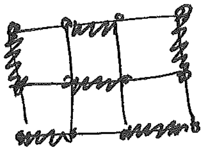
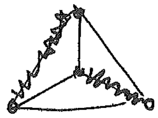
(75)

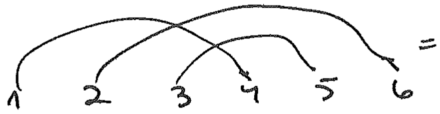
Pfaffians and matchings (Ardila §3.1.5)

DEFN: In a graph $G=(V,E)$, a (perfect) matching $M \subset E$ (1-factor) is a set of edges for which $\deg_M(v)=1 \ \forall v \in V$

e.g. $G=$  $M=$ ~~~ edges

$G=$ 

A matching M in $K_{2n} = (\overset{V}{\overset{\shortparallel}{[2n]}}, \underset{\{\text{all pairs } \{i,j\}: 1\le i<j\le n\}}{\underset{\shortparallel}{E}})$ will be depicted by V on a line with arcs in upper half-plane:

e.g.  $=M$

Its crossing number $cr(M) := \#$ crossing of arcs (drawn generically; none of this )

$= \#\{1 \le i < j < k < l : \{i,k\},\{j,l\} \in M\}$

PROP: The generic skew symmetric matrix $A = \begin{bmatrix} 0 & a_{12} & a_{13} \cdots & a_{1N} \\ -a_{12} & 0 & a_{23} & \\ -a_{13} & -a_{23} & \ddots & \\ \vdots & & & \\ -a_{1N} & & & 0 \end{bmatrix} = -A^T$

has $\det A = 0$ if N is odd

$\det A = \mathrm{Pf}(A)^2$ if N is even

where $\mathrm{Pf}(A) := \sum_{\substack{\text{matchings } M \\ \text{of } K_{2n}}} (-1)^{cr(M)} \prod_{\substack{\{i,j\} \in M \\ 1 \le i<j \le n}} a_{ij}$

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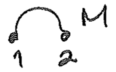
e.g. $N=2$ $\quad \det\begin{bmatrix} 0 & +a_{12} \\ -a_{12} & 0 \end{bmatrix} = a_{12}^2 \qquad \mathrm{Pf}(A) = a_{12}$

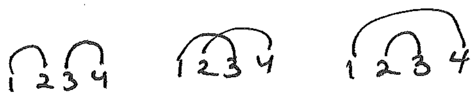


$N=3$ $\quad \det\begin{bmatrix} 0 & +a_{12} & +a_{13} \\ -a_{12} & 0 & +a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = a_{12}\det\begin{bmatrix} a_{12} & a_{13} \\ -a_{23} & \end{bmatrix} - a_{13}\det\begin{bmatrix} a_{12} & \\ 0 & a_{23} \end{bmatrix}$

$= a_{12}a_{13}a_{23} - a_{13}a_{12}a_{23}$

$= 0$

$N=4$ $\quad \det\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} = (\overbrace{a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}}^{\mathrm{Pf}(A)})^2$



If N is odd, then $\det(A) = \det(A^t) = \det(-A) = (-1)^N\det(A) = -\det(A) \Rightarrow \det(A) = 0$.

<u>proof:</u> Want $\det A = \sum_{w\in\mathfrak{S}_{2n}} \mathrm{sgn}(\sigma)\prod_{i=1}^{2n} a_{i,w(i)} \overset{?}{=} \mathrm{Pf}(A)^2$

for $\underset{2n}{\overset{\shortparallel}{N}}$ even

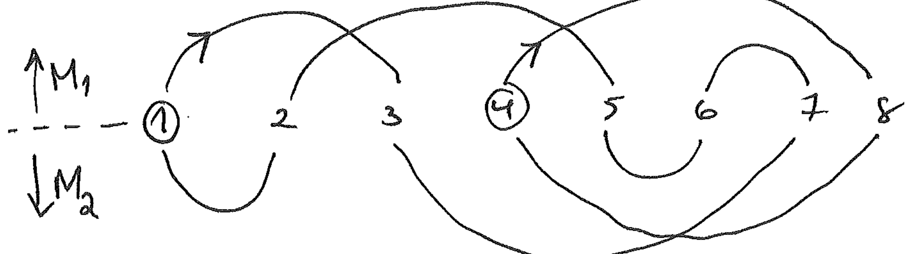
$$\mathrm{Pf}(A)^2 = \sum_{\substack{\text{matchings}\\ (M_1,M_2)\\ \text{in } K_{2n}}} (-1)^{cr(M_1)+cr(M_2)} \prod_{\substack{\{i,j\}\in M_1\sqcup M_2\\ 1\le i<j\le 2n}} a_{ij}$$

with convention $a_{j,i} = -a_{i,j}$ if $i<j$

[NOTE: $(2n)!$ terms in LHS $> ((2n-1)(2n-3)\cdots5\cdot3\cdot1)^2$ terms in RHS

$(2n)! = ((2n-1)\cdots5\cdot3\cdot1)\cdot((2n)(2n-2)\cdots6\cdot4\cdot2)$]

A pair (M_1,M_2) of matchings gives rise to a $w\in\mathfrak{S}_{2n}$ by orienting the (even!) cycles in $M_1\sqcup M_2$:



$\rightsquigarrow w = (\overset{M_1}{\frown}\!\!\!\!\!\textcircled{1}3\,7\,6\,5\,2)(\overset{M_1}{\frown}\!\!\!\!\!\textcircled{4}8)$

11/06/2015 We ~~CLAIM:~~ $(-1)^{cr(M_1)+cr(M_2)}\prod_{\{i,j\}\in M_1\sqcup M_2} a_{ij} = \mathrm{sgn}(\sigma)\prod_{i=1}^{2n} a_{i,w(i)}$

e.g. $(-1)^{2+1}a_{13}a_{37}a_{67}a_{56}a_{25}\cdot a_{48}^2 = (-1)^2 a_{13}a_{37}a_{76}a_{65}a_{52}a_{21}a_{48}a_{84}$ ✓

(77) Note the CLAIM is equivalent to $(-1)^{cr(M_1)+cr(M_2)} \overset{?}{=} \mathrm{sgn}(w)\cdot(-1)^{exc(w)}$

which one can prove by noting that

(i) left and right sides both change by ±1 in same way if one conjugates w by an adjacent transposition $(i,i+1)$
(check cases - easy to see both change by $\begin{cases} -1 & \text{if } (i,i+1) \text{ matched in } M_1 \text{ or } M_2 \\ +1 & \text{else} \end{cases}$)

(ii) so by conjugating ~~[illegible]~~ one can make w some canonical permutation of a given cycle type λ, where it is easy to check,

e.g. $\lambda = 442$

$\leadsto w = (1234)(5678)(9\,10)$

$$(-1)^{cr(M_1)+cr(M_2)} \overset{?}{\underset{\checkmark}{=}} \mathrm{sgn}(w)\cdot(-1)^{exc(w)}$$

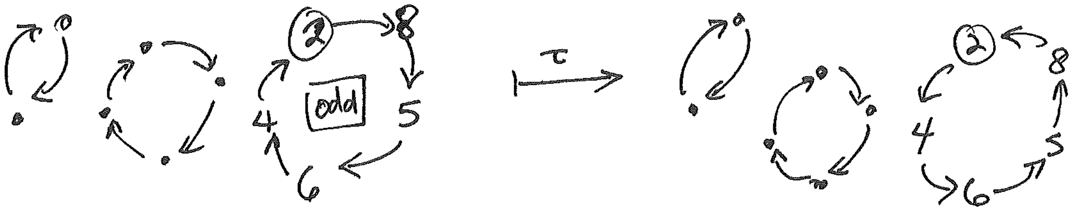
$$(-1)^{cr(M_1)+cr(M_2)} = (-1)^0 = +1$$

$$\mathrm{sgn}(w)\cdot(-1)^{exc(w)} = (-1)^{1+1}\cdot(-1)^{1+1}\cdot(-1)^{n+1} = +1$$

Need only define a sign-reversing involution $\tau: X \to X$, $X = \mathfrak{S}_{2n}$

that cancels w having <u>at least one odd cycle</u>:

find the odd cycle in w with smallest entry, and reverse its arrows:



w $\quad a_{28}a_{85}a_{56}a_{64}a_{42}$

$\tau(w)$ $\quad a_{82}a_{58}a_{65}a_{46}a_{24}$

$= (-1)^5 a_{28}a_{85}a_{56}a_{64}a_{42}$

∎

RMK: Pfaffians <u>generalize</u> <u>determinants</u> also:

$$\det\left[\begin{array}{c|c} 0 & A \\ \hline -A^t & 0 \end{array}\right] = (\det A)^2 \quad \text{and} \quad \mathrm{Pf}\left[\begin{array}{c|c} \overset{a_1\cdots a_n}{0} & \overset{b_1\cdots b_n}{A} \\ \hline -A^t & 0 \end{array}\right] = \det A$$

consistent with

=M having cr(M)=sgn

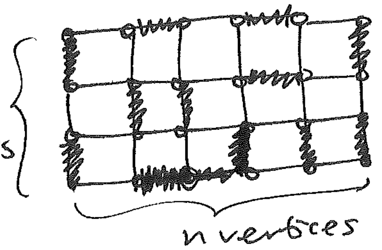
if w= (a₁ - - - - aₙ

(78) Kasteleyn's (1961) method for the dimer problem – the "Pfaffian-Hafnian" (& "permanent-determinant") method

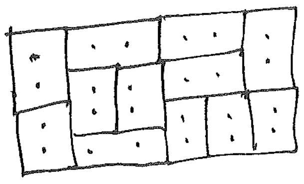
(Ardila §3.1.5, Loehr "Bijective Comb." §12.12, B)

Kasteleyn wanted to count the number of perfect matchings

in $G_{m,n}$:=



bijects with domino tilings of rectangles



and other graphs;

called the dimer problem for G

WLOG m is even (else $|V| = mn$ is odd if both m, n odd)

e.g. we saw already for $m=2$ one gets Fibonacci #'s

$n=2$:

2

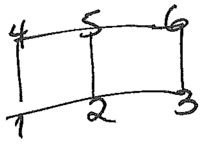
$n=3$:

3

His idea was to start with $A_G = \begin{cases} a_{ij} = -a_{ji} & \text{if } i<j \text{ and } \{i,j\} \in G \\ 0 & \text{if } \{i,j\} \notin G \end{cases}$

and its Pfaffian; which counts the matchings with unwanted signs ...

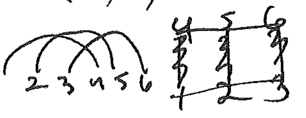
e.g. $G =$



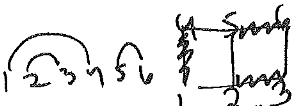
$$\rightsquigarrow A_G = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & a_{12} & & a_{14} & & \\ 2 & -a_{12} & 0 & a_{23} & & a_{25} & \\ 3 & & -a_{23} & 0 & & & a_{36} \\ 4 & -a_{14} & & & 0 & a_{45} & \\ 5 & & -a_{25} & & -a_{45} & 0 & a_{56} \\ 6 & & & -a_{36} & & -a_{56} & 0 \end{array}$$

has $Pf(A_G) \left(= \pm\sqrt{\det(A_G)} \right)$

$= -a_{14}a_{25}a_{36}$



$+a_{14}a_{23}a_{56}$



$+a_{12}a_{36}a_{45}$

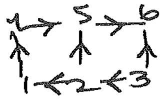
But it would be fixed, i.e. all signs the same, if we swapped $+a_{12}$ for $-a_{12}$, $+a_{23}$ for $-a_{23}$

(79)

<u>DEF'N</u>: Given $G=(V,E)$ undirected and $D=(V,A)$ <u>directing</u> E (i.e. an orientation of G)

create S_D skew-symmetric $|V|\times|V|$ with ~~above~~ diagonal entries

$$(S_D)_{ij} = \begin{cases} +a_{ij} \text{ if } i<j \text{ and } \overset{i}{\circ}\to\overset{j}{\circ} \text{ in } D \\ (=-a_{ji}) \\ -a_{ij} \text{ if } i<j \text{ and } \overset{i}{\circ}\leftarrow\overset{j}{\circ} \text{ in } D \\ (=+a_{ji}) \end{cases}$$

e.g. $D=$ 

gives

$$S_D = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & -a_{12} & & +a_{14} & & \\ 2 & +a_{12} & 0 & -a_{13} & & +a_{25} & \\ 3 & & +a_{13} & 0 & & & +a_{36} \\ 4 & -a_{14} & & & 0 & +a_{45} & \\ 5 & & -a_{25} & & -a_{45} & 0 & +a_{56} \\ 6 & & & -a_{36} & & -a_{56} & 0 \end{array}$$

and $\mathrm{Pf}(S_D) = -(a_{14}a_{25}a_{36} + a_{14}a_{23}a_{56} + a_{12}a_{36}a_{45})$

all same sign

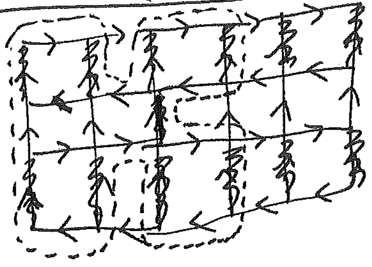
11/09/2015 >

<u>THM</u> (Kasteleyn):

(a) Given $G=(V,E)$, an orientation D makes all terms have same sign in $\mathrm{Pf}(S_D)$ (sometimes called a <u>Pfaffian orientation of G</u>)

$\iff$ every circuit C of G that <u>alternates</u> <u>edges</u> in some perfect matching M of G has an <u>odd</u> <u>number</u> of C's edges agreeing/disagreeing with D.

(b) <u>Every planar graph with a perfect matching has a Pfaffian orientation</u>

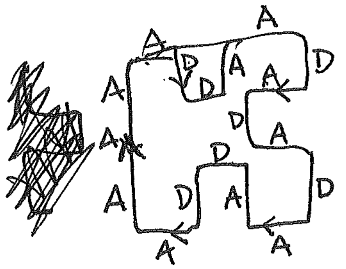
e.g. 

- up in cols
- alternate right/left in rows

turns out to work for grids (not obvious – requires proof!)

M = 〰

C = ⬚



11 A's (agrees)
7 D's (disagees

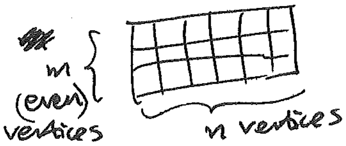
RMK: This shows one can count perfect matchings in planar graphs $G=(V,E)$ (or graphs with a Pfaffian orientation D) in $\le c|V|^3$ steps, by computing

$$\left|\mathrm{Pf}\left([S_D]_{a_{ij}=1}\right)\right| = \left|\sqrt{\underbrace{\det\left([S_D]_{a_{ij}=1}\right)}_{\text{computed via Gaussian elimination}}}\right|$$

Thus these dimer problems become tractable, like spanning trees (via Kirchoff) Euler tours (via B.E.S.T.).

THM (Kasteleyn) essentially - but

$$\sum_{\substack{\text{matchings } M\\ \text{in}}}' x^{\#\text{vertical edges in } M}\, y^{\#\text{horizontal edges in } M} = 2^{\frac{mn}{2}} \prod_{j=1}^{\frac{m}{2}} \prod_{k=1}^{n} \sqrt{x^2\cos^2\left(\frac{j\pi}{m+1}\right)+y^2\cos^2\left(\frac{k\pi}{n+1}\right)}$$



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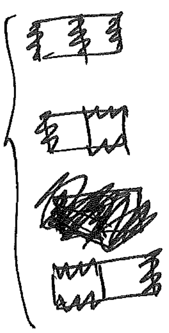
e.g. for $m=2$, $n=3$



$$\text{LHS} = \left\{ \begin{array}{l} x^3 \\ +xy^2 \\ +xy^2 \end{array} \right\} = x^3+2xy^2 \quad \text{vs.}$$

$$\text{RHS} = 2^{\frac{2\cdot 3}{2}} \prod_{j=1}^{\frac{2}{2}} \prod_{k=1}^{3} \sqrt{x^2\cos^2\left(\frac{j\pi}{3}\right)+y^2\cos^2\left(\frac{k\pi}{4}\right)}$$

$$= 8 \prod_{k=1}^{3} \sqrt{x^2\left(\frac{1}{4}\right)+y^2\cos^2\left(\frac{k\pi}{4}\right)}$$

$$= 8\sqrt{\frac{x^2}{4}+\frac{y^2}{2}} \cdot \sqrt{\frac{x^2}{4}+0}\ \sqrt{\frac{x^2}{4}+\frac{y^2}{2}}$$

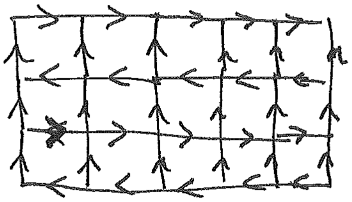
$$= 8\left(\frac{x^2}{4}+\frac{y^2}{2}\right)\left(\frac{x}{2}\right)$$

$$= x^3+2xy^2 \ \checkmark \quad (!)$$

(81)

<u>proof</u> (sketch): Choose D orienting G as above

See Loehr for details



and set $a_{ij} = \begin{cases} x & \text{if } \{i,j\} \text{ vertical } \begin{smallmatrix} \circ\, i \\ | \\ \circ\, j \end{smallmatrix} \\ y & \text{if } \{i,j\} \text{ horizontal } \begin{smallmatrix} \circ - \circ \\ i \quad j \end{smallmatrix} \end{cases}$

Then check that

$$S_D \quad \left(\underset{\text{for } m=2,\ n=3}{=\!=} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \left[\begin{array}{ccc|ccc} 0 & -y & & x & & \\ +y & 0 & -y & & x & \\ & +y & 0 & & & x \\ \hline -x & & & 0 & +y & \\ & -x & & -y & 0 & +y \\ & & -x & & -y & 0 \end{array} \right] \right)$$

can be re-expressed as

$$S_D = x\left(I_n \otimes Q_m\right) + y\left(Q_n \otimes F_m\right)$$

where $\underbrace{I_n := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n}$, $F_m := \underbrace{\begin{bmatrix} -1 & & & \\ & +1 & & \\ & & -1 & \\ & & & +1 \ddots \end{bmatrix}}_{m \text{ (even)}}$, $Q_n := \begin{bmatrix} 0 & +1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & +1 \\ 0 & & -1 & 0 \end{bmatrix}$.

Then Q_n has <u>explicit</u> eigenvectors and eigenvalues

$$\uparrow \left\{ 2i \cos\left(\frac{j\pi}{n+1}\right) \right\}_{j=1,\dots,n}$$

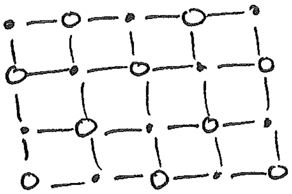
(related to discrete Laplacian for a <u>path</u>: $\circ - \circ \cdots - \circ - \circ$)

from which one can deduce the eigenvectors/ eigenvalues of S_D,

and compute $\sqrt{|\det S_D|} = \mathrm{Pf}(S_D)$) ∎

<u>REMARKS</u>

① If G is bi<u>partite</u>, as with the grids $G_{m,n}$ =



then one can write

$$S_D = \begin{array}{c} \text{black} \\ \text{white} \end{array} \left[\begin{array}{c|c} \overset{\text{black}}{0} & \overset{\text{white}}{A} \\ \hline -A^t & 0 \end{array} \right] \quad \text{so} \quad \mathrm{Pf}(S_D) = \det(A)$$

② Why "Pfaffian-Hafnian", "Permanent-determinant"?

$\mathrm{Per}(M) := \sum \left(+ \prod^{n} m_{i,w(i)}\right) \left(= \det(M') \right)$, $\underset{\text{Hafnian}}{\mathrm{Haf}(A)} := \sum_{\substack{\text{matchings} \\ M \text{ of } [2n]}} \left(+ \prod_{\substack{\{i,j\} \in M \\ 1 \le i < j \le 2n}} a_{ij} \right) \left(= \mathrm{Pf}(A')\right)$

↑ if ... can luckily

(82)

11/11/2015

MacMahon's "Master Theorem"

—another way to convert a permanent to a determinant...

Given $A = (a_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$ if $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ are related

by $\underline{y} = A\underline{x}$, then one can view

$$\mathrm{per}(A) := \sum_{w \in \mathfrak{S}_n} a_{1,w(1)} \cdots a_{n,w(n)} = [x_1^1 \cdots x_n^1](a_{11}x_1 + \cdots + a_{1n}x_n)^1 (a_{21}x_1 + \cdots + a_{2n}x_n)^1 \cdots (a_{n1}x_1 + \cdots + a_{nn}x_n)^1$$

$$= [x_1^1 \cdots x_n^1]\, y_1^1 \cdots y_n^1$$

$$\boxed{\begin{array}{l} \text{e.g. } [x_1^1 x_2^1](ax_1+bx_2)(cx_1+dx_2) \\ \boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \quad = [x_1x_2](acx_1^2 + bdx_2^2 + (ad+bc)x_1x_2) \\ \qquad\qquad\qquad = ad+bc \end{array}}$$

and more generally one might consider for $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$

the generalized permanents $\mathrm{per}_{\underline{k}}(A) := [x_1^{k} \cdots x_n^{k}]\, y_1^{k} \cdots y_n^{k}$

$$= [\underline{x}^{\underline{k}}]\, \underline{y}^{\underline{k}}$$

MacMahon's "Master THM": If $\underline{y} = A\underline{x}$ then in $\mathbb{Z}[a_{ij}][[t_1, \ldots, t_n]]$

$$\sum_{\underline{k} \in \mathbb{N}^n} \underbrace{[\underline{x}^{\underline{k}}]\underline{y}^{\underline{k}}}_{\mathrm{per}_{\underline{k}}(A)} \cdot \underline{t}^{\underline{k}} = \frac{1}{\det(I_n - TA)} \quad \text{where } T = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}$$

proof: Let's prove the version with $t_1 = \cdots = t_n = 1$

$$\sum_{\underline{k} \in \mathbb{N}^n} [\underline{x}^{\underline{k}}]\, \underline{y}^{\underline{k}} \overset{(*)}{=} \frac{1}{\det(I_n - A)} \quad \text{as an identity in } \mathbb{Z}[[a_{ij}]]$$

since we can then apply the map $\mathbb{Z}[[a_{ij}]] \xrightarrow{\varphi} \mathbb{Z}[[a_{ij}, t_i]]$

$$a_{ij} \longmapsto a_{ij}t_i$$

$$A \longmapsto \begin{bmatrix} t_1a_{11} & t_1a_{12} & \cdots \\ t_2a_{21} & t_2a_{22} & \\ \vdots & & \ddots \end{bmatrix} = TA$$

$$y_i \longmapsto y_it_i$$

and then $(*)$ becomes the MMT.

(83) For $(*)$, $\displaystyle\sum_{\underline{k}\in\mathbb{N}^n} [\underline{x}^{\underline{k}}]\,\underline{y}^{\underline{k}} \overset{?}{=} \frac{1}{\det(I_n - A)}$ if $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \underline{y} = A\underline{x}$ where $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

let's reinterpret the LHS in terms of A as a map on polynomials $K[\underline{x}] := K[x_1, \ldots, x_n]$ over some algebraically closed field K that contains the $\{a_{ij}\}$, e.g. $K := \overline{\mathbb{C}(a_{ij})}$ = alg. closure of rational fns in $\{a_{ij}\}$

One has
$$\begin{array}{ccc} K[\underline{x}] & \xrightarrow{A} & K[\underline{x}] \\ x_i & \longmapsto & y_i = a_{i1}x_1 + \ldots + a_{in}x_n \end{array}$$

and A preserves the degree of polynomials

so it maps $K[\underline{x}]_d \xrightarrow{A} K[\underline{x}]_d$
$$K[\underline{x}]_d = K\text{-span of the basis } \{\underline{x}^{\underline{k}}\}_{|\underline{k}|=d}, \quad |\underline{k}| = k_1 + \ldots + k_n$$

Then in $(*)$ one has
$$\text{LHS} = \sum_{d=0}^{\infty} \left(\sum_{\substack{\underline{k}\in\mathbb{N}^n: \\ |\underline{k}|=d}} [\underline{x}^{\underline{k}}]\,\underline{y}^{\underline{k}} \right) = \sum_{d=0}^{\infty} \operatorname{Trace}\left(A : K[\underline{x}]_d \to K[\underline{x}]_d\right)$$

Since K is alg. closed, we can pick some change-of-basis in $Kx_1 + \ldots + Kx_n = K[\underline{x}]_1$ to get an A-eigenvector basis $z_1, \ldots, z_n$, say with eigenvalues $\lambda_1, \ldots, \lambda_n \in K$

LYING: a triangular basis is good enough! $PAP^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then $A\,\underline{z}^{\underline{k}} = \underline{\lambda}^{\underline{k}}\,\underline{z}^{\underline{k}}$ and $\{\underline{z}^{\underline{k}}\}_{|\underline{k}|=d}$ gives an A-eigenbasis for $K[\underline{x}]_d$,

so $\displaystyle\operatorname{Trace}\left(A : K[\underline{x}]_d \to K[\underline{x}]_d\right) = \sum_{\substack{\underline{k}\in\mathbb{N}^n: \\ |\underline{k}|=d}} \underline{\lambda}^{\underline{k}}.$

Hence
$$\text{LHS} = \sum_{d=0}^{\infty} \left(\sum_{\substack{\underline{k}\in\mathbb{N}^n: \\ |\underline{k}|=d}} \underline{\lambda}^{\underline{k}} \right) = \prod_{i=1}^{n} \left(\sum_{k_i=0}^{\infty} \lambda_i^{k_i} \right) = \frac{1}{\prod_{i=1}^{n}(1-\lambda_i)}$$
$$= \frac{1}{\det\begin{bmatrix} 1-\lambda_1 & & \circledast \\ & \ddots & \\ 0 & & \end{bmatrix}} = \frac{1}{\det\left(I - \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)} = \frac{1}{\det(I - A)} \quad \blacksquare$$

EXAMPLE: We know $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$. What about $\sum_k(-1)^k\binom{n}{k}^2$?

$\sum_k(-1)^k\binom{n}{k}^3$?

e.g. $n=3$

$$\left.\begin{array}{l} 1-3+3-1=0 \\ 1^2-3^2+3^2-1^2=0 \\ 1^3-3^3+3^3-1^3=0 \\ \vdots \end{array}\right\} \text{ Clearly } \sum_k(-1)^k\binom{n}{k}^d=0 \text{ if } n \text{ odd } \forall d=1,2,3,\ldots$$

$n=4$

$$1-4+6-4+1=0 \checkmark$$

$$1^2-4^2+6^2-4^2+1^2=36-32+2=6\left(=\binom{4}{2}\right)$$

$$1^3-4^3+6^3-4^3+1^3=216-128+2=90\left(=-\binom{6}{2,2,2}\right)$$

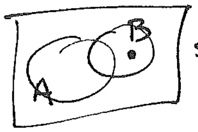
PROP: (d=2)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=\begin{cases}0 & \text{if } n \text{ odd}\\ (-1)^m\binom{2m}{m} & \text{if } n=2m \text{ even}\end{cases}$$

$$\|$$

proof:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n}{n-k}=\sum_{\substack{(A,B)\\ A,B\subseteq[n]\\ |A|+|B|=n}}(-1)^{|A|}=\sum_{\substack{\text{such }(A,B)\\ \text{with } A=B}}(-1)^{|A|}=\begin{cases}(-1)^{n/2}\binom{n}{n/2} & n \text{ even}\\ 0 & \text{else}\end{cases}$$



sign-rev. invol'n that swaps smallest

i∈ $A\triangle B=(A\setminus B)\sqcup(B\setminus A)$

from A to B

or B to A

if $A\neq B$ ∎

THM (d=3) (Dixon's Identity 1890)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\begin{cases}0 & \text{if } n \text{ odd}\\ (-1)^m\binom{3m}{m,m,m} & \text{if } \underset{\swarrow 2m}{n} \text{ even}\end{cases}$$

MacMahon (1915) used his master thm; Good (1960) pointed out you could do more:

THM (Fjeldstad 1954)

$$\sum_{l\in\mathbb{Z}}(-1)^l\binom{b+c}{b+l}\binom{c+a}{c+l}\binom{a+b}{a+l}=\binom{a+b+c}{a,b,c} \quad \text{for } a,b,c\in\mathbb{N}$$

$$\Big\downarrow a=b=c=m$$

$$\sum_{l\in\mathbb{Z}}(-1)^l\binom{2m}{m+l}^3=\binom{3m}{m,m,m}$$

$l:=m-k$
$k:=m-l$

$$\|$$

$$\sum_k(-1)^{3(m-k)}\binom{2m}{k}^3 \qquad \text{i.e. } \sum_k(-1)^k\binom{2m}{k}^3=(-1)^m\binom{3m}{m,m,m}$$

(55)

11/13/2015 (Good's) proof: Take $A = \begin{bmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{bmatrix}$ and compute in 2 ways

$$\left[x_1^{b+c} x_2^{a+c} x_3^{a+b}\right] \underbrace{\underline{y}^{(b+c,a+c,a+b)}}_{} \quad \text{where } \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\underline{x} = \begin{bmatrix} x_2 - x_3 \\ x_3 - x_1 \\ x_1 - x_2 \end{bmatrix}$$

$$= \underbrace{(x_2 - x_3)^{b+c} (x_3 - x_1)^{a+c} (x_1 - x_2)^{a+b}}_{}$$

$\boxed{1^{st} \text{ way}}$: 3 binomial thms

$$\overset{\downarrow}{=} \sum_{(i,j,k)} \binom{b+c}{i} \binom{a+c}{j} \binom{a+b}{k} (-1)^{i+j+k} x_1^{a+b-k+j} x_2^{b+c-i+k} x_3^{a+c-j+i}$$

extracting $[x_1^{b+c} x_2^{a+c} x_3^{a+b}]$ needs
$\cancel{b}+c = a+\cancel{b}-k+j \Leftrightarrow j-c = k-a$
$a+\cancel{c} = b+\cancel{c}-i+k \Leftrightarrow k-a = i-b$
$\cancel{a}+b = \cancel{a}+c-j+i \Leftrightarrow j-c = i-b$

$$\downarrow$$

$$\sum_{\ell} \binom{b+c}{b+\ell} \binom{a+c}{c+\ell} \binom{a+b}{a+\ell} (-1)^{3\ell + (a+b+c)}$$

so let $\ell := j-c = k-a = i-b$

$\boxed{2^{nd} \text{ way}}$: using MMT, get

$$\left[t_1^{b+c} t_2^{a+c} t_3^{a+b}\right] \underbrace{\frac{1}{\det(I_3 - TA)}}_{}$$

$$= \frac{1}{\det \begin{bmatrix} 1 & -t_1 & t_1 \\ t_2 & 1 & -t_2 \\ -t_3 & t_3 & 1 \end{bmatrix}} = \frac{1}{1\begin{vmatrix} 1 & -t_2 \\ t_3 & 1 \end{vmatrix} - t_2 \begin{vmatrix} -t_1 & t_1 \\ t_3 & 1 \end{vmatrix} - t_3 \begin{vmatrix} -t_1 & t_1 \\ 1 & -t_2 \end{vmatrix}}$$

$$= \frac{1}{1 + t_2 t_3 - t_2(-t_1 - t_1 t_3) - t_3(t_1 t_2 - t_1)}$$

$$= \frac{1}{1 + t_2 t_3 + t_1 t_2 + \cancel{t_1 t_2 t_3} - \cancel{t_1 t_2 t_3} + t_1 t_3}$$

$$= \frac{1}{1 + t_1 t_2 + t_1 t_3 + t_2 t_3}$$

$$= \sum_{m \geq 0} (-1)^m (t_1 t_2 + t_1 t_3 + t_2 t_3)^m$$

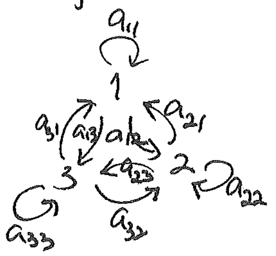
$$= \sum_{m \geq 0} (-1)^m \left(\sum_{\substack{(\alpha,\beta,\gamma): \\ \alpha+\beta+\gamma = m}} \binom{m}{\alpha,\beta,\gamma} t_1^{\alpha+\beta} t_2^{\alpha+\gamma} t_3^{\beta+\gamma} \right)$$

So need $\left\{ \begin{matrix} \alpha+\beta = b+c \\ \alpha+\gamma = a+c \\ \beta+\gamma = a+b \end{matrix} \right\} \Leftrightarrow \begin{matrix} (\alpha,\beta,\gamma) \\ = (c,b,a) \end{matrix}$, giving $(-1)^{a+b+c} \binom{a+b+c}{c,b,a}$. Hence $\sum_{\ell} \binom{b+c}{b+\ell} \binom{a+c}{c+\ell} \binom{a+b}{b+\ell} (-1)^{\ell} = \binom{a+b+c}{a,b,c}$ ■

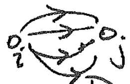
The transfer matrix method (Stanley §4.7, Ardila §3.1.2)

– ostensibly for counting walks in digraphs, based on…

THM: Given a square matrix $A=(a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$, think of a_{ij} as labelling arcs $\overset{i}{\circ}\to\overset{j}{\circ}$ in a complete digraph on $[n]$ (with loops), e.g. $n=3$



(–OR– as giving the # of arcs



in some D)

and then:

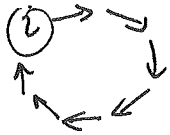
$$\text{(a)}\quad \sum_{\substack{\text{(directed)}\\ \text{walks } P \text{ of length } \ell\\ i=:i_0\to i_1\to\dots\to i_{\ell-1}\to i_\ell=:j}} \overbrace{\text{wt}(P)}^{\prod_{k=1}^{\ell} a_{i_{k-1},i_k}} = (A^\ell)_{i,j} \text{ for } \ell\ge 0$$

a_{ij} = #arcs $i\to j$ ⟶ total # of such paths P in D

$$\text{(b)}\quad \sum_{\text{walks } P \text{ from } i \text{ to } j} t^{\text{length}(P)}\,\text{wt}(P) = \frac{(-1)^{i+j}\det\left((I_n-tA)_{\text{with } i^{\text{th}} \text{ row},\ j^{\text{th}} \text{ col removed}}\right)}{\det(I_n-tA)}$$

$$\text{(c)}\quad \sum_{\text{closed walks } P} t^{\text{length}(P)}\,\text{wt}(P) = \frac{-t\frac{d}{dt}\det(I_n-tA)}{\det(I_n-tA)} = \sum_{\ell\ge0} t^\ell(\lambda_1^\ell+\dots+\lambda_n^\ell)$$

if A has eigenvalues $\lambda_1,\dots,\lambda_n$



i.e. $\sum_{\substack{\text{closed walks } P\\ \text{of length } \ell}} \text{wt}(P) = \lambda_1^\ell+\dots+\lambda_n^\ell$

---

proof: (a) is clear from matrix multiplication:

$$(A^\ell)_{i,j} = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_{\ell-1}=1}^{n} a_{i,i_1}a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{\ell-2},i_{\ell-1}}a_{i_{\ell-1},j} = \text{RHS of (a)} ✓$$

For (b), LHS $\overset{\text{by (a)}}{=} \sum_{\ell\ge0} t^\ell (A^\ell)_{ij} \overset{\text{in } \mathbb{C}[[t]]^{n\times n}}{=} \left(\sum_{\ell\ge0} t^\ell A^\ell\right)_{i,j}$

$$= (I_n + tA + t^2A^2 + t^3A^3 + \dots)_{i,j}$$

$$= \left[(I_n - tA)^{-1}\right]_{i,j}$$

$$\overset{\text{CRAMER'S RULE}}{=} \frac{(-1)^{i+j}\det\left((I_n-tA)_{\text{w/o } i^{\text{th}} \text{ row},\ j^{\text{th}} \text{ col}}\right)}{\det(I_n-tA)}$$

CRAMER'S RULE

in $\mathbb{C}(a_{ij},t)^{n\times n} = k^n$:

$\text{adj}B\cdot B = \det B\cdot I_n$

if $B\in k^n$, where

$(\text{adj}B)_{ij} := (-1)^{i+j}\det(B - {}^{i^{\text{th}}\text{ row}}_{j^{\text{th}}\text{ col}})$

For (c), $\text{LHS} \overset{\text{by (a)}}{=} \sum_{\ell \geq 1} \sum_{i=1}^{n} t^{\ell} (A^{\ell})_{i,i} = \sum_{\ell \geq 1} t^{\ell} \, \text{Trace}(A^{\ell})$

$= \sum_{\ell \geq 1} t^{\ell} (\lambda_1^{\ell} + \cdots + \lambda_n^{\ell})$ if A has eigenvalues $\lambda_1, \ldots, \lambda_\ell$ in $K = \overline{\mathbb{C}(a_{ij})}$,

since $PAP^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$PA^{\ell}P^{-1} = \begin{bmatrix} \lambda_1^{\ell} & & * \\ & \ddots & \\ 0 & & \lambda_n^{\ell} \end{bmatrix}$

$$= \frac{\lambda_1 t}{1-\lambda_1 t} + \cdots + \frac{\lambda_n t}{1-\lambda_n t}$$

$$= \frac{t \sum_{k=1}^{n} \lambda_k (1-\lambda_1 t) \cdots \widehat{(1-\lambda_k t)} \cdots (1-\lambda_n t)}{\prod_{k=1}^{n} (1-\lambda_k t)}$$

$$= \frac{-t \frac{d}{dt} \prod_{k=1}^{n} (1-\lambda_k t)}{\prod_{k=1}^{n} (1-\lambda_k t)} = \frac{-t \frac{d}{dt} \det(I_n - tA)}{\det(I_n - tA)} \quad ■$$

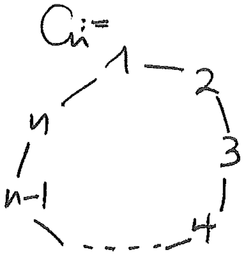
11/23/2015

EXAMPLES

① Let $f(n,k) :=$ # of proper vertex-colorings of $C_n :=$ with k colors

(Stanley Ex 4.7.5)
(Ardila 3.1.2 #1)

proper: no adjacent vertices get same color



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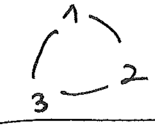
e.g. $n=2$

$f(2,k) = k(k-1) \quad = (k-1) \cdot k$

k: color 1 first k ways

$(k-1)$: color 2 differently

$n=3$



$f(3,k) = k(k-1)(k-2) \quad = (k-1)(k^2-2)$

k: color 1; $(k-1)$: color 2; $(k-2)$: color 3

$n=4$

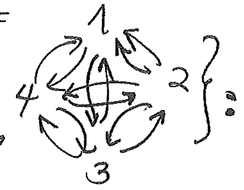
$$f(4,k) = \underbrace{k(k-1)(k-2)(k-2)}_{\text{all 4 vertices get different colors}} + \underbrace{k(k-1)(k-1)}_{\text{vertices 2,4 get same color}}$$

first term: k: color 1; $(k-1)$: color 2; $(k-2)$: color 4; $(k-2)$: color 3

second term: k: color 1; $(k-1)$: color 2,4 same; $(k-1)$: color 3

$= k(k-1)\left((k-2)^2 + k-1 \right)$

But note that ~~ex~~{proper k-colorings of C_n} $\longleftrightarrow$ {closed walks of length n from in $\overleftrightarrow{K}_k$ =

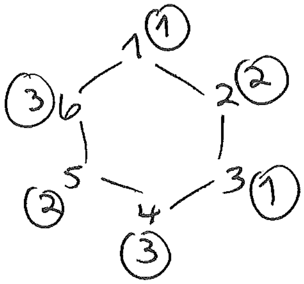


}:

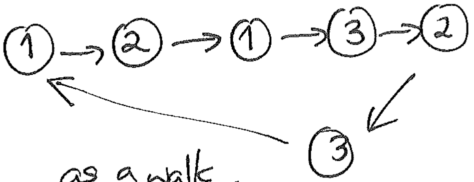
if the coloring assigns vertex $\overset{[n]}{\overset{\Downarrow}{i}}$ to color $j^{\in [k]}$
then the walk visits vertex j of $\overleftrightarrow{K}_k$ at its i^{th} step

e.g. $n=6$
$k=3$

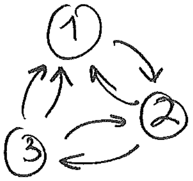
C_n =



$\longleftrightarrow$



as a walk in $\overleftrightarrow{K}_k = \overleftrightarrow{K}_3$



$$\text{So taking } A = \begin{array}{c|ccccc} & ① & ② & ③ & \cdots & ⓚ \\ \hline ① & 0 & 1 & 1 & \cdots & 1 \\ ② & 1 & 0 & \ddots & & \vdots \\ \vdots & 1 & 1 & \ddots & \ddots & \\ & \vdots & & \ddots & \ddots & 1 \\ ⓚ & 1 & - & - & 1 & 0 \end{array} = \mathbb{1}_k - I_k$$

which has eigenvalues $(\lambda_1, \ldots, \lambda_k) = (k-1, \overbrace{-1, \ldots, -1}^{k-1 \text{ times}})$

(since we already saw $\mathbb{1}_k$ has eigenvalues $(k, \overbrace{0, 0, \ldots, 0}^{k-1 \text{ times}})$

one finds that $f(n,k) = \lambda_1^n + \lambda_2^n + \ldots + \lambda_k^n$

$$= (k-1)^n + (-1)^n + \ldots + (-1)^n = (k-1)^n + (k-1)(-1)^n$$

$$= (k-1)\left((k-1)^{n-1} + (-1)^n\right)$$

e.g. $n=2$ $f(2,k) = (k-1) \cdot (\underbrace{k-1+1}_{k \checkmark})$

$n=3$ $f(3,k) = (k-1)(\underbrace{(k-1)^2 - 1}_{k^2-2k \checkmark})$

$n=4$ $f(4,k) = (k-1)(\underbrace{(k-1)^3 + 1}_{k^3-3k^2+3k \checkmark})$

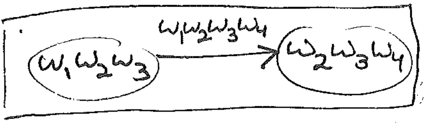
② How many words $(w_1, w_2, \ldots, w_n)$ with n letters from $\{a,b\}$ avoid aa and abba as consecutive subwords, and have $w_1 = w_n$? Call this g_n.

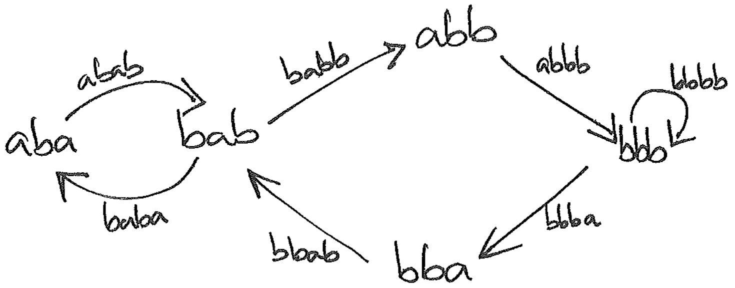
(Ardila 3.1.2#3)

__Can't__ model it as closed walks in a digraph on vertices

or even on vertices

but you __can__ do it ~~with~~ vertices which are __3-letter__ words:





$$A = \begin{array}{c|ccccc} & aba & bba & bab & abb & bbb \\ \hline aba & 0 & 0 & 1 & 0 & 0 \\ bba & 0 & 0 & 0 & 0 & 1 \\ bab & 1 & 1 & 0 & 0 & 0 \\ abb & 0 & 0 & 1 & 0 & 0 \\ bbb & 0 & 0 & 0 & 1 & 1 \end{array}$$

$$\Rightarrow \det(I_5 - tA) = \det\begin{bmatrix} 1 & 0 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 & -t \\ -t & -t & 1 & 0 & 0 \\ 0 & 0 & -t & 1 & 0 \\ 0 & 0 & 0 & -t & 1-t \end{bmatrix} = -t^4 + t^3 - t^2 - t + 1$$

$$\text{So } \sum_{n\geq 0}' g_n t^n = \frac{-t\frac{d}{dt}\det(I_5 - tA)}{\det(I_5 - tA)} = \frac{t + 2t^2 - 3t^3 + 4t^4}{1 - t - t^2 + t^3 - t^4} = t^1 + 3t^2 + t^3 + 7t^4 + 6t^5 + 15t^6$$

bbbb	bbbbb	bbbbbb	bbbbbbb = (bbbb)
"	ababa	"	abababa = (abab)
(b)	babab	(bbb)	bababab = (baba)
	"		babbbab = (babb)
	(bb)		abbbabb = (abbb)
	(ab)		(bbba)
	(ba)		(bbab)
1	3	1	7