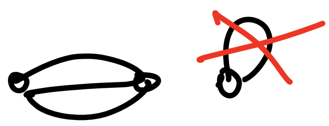
Math 5707 Spring 2023 Edge Coloring

(Bondy-Murty Chap. 6)

DEF'N: Given $G=(V,E)$ a loopless multigraph, an assignment



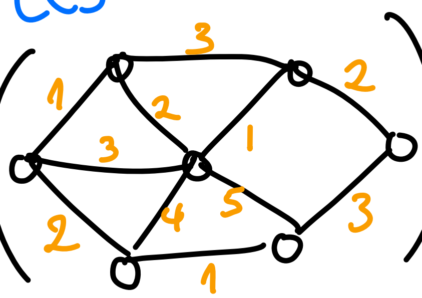
$f: E \to \{1,2,\ldots,k\}$

is called a proper edge-coloring if $f(e)\neq f(e')$ $\forall e,e'\in E$ that are incident at some vertex $v\in V$.

$\chi'(G) := \min\{k : \exists$ a proper edge-k-coloring of $G\}$

edge-chromatic # of G

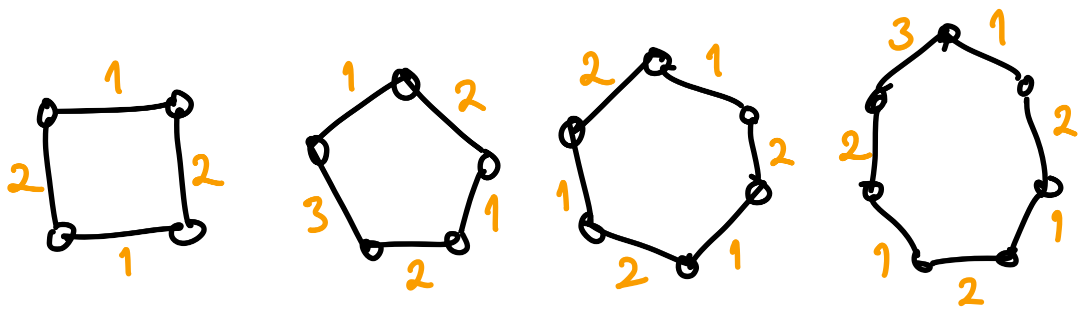
EXAMPLES:

(1) $\chi'\Big($  $\Big) \leq 5$

$= 5$

More generally, $\chi'(G) \geq \Delta(G)$ = max vertex-degree in G

(2) cycles C_n



C_4 C_5 C_6 C_7

$$\underset{2}{\underset{\|}{\Delta(C_n)}} \le \chi'(C_n) \le \underset{3}{\underset{\|}{1+\Delta(C_n)}}$$

since $\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$

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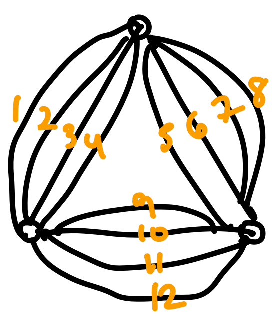
(3) G_μ



$\mu = 4$

has $\Delta(G_\mu) = 2\mu$

and $\chi'(G_\mu) = 3\mu$

Computing $\chi'(G)$, or deciding $\chi'(G) \le k$ is again an NP-complete problem.

Too bad! But it's even more frustrating for simple graphs G, because of a famous result:

THEOREM (Vizing's Thm. 1964)

not too hard; see Bondy-Murty §6.2

For any simple graph G,

$$\Delta(G) \le \chi'(G) \le 1 + \Delta(G)$$

(call a simple graph "class one" if $\chi'(G) = \Delta(G)$

"class two" if $\chi'(G) = 1 + \Delta(G)$.

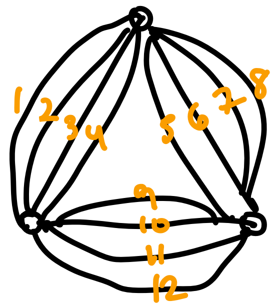
It's even NP-complete to decide for a simple graph G whether it is class one or two!)

For multigraphs, if $\mu(G) =$ largest edge multiplicity

then $\Delta(G) \le \chi'(G) \le \mu(G) + \Delta(G)$

e.g.

G_μ



$\mu = 4$

has $\Delta(G_\mu) = 2\mu$

and $\chi'(G_\mu) = 3\mu$

so has $\underset{\substack{\| \\ 3\mu}}{\chi'(G_\mu)} = \underbrace{\mu(G)}_{=\mu} + \underbrace{\Delta(G)}_{=2\mu}$

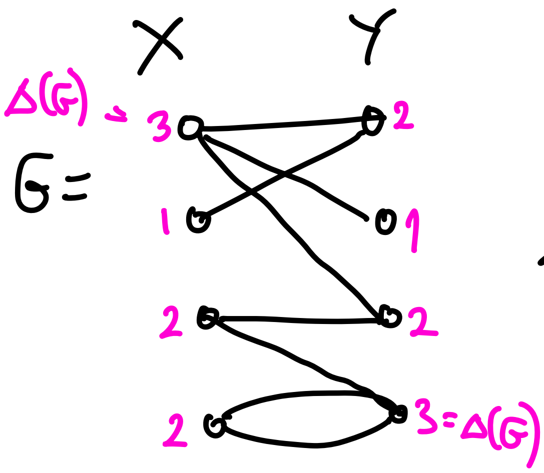
On the other hand, bipartite graphs behave better:

THEOREM (König's "Line-coloring theorem") 1931

For G a bipartite multigraph,

$$\chi'(G) = \Delta(G).$$

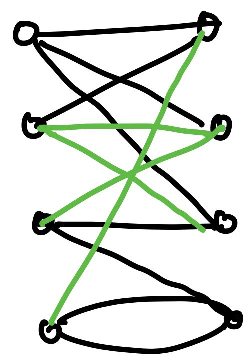
proof: Given $G = (X \sqcup Y, E)$ a bipartite multigraph with $\Delta(G)$ its max vertex degree, one can add edges to obtain $G^+ = (X \sqcup Y, E^+)$ which is degree $\Delta(G)$-regular:



G bipartite

$\Delta(G) = 3$

G^+

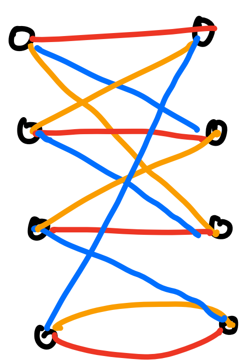


G^+ bipartite and $\underset{\Delta(G)}{\overset{\shortparallel}{3}}$-regular

by a result already proven in matching theory using Hall's Theorem, can decompose

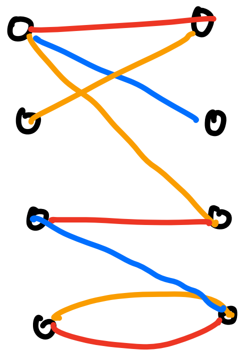
$$E(G^+) = M_1 \sqcup M_2 \sqcup \dots \sqcup M_{\Delta(G)}$$

for matchings M_i



$E(G^+) = M_1 \sqcup M_2 \sqcup M_3$ $\quad 3 = \Delta(G)$

erase the edges in $E(G^+) - E(G)$,
giving...

$G =$  ... with a proper edge-$\Delta(G)$-coloring. $\blacksquare$