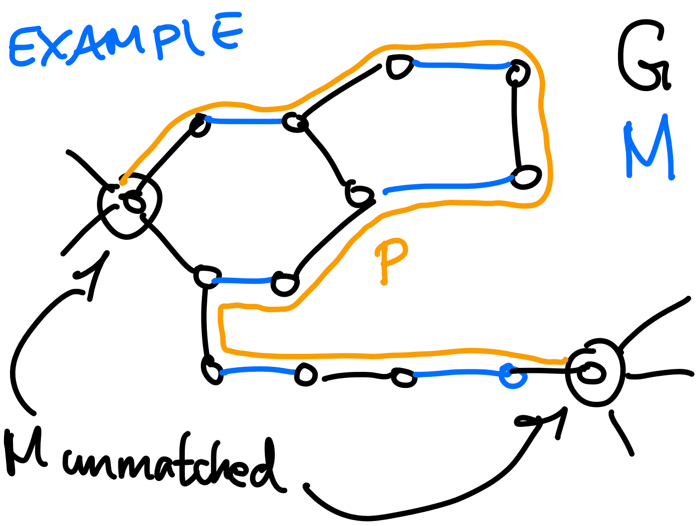
Math 5707 Spring 2023

Matching Theory Snippet 4: Max size non bipartite matching via Blossom algorithm (Schrijver §5.2)

Q: How to find a max-size matching M (so $|M| = \nu(G)$) when G is not bipartite?

We know it comes down to reliably finding M-augmenting paths P in G.

EXAMPLE



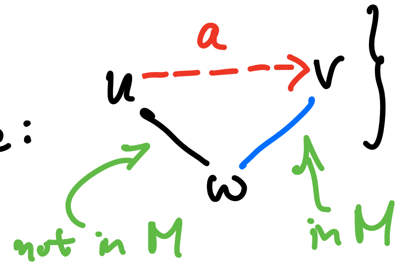
Revised Q: Can we find M-aug paths reliably (and quickly) in the non-bipartite case, where we don't know how to orient G to get a digraph D like before?

Edmonds (1961) had a good idea for another relevant digraph to associate with G, M:

Given $G=(V,E)$ and matching $M \subset E$,

let $D=(V,A)$

$A = \{$ arcs a as shown here:



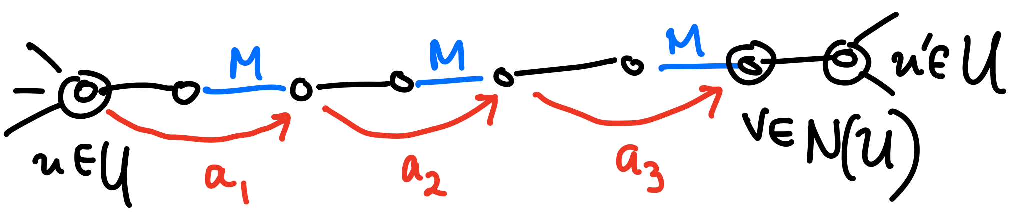
$\}$

Let $U :=$ M-unmatched vertices in G

$N(U) :=$ neighbors of U in G

$= \{ v \in V : \exists \{u,v\} \in E$ for some $u \in U \}$

He considered what happens when one searches for shortest directed paths P in D from U to $N(U)$, that is paths P from any $u \in U$ to any $v \in N(U)$

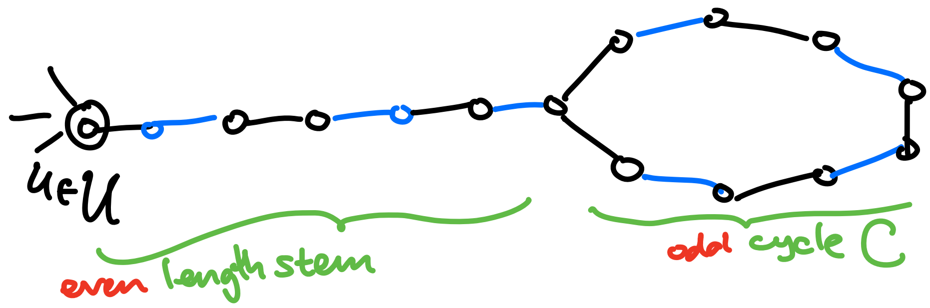


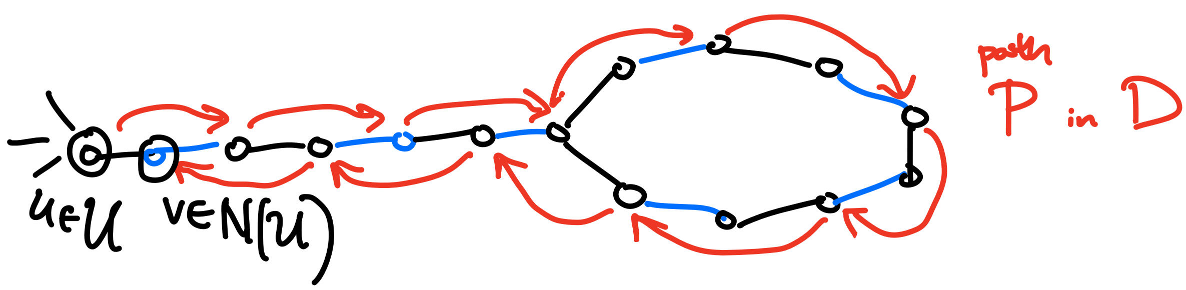
PROPOSITION:
(not hard; proof omitted)

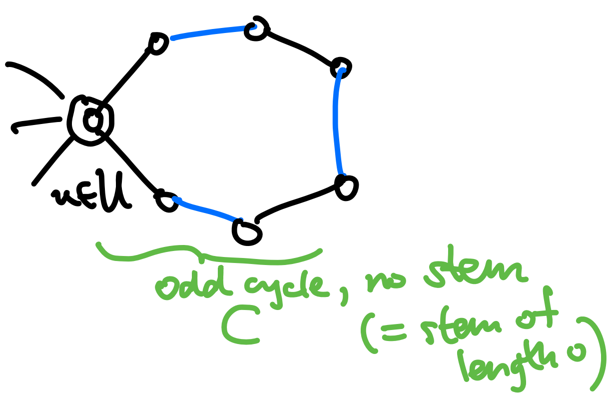
Shortest paths P from $u \in \mathcal{U}$ to $v \in N(\mathcal{U})$ in D are either M-augmenting paths P in G (good!) if they never revisit any vertices,

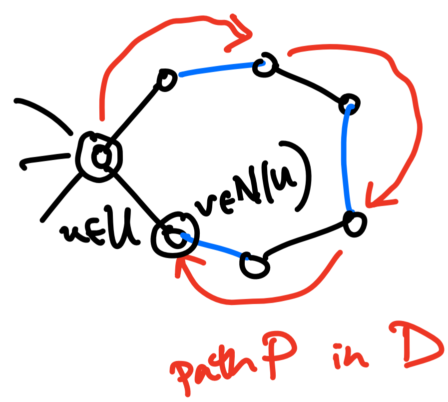
— OR —

what he called a blossom:







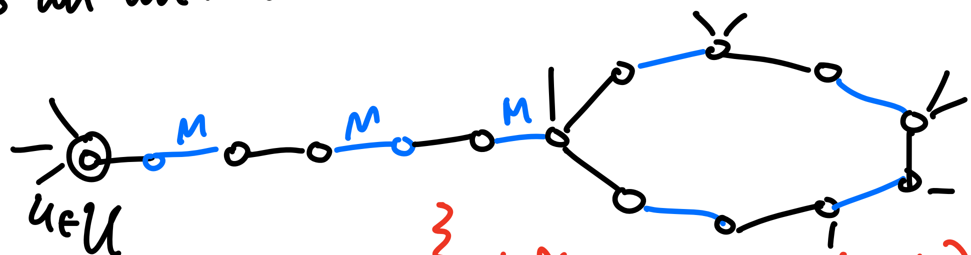


The M-augmenting paths we like.

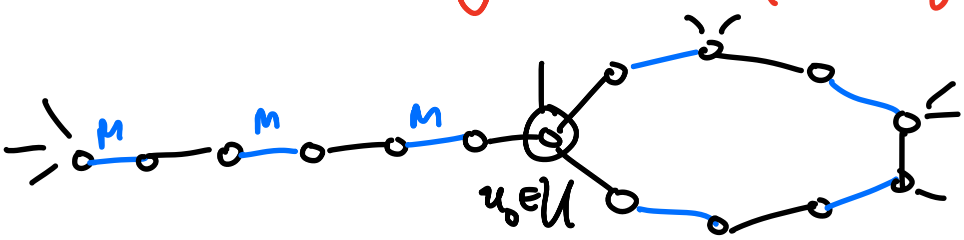
$\underline{\underline{Q}}$: How to deal with the blossoms ?

Edmonds explains how to contract them away:

$\underline{\underline{1^{st}}}$: If the stem has length ≥ 2, then shift the M edges in the stem so that the cycle C has an unmatched vertex u_0 where the stem enters it:

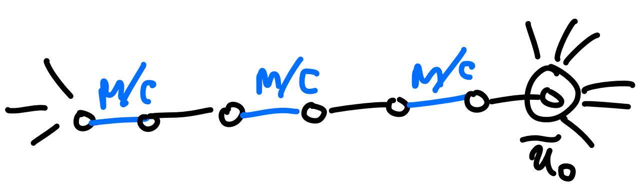


shift M on (even length) stem



$\underline{\underline{2^{nd}}}$: Contract all of the edges in C, leaving a single vertex $\bar{u}_0$ in $G/C = (V/C, E-C)$

with matching M/C



How did this help?

PROPOSITION: G has an M-augmenting path P

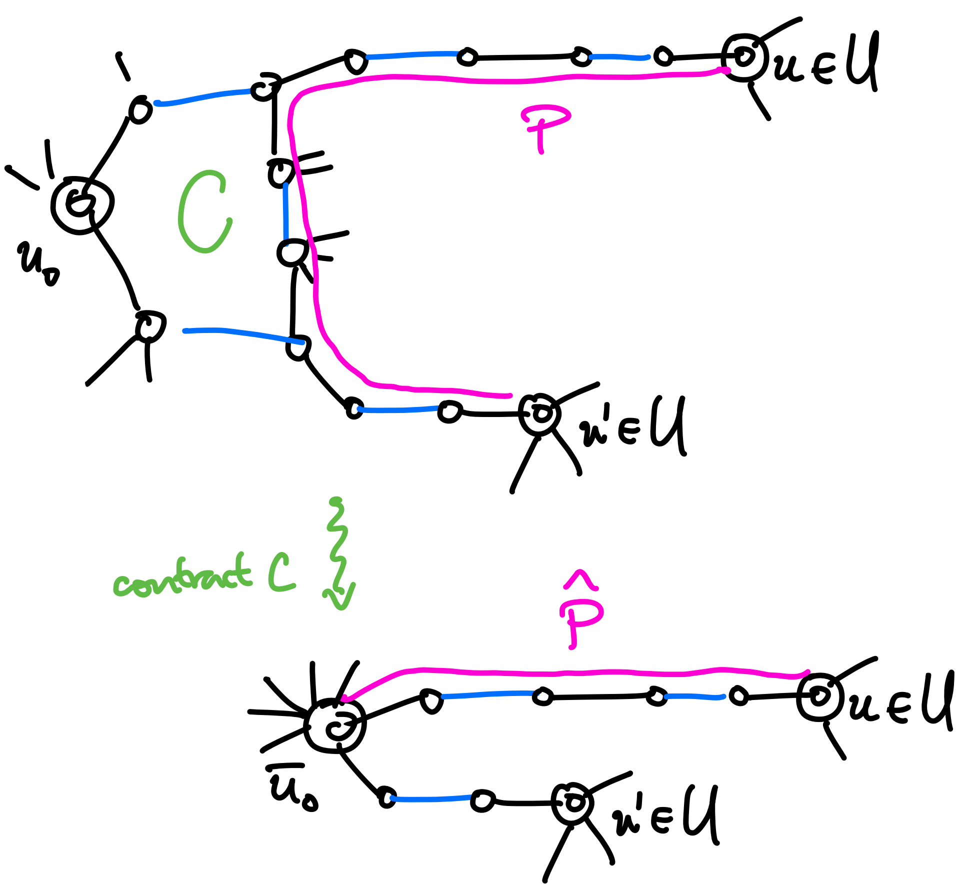
$\iff$ G/C has an M/C-augmenting path $\hat{P}$

proof: ($\Rightarrow$): An M-aug path P either

- misses C entirely, so it persists in G/C, i.e. $\hat{P} = P$,

or

- P enters C along a non-M edge, and gives rise to an M/C-aug path in G/C ending at $\bar{u}_0$:

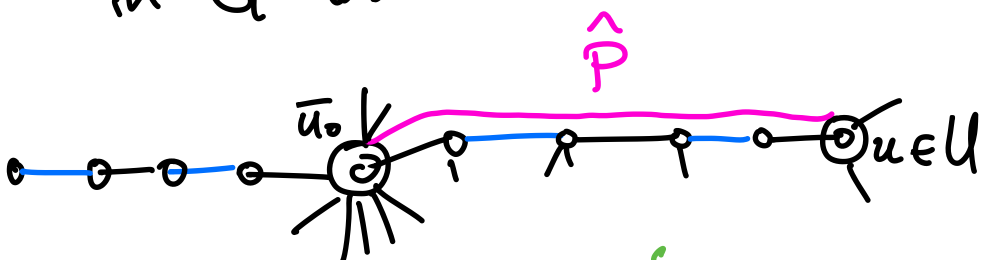


($\Leftarrow$): An M/C-aug path $\hat{P}$ in G/C either

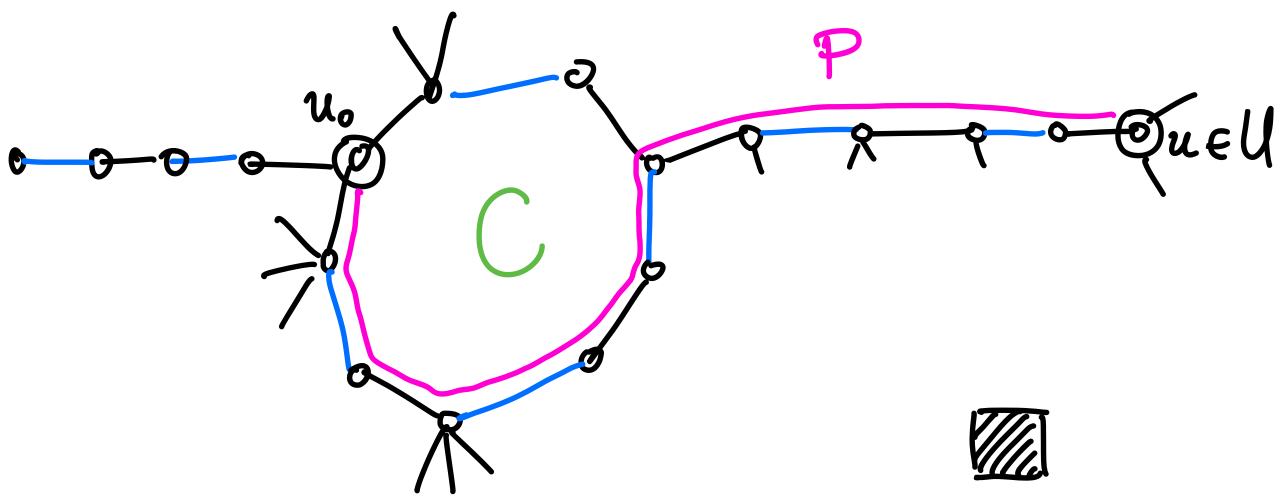
- misses $\bar{u}_0$ entirely, so it persists in G, i.e. $P = \hat{P}$,

  or

- $\hat{P}$ ends at $\bar{u}_0$ (since $\bar{u}_0$ is M/C-unmatched), and then there is exactly one way to expand $\hat{P}$ inside the cycle C to give an M-aug path P in G that ends at u_0:



expand $\bar{v}_0$ to C



$\blacksquare$

This gives Edmonds' Blossom algorithm for finding a max-sized matching M in nonbipartite graphs: keep looking for M-aug paths P using the digraphs D, and contracting blossoms C whenever D finds them, so as to work in smaller graphs G/C, where one has already found M/C-aug paths $\hat{P}$ (or shown that none exists).

REMARK:

Edmonds (1965) also produced a fast (polynomial-time) algorithm to find a maximum weight matching M in nonbipartite graphs $G=(V,E)$ with edge weights $w: E \to \mathbb{R}_{\geq 0}$, generalizing Kuhn's algorithm.

(See Schrijver §5.3)