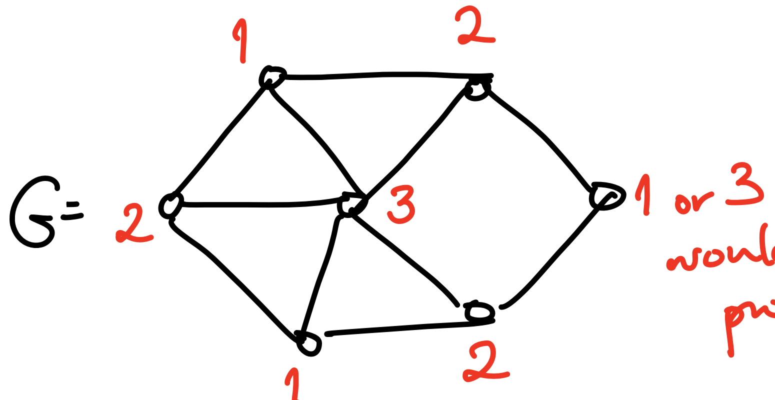
Math 5707 Spring 2023 Vertex-coloring

Vertex-coloring (Bondy–Murty Ch. 8)

DEF'N: Given $G=(V,E)$ a (simple) graph, an assignment

$$f: V \longrightarrow \{1,2,\dots,k\}$$

is called a proper vertex-coloring of G if $f(x)\neq f(y)$ $\forall$ edges $\{x,y\}\in E$.



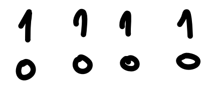
would both give proper colorings

$k=3$

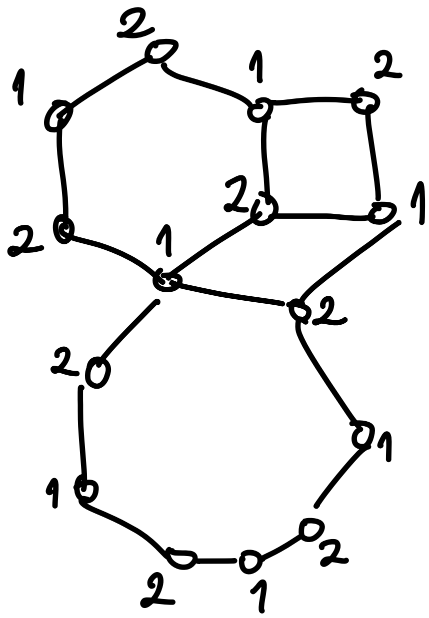
G has no proper 2-colorings

DEF'N: $\chi(G) :=$ chromatic number of G
$:= \min\{k : \exists$ at least one proper vertex-coloring of G with k colors$\}$

EXAMPLES:

- $\chi(G) = 1 \iff G$ has no edges i.e. $G =$ 

- $\chi(G) = 2 \iff G = (X \sqcup Y, E)$ is bipartite



$k=2$

since can make

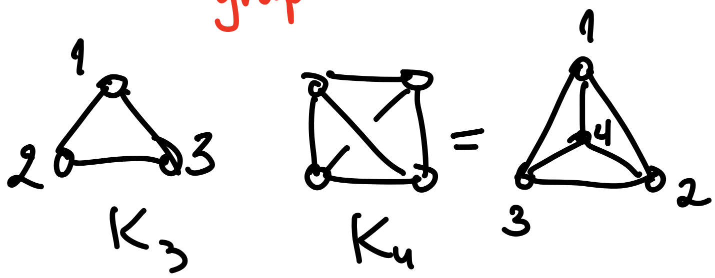
$X := f^{-1}(1)$

$Y := f^{-1}(2)$

- $\chi(G) = 3$ is hard to characterize – there are no simple necessary and sufficient conditions for it, and deciding whether $\chi(G) \leq 3$ is an NP-complete decision problem.

- $\chi(K_n) = n$

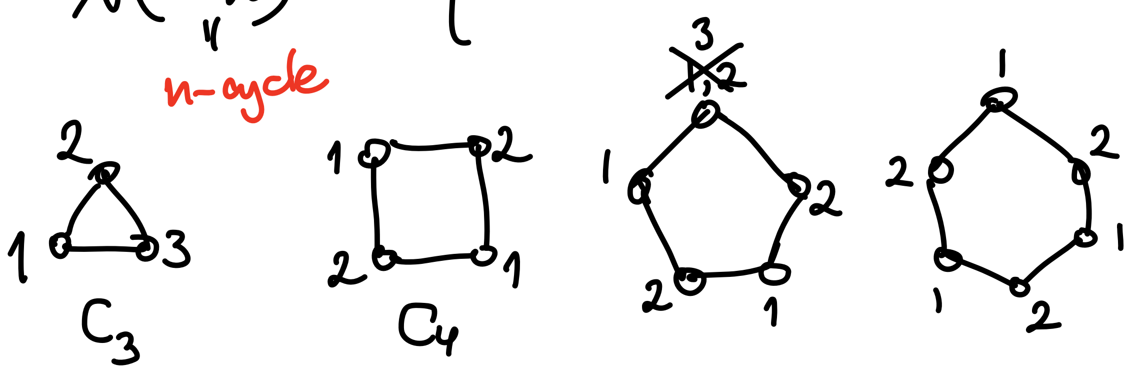
  K_n = complete graphs



$\chi(K_3)=3$

- $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$

  C_n = n-cycle



$\chi(C_3)=3$ $\chi(C_4)=2$ $\chi(C_5)=3$ $\chi(C_6)=2$

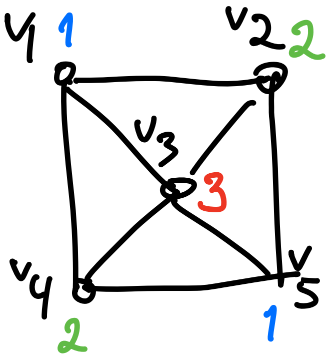
- If we pick $V' \subset V$ for $G=(V,E)$ and form the vertex-induced subgraph

$$G[V'] := (V', E'), \quad E' = \{\{x,y\} \in E : x,y \in V'\}$$

then $\chi(G[V']) \leq \chi(G)$

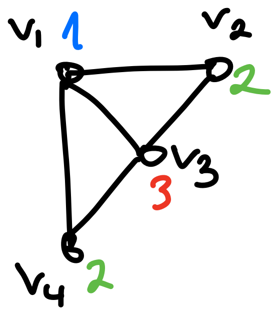
e.g.

$G=$



$V' = \{v_1, v_2, v_3, v_4\}$

$\rightsquigarrow$ $G[V'] =$



In particular,

$$\chi(G) \geq \max\{k : \exists \text{ a vertex-induced subgraph of } G \text{ isomorphic to } K_k\}$$

called a <u>k-clique</u> in G

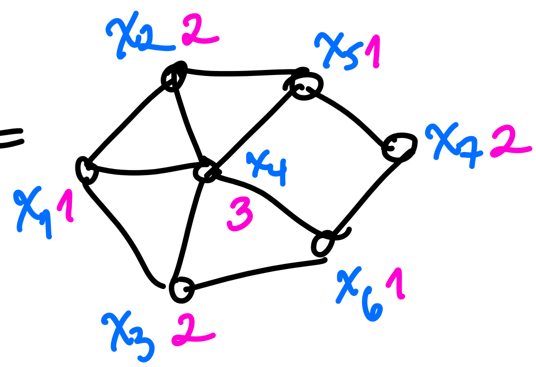
One can get an easy upper bound for $\chi(G)$ in terms of vertex degrees, from the greedy coloring algorithm:

Order $V = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$

and then for each $i = 1, 2, \ldots, n$

assign vertex x_i the color

$$f(x_i) = \min\Big\{ \{1,2,3,\ldots\} - \{ f(x_j) : j \in \{1,2,\ldots,i-1\} \text{ and } \{x_i, x_j\} \in E \} \Big\}$$

e.g. $G =$ 

CORROLLARY:

Let $\Delta(G) :=$ max vertex degree in G

$= \max\{ \deg_G(v) : v \in V \}$

Then $\chi(G) \leq 1 + \Delta(G)$

proof: When $k = 1 + \Delta(G)$, as you

do the greedy coloring, there is always a color avaible for $f(x_i)$ since x_i has $\leq \Delta(G)$ previously colored neighbors. ∎

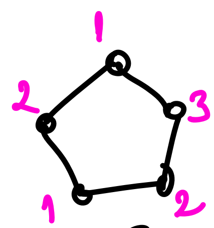
REMARK:

We really showed

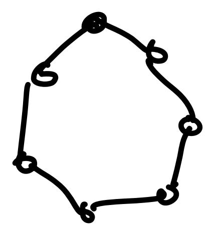
$$\chi(G) \leq 1 + \max\{\deg_{G[x_1,x_2,\dots,x_i]}(x_i) : i = 1,2,\dots,n\}$$

$$\leq 1 + \Delta(G)$$

EXAMPLES:



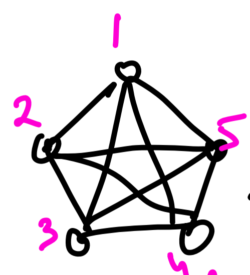
C_5



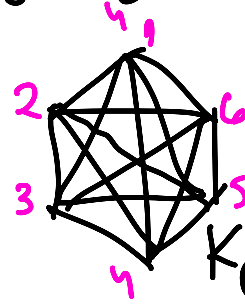
C_7

Cycles C_n with n odd have

$$\underbrace{\chi(C_n)}_{=3} = 1 + \underbrace{\Delta(C_n)}_{=2} = 3$$



K_5



K_6

K_n has $\underbrace{\chi(K_n)}_{=n} = 1 + \underbrace{\Delta(K_n)}_{=n-1} = n$

THEOREM (Brooks 1941) For a connected simple graph G, unless $G = C_n$ or K_n
$\quad$ (C_n: n odd; K_n: complete)

one has $\chi(G) \leq \Delta(G)$ $\quad$ $\Delta(G)$ = max degree in G

NOTE: Brooks does not say that

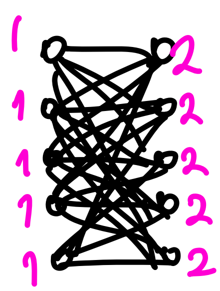
$$G \neq \begin{cases} C_n & n \text{ odd} \\ K_n \end{cases}$$

then $\chi(G) = \Delta(G)$

e.g. $K_{m,m}$ for m large has

$$\underbrace{\chi(K_{m,m})}_{2} < \underbrace{\Delta(K_{m,m})}_{m}$$

$m=5$



$\chi=2$

$\Delta=5$