in one of the following two subcases:~~

~~SUBCASE 2.1: $c \in A$.~~
~~SUBCASE 2.2: $c \in A^{-1}$.~~

~~Let us consider Subcase 2.2. Thus, $c \in A^{-1}$.~~
So here is how the two arcs b and c look like in $(V, A \cup A^{-1})$:



And here is how the corresponding arcs b^{-1} and c^{-1} look like in (V, A):



Thus, $\gamma(b^{-1}) = -\rho$ and $\gamma(c^{-1}) = -\rho$.
All other arcs in A having source or target v are sent to 0 by γ.
Thus, $\gamma^{-}(v) = -\rho$ and $\gamma^{+}(v) = -\rho$.
So Lem. 6 (a) holds in Subcase 2.2.
All the other three subcases are similar.

(b) This requires a case analysis like for (a), but this time it is simpler, since there is no c (because the path $\mathbf{p}$ starts at s). $\square$

Let us now prove Lem. 4:

Proof of Lem. 4. (a) Fix a path $\mathbf{p}$ from s to t in D_f.

Thus, $\mathbf{p}$ is a path in the digraph $(V, A \cup A^{-1})$, where $A^{-1} = \{a^{-1} \mid a \in A\}$ (since $A_f \subseteq A \cup A^{-1}$).

~~For each arc b of $\mathbf{p}$~~

~~Define $\rho_b = \max\{c$~~

$$\cancel{\rho_b = \begin{cases} c(b) - f(b), & \text{if } b \in A; \\ f(b^{-1}), & \text{if } b \in A^{-1} \setminus A \end{cases}}$$

~~for each $b \in A \cup A^{-1}$. Notice that $\rho_b > 0$~~

Let ~~p~~ P be the set of arcs ~~[illegible]~~ of $\mathbf{p}$.

We color each arc $b \in P$ either red or blue, by the following rules:

- If $b \in A$ and $f(b) < c(b)$, then we color b red.
- Otherwise, if $b \in A^{-1}$ and $f(b^{-1}) > 0$, then we color b blue.
- Otherwise, we rejoice, as we have found a bug in the universe ($b \in P \subseteq A_f = \{a \in A \mid f(a) < c(a)\} \cup \{a^{-1} \mid a \in A \text{ and } f(a) > 0\}$, so we must have one of the two cases above).

For each arc $b \in P$, define ~~an integer~~ a number ρ_b by

$$\rho_b = \begin{cases} c(b) - f(b), & \text{if } b \text{ is red;} \\ f(b^{-1}), & \text{if } b \text{ is blue.} \end{cases}$$

By the construction of the coloring, this is a positive ~~[illegible]~~ number.

Now, set $\rho = \min \{\rho_b \mid b \in P\}$. Then, ρ is positive (since all ρ_b are positive).

Now, define a map $\gamma : A \to \mathbb{R}$ as in Lem. 6.
The maps f and γ both satisfy the conservation constraints (for γ, this follows from Lem. 6 (a)). Thus, so does their sum $f+\gamma$.

But $f+\gamma$ also satisfies the capacity constraints — i.e., we have

(1) $$0 \leq (f+\gamma)(a) \leq c(a) \quad \forall a \in A.$$

Let us prove this.
[<u>Proof of (1):</u> Fix $a \in A$. Recall that f is a flow, whence $0 \leq f(a) \leq c(a)$.
Now, we are in one of the following three cases:

CASE 1: $a \in P$, and a is red;
CASE 2: $a^{-1} \in P$, and a is blue;
CASE 3: Neither of these two.

In Case 3, all is clear (because the definition of γ yields $\gamma(a)=0$ in Case 3, hence $(f+\gamma)(a) = f(a) + \underbrace{\gamma(a)}_{=0} = f(a)$, and thus (1) follows from $0 \leq f(a) \leq \gamma(a)$).

Remains to deal with Cases 1 and 2.

I shall do Case 2; Case 1 is similar.

So, in Case 2, we have $a^{-1} \in P$, and a is blue. Hence, $\gamma(a) = -\rho$ (by definition of γ). But the definition of ρ yields $\rho \leq \rho_{a^{-1}}$ (since $\rho_{a^{-1}}$ is an element of the set whose ~~maximum is ρ)~~ minimum is ρ).

The definition of $\rho_{a^{-1}}$ yields $\rho_{a^{-1}} = f((a^{-1})^{-1})$ $= f(a)$. Hence $\rho \leq \rho_{a^{-1}} = f(a)$, so

$$(f+\gamma)(a) = f(a) + \underbrace{\gamma(a)}_{= -\rho} = f(a) - \underbrace{\rho}_{\leq f(a)}$$

$$\geq f(a) - f(a) = 0.$$

Combined with

$$(f+\gamma)(a) = f(a) - \underbrace{\rho}_{>0} < f(a) \leq c(a),$$

this completes the proof of (1) in Case 2. As I said, Case 1 is similar.]

Thus, $f+\gamma$ is a flow. Its value is

$$|f+\gamma| = |f| + \underbrace{|\gamma|}_{\substack{=\rho \\ \text{(by Lem. 6 (b))}}} = |f| + \underbrace{\rho}_{>0} > |f|.$$

So $f+\gamma$ has larger value than f.
Thus, Lem. 4 (a) is proven (just set $f' = f+\gamma$).

(b) Assume that the digraph D_f has no path from s to t.
Set

$$S = \{v \in V \mid \text{the digraph } D_f \text{ has a path } s \to v\}.$$

Then, $S \subseteq V$ and $s \in S$ (since there is a trivial path $s \to s$ in D_f) and $t \notin S$ (since D_f has no path $s \to t$, by assumption).
Remains to show that $c(S, \bar{S}) = |f|$.

Observe that

(2) any $(u,v) \in A_f$ satisfying $u \in S$ must satisfy $v \in S$

(because $u \in S$ shows that D_f has a path $s \to u$, and thus we can extend this path by the arc $(u,v) \in A_f$ to obtain a walk $s \to v$ in

D_f; but from this walk we obtain a path $s \to v$ in D_f; hence, such a path exists, so that $v \in S$).

Next, observe that

(3) $\quad f(a) = 0 \quad \forall a \in [\bar{S}, S]$.

[Proof: Let $a \in [\bar{S}, S]$. Then, $a \in A$ and $a = (v, u)$ for some $u \in S$ and $v \in \bar{S}$. Now, assume for contradiction that (3) is false. Thus, $f(a) \neq 0$, hence $f(a) > 0$. By the definition of A_f, we thus have $a^{-1} \in A_f$. Hence, $(u, v) = a^{-1} \in A_f$. Thus, (2) yields $v \in S$, contradicting $v \in \bar{S}$. Hence (3) must be true after all.]

Similarly,

(4) $\quad f(a) = c(a) \quad \forall a \in [S, \bar{S}]$.

[Proof: Similar to (3), but this time you write a as $a = (u, v)$, not as $a = (v, u)$. And this time, you argue that a (not a^{-1}) lies in A_f.]

Now, $f^-(S) = \sum\limits_{a \in [\bar{S}, S]} \underbrace{f(a)}_{\substack{=0 \\ (\text{by } (3))}} = 0$

and

$$f^+(S) = \sum_{a \in [S,\bar{S}]} \underbrace{f(a)}_{\substack{= c(a) \\ (\text{by } (4))}} = \sum_{a \in [S,\bar{S}]} c(a) = c(S,\bar{S}).$$

Now, Prop. 2(c) yields

$$|f| = \underbrace{f^+(S)}_{= c(S,\bar{S})} - \underbrace{f^-(S)}_{= 0} = c(S,\bar{S}),$$

and thus Lem. 4 (b) is proven.

<u>Proof of Thm. 3.</u> We start with (b).

(b) ~~We shall~~ Here is an algorithm that finds a flow f and an $S \subseteq V$ satisfying $s \in S$, $t \notin S$ and $c(S,\bar{S}) = |f|$:

<u>FORD-FULKERSON ALGORITHM:</u>

<u>INPUT:</u> N, V, A, s, t, c as above.

<u>OUTPUT:</u> Flow f and subset $S \subseteq V$ with $s \in S$, $t \notin S$ and $c(S,\bar{S}) = |f|$.

<u>ALGORITHM:</u>

- Define $f : A \to \mathbb{N}$ by $f(a) = 0$ $\forall a \in A$.

(This is called the zero flow.)

- **While** the digraph D_f has a path from s to t (this can be checked using, e.g., the Dijkstra algorithm), **do** the following:

  - Apply Lemma 4 (a) to find a flow $f' : A \to \mathbb{N}$ with a larger value than f.
  - Replace f by f'. (Thus, f is now this larger flow.)

- Now, the digraph D_f has no path from s to t. Apply Lemma 4 (b) to find a subset $S \subseteq V$ satisfying $s \in S$, $t \notin S$ and $c(S, \overline{S}) = |f|$.

- Output (the current values of) f and S.

Why does this algorithm work?
By now, this is pretty clear; we just have to check one thing:

Claim 1: The algorithm cannot get stuck in the "while"-loop (i.e., it cannot happen that it keeps running ~~there~~ this loop forever).

[<u>Proof:</u> Each iteration of the "while"-loop increases the value $|f|$ (since f' has larger value than f), and thus increases it by at least 1 (since $|f| \in \mathbb{N}$ (because everything we are currently doing is over $\mathbb{N}$)). Hence, if the "while"-loop runs ~~for longer than~~ through K iterations, then $|f|$ must be $\geq K$ after that. But we know that $|f|$ is bounded from above by $c(\{s\}, \overline{\{s\}})$ (indeed, Prop. 2 (c) (applied to $\{s\}$ instead of S) shows that $|f| \leq c(\{s\}, \overline{\{s\}})$, whatever the flow f is). Hence, $|f|$ can never become $\geq c(\{s\}, \overline{\{s\}}) + 1$. Thus, the "while"-loop cannot run through $c(\{s\}, \overline{\{s\}}) + 1$ iterations. Hence, the algorithm eventually leaves it.]

So the algorithm works. Hence, it gives us <u>one</u> particular flow f and <u>one</u> particular $S \subseteq V$ satisfying $s \in S$, $t \notin S$ and $c(S, \overline{S}) = |f|$. Thus,

$\max \{ |f| \mid f \text{ is a flow} \}$

$\geq$ THIS PARTICULAR $|f|$

$=$ THIS PARTICULAR $c(S, \overline{S})$

$\geq \min \{ c(S, \overline{S}) \mid S \subseteq V;\ s \in S;\ t \notin S \}$.

Thus, we have proven the $\geq$ sign in the inequality we want. It thus remains to prove the $\leq$ sign. In other words, we must prove that

$$\max \{ |f| \mid f \text{ is a flow} \}$$
$$\leq \min \{ c(S, \overline{S}) \mid S \subseteq V;\ s \in S;\ t \notin S \}.$$

In other words, we must prove that $|f| \leq c(S, \overline{S})$ $\forall$ flow f $\forall$ $S \subseteq V$ satisfying $s \in S$ and $t \notin S$. But this is part of Prop. 2 (c).

So Thm. 3 (b) is proven.

(a) Proceed as in part (b), but the proof of Claim 1 becomes slightly subtler: We need to argue that if N is the lowest common denominator of the values $c(a)$ with $a \in A$, then the flow f satisfies

$$N \cdot f(a) \in \mathbb{N} \quad \forall a \in A$$

throughout the execution of the algorithm. This is easy once you revisit the proof of Lem. 4.

(c) This is harder.
Good news: We won't need it.
So, omitted. $\square$