COUNTING FACTORIZATIONS OF COXETER ELEMENTS INTO PRODUCTS OF REFLECTIONS

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ABSTRACT. In this paper, we count factorizations of Coxeter elements in well-generated complex reflection groups into products of reflections. We obtain a simple product formula for the exponential generating function of such factorizations, which is expressed uniformly in terms of natural parameters of the group. In the case of factorizations of minimal length, we recover a formula due to P. Deligne, J. Tits and D. Zagier in the real case and to D. Bessis in the complex case. For the symmetric group, our formula specializes to a formula of B. Shapiro, M. Shapiro and A. Vainshtein.

CONTENTS

1. INTRODUCTION

One of the many equivalent forms of Cayley's formula [Cay89] counting labeled trees asserts that the number of factorizations of the long cycle $(1, 2, \dots, N)$ as a product of $(N - 1)$ transpositions is

$$\#\big\{\tau_1 \tau_2 \cdots \tau_{N-1} = (1, 2, \dots, N)\big\} = N^{N-2}.$$

A widely considered generalization of this formula comes from the field of map enumeration. Instead of considering minimal factorizations, one can as well consider *higher genus factorizations*, which are factorizations of the long cycle into $N-1+2g$ transpositions. The term *genus* and the letter g come from the fact that this g is indeed the genus of a surface associated to a natural embedded graph representing the factorization, see for example [LZ04] for further details. The following exponential generating function identity, proved in [SSV97], provides a beautiful and compact way to count such factorizations,

$$\sum_{g \geq 0} \frac{t^{N-1+2g}}{(N-1+2g)!} \#\left\{\tau_1 \tau_2 \ldots \tau_{N-1+2g} = (1, 2, \ldots, N)\right\} = \frac{1}{N!} \left(e^{t\frac{N}{2}} - e^{-t\frac{N}{2}}\right)^{N-1}.$$

Observe that near $t = 0$ this formula has the expansion $\frac{t^{N-1}}{(N-1)!} N^{N-2}$, in agreement with Cayley's formula.

Another way to generalize Cayley's formula is to count factorizations of Coxeter elements in real reflection groups, and, even more generally, in well-generated complex reflection groups. If we replace the symmetric group by any well-generated complex reflection group W of rank n with Coxeter number h, the number of factorizations of a fixed Coxeter element $c \in W$ into a product of n reflections is given by the following formula

$$\#\left\{\tau_1 \tau_2 \ldots \tau_n = c\right\} = \frac{n!}{|W|} h^n. \tag{1}$$

In the case of real reflection groups, this formula was proved in a letter from P. Deligne to E. Looijenga [Del] crediting discussions with J. Tits and D. Zagier. The remaining cases were then proven by D. Bessis in [Bes07, Proposition 7.5], using the geometry of braid groups. Formula (1) was as well rediscovered for real reflection groups in the context of the noncrossing partition lattice, see [Cha05, Proposition 9], [Arm09, Cor. 3.6.10], and [Rea08, Thm. 3.6].

For the symmetric group $\mathcal{S}_N$ the rank is given by $n = N - 1$ and the Coxeter number is $h = N$, we thus again get back Cayley's formula.

In this paper, we provide a uniform generalization of both results by counting "higher genus" factorizations of Coxeter elements in well-generated complex reflection groups into products of reflections. The main result is the following identity for their exponential generating function. We refer to Section 2 for the definitions used here.

Theorem 1.1. *Let W be an irreducible well-generated complex reflection group of rank n. Let c be a Coxeter element in W, let $\mathcal{R}$ be the set of all reflections in W, and let $\mathcal{R}^*$ be the set of all reflecting hyperplanes. Define*

$$\mathrm{FAC}_W(t) := \sum_{\ell \geq 0} \frac{t^\ell}{\ell!} \#\left\{(\tau_1, \tau_2, \ldots, \tau_\ell) \in \mathcal{R}^\ell \ , \ \tau_1 \tau_2 \ldots \tau_\ell = c\right\}$$

to be the exponential generating function of factorizations of c into a product of reflections. Then $\mathrm{FAC}_W(t)$ is given by the product formula

$$\mathrm{FAC}_W(t) = \frac{1}{|W|} \left(e^{t|\mathcal{R}|/n} - e^{-t|\mathcal{R}^*|/n}\right)^n. \tag{2}$$

The Coxeter number h is given by $(|\mathcal{R}| + |\mathcal{R}^*|)/n$, see Formula (5) in Section 2 below. Therefore the expansion of (2) near $t = 0$ gives back Formula (1). Moreover, in the case of real reflection groups one has $|\mathcal{R}| = |\mathcal{R}^*|$, so the exponents in the righthand side of (2) are given by $th/2$ and $-th/2$. For real reflection groups, Theorem 1.1 has thus the following form.

Corollary 1.2. *Let W be an irreducible real reflection group of rank n and Coxeter number h. Then one has*

$$\mathrm{FAC}_W(t) = \frac{1}{|W|} \left(e^{th/2} - e^{-th/2} \right)^n. \tag{3}$$

Remark 1.3. There are several conventions in the literature for the definition of Coxeter elements in well-generated complex reflection groups. For the definition we use here, there might be several conjugacy classes $\mathcal{C}_\zeta$ of Coxeter elements for the various primitive h-th roots of unity ζ. One of these conjugacy classes corresponds to the Coxeter elements in the sense of the more restrictive definition used for example in [Bes07]. Notice that, strictly speaking, the definition of $\mathrm{FAC}_W(t)$ depends a priori on the conjugacy class of the chosen Coxeter element. Therefore, Theorem 1.1 implies as well that the number of factorizations is independent of this choice, which is why we suppress the information of the conjugacy class in the notation $\mathrm{FAC}_W(t)$. This seems to illustrate a much more general phenomenon, namely that all enumerative properties of Coxeter elements seem to be valid for the more general definition. For the infinite familes $G(r, 1, n)$ and $G(r, r, n)$ defined in Section 2, this phenomenon can easily be explained by the existence of automorphisms of the group, inherited from automorphisms of the base field $\mathbb{Q}[\zeta]$, that preserve reflections and send one class of Coxeter elements to the other. For exceptional groups, however, we do not know of such a simple explanation[1]. We refer the reader to Section 2 for more details on our definition and for a comparison with other definitions. Finally, we want to emphasize again that Theorem 1.1 is *in particular* true with the more restrictive definition of Coxeter elements.

Remark 1.4. There is no loss of generality in considering irreducible groups in Theorem 1.1 and Corollary 1.2. Indeed, let W be a well-generated non-irreducible complex reflection group and write $W = W_1 \times \cdots \times W_k$, where the W_i are the irreducible components. Since the Coxeter elements in W are, by definition, the product of the Coxeter elements of each W_i, and the set of reflections in W is the union of the set of reflections in all W_i, one obtains immediately that $\mathrm{FAC}_W(t) = \mathrm{FAC}_{W_1}(t) \cdots \mathrm{FAC}_{W_k}(t)$.

In Section 2, we recall needed background on complex reflection groups. In Section 3, we prove the main theorem for the dihedral groups $I_2(m)$ by explicit computations. In Section 4, we recall a general approach to the enumeration of factorizations in groups via representation theory. In Section 5, we prove the main theorem for the two infinite families of well-generated complex reflection groups. In Section 6, we then present computer verifications for the exceptional well-generated complex reflection groups. Since we were not able to provide references for explicit descriptions of the irreducible characters of the two infinite families of irreducible

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[1]In the same vein, we observe that the various choices of Coxeter elements lead to isomorphic non-crossing partition lattices, see e.g. [BR11] for the definition. For the infinite families, this follows from the same argument as above, whereas we have used a computer to verify this fact for the exceptional groups.

well-generated complex reflection groups, we describe them explicitly in Appendix A. In the final Appendix B, we provide parts of the data that we have used in the computer verification of the main theorem for exceptional groups, namely the evaluations of the irreducible characters of exceptional well-generated complex reflection groups at reflections and at one conjugacy class of Coxeter elements.

2. Background on complex reflection groups

In this section, we recall some background on complex reflection groups. All these results can be found for example in [LT09] to which we also refer for further details. We also adapted some notation from [BR11].

Let $V = \mathbb{C}^n$ be a complex vector space of dimension n. A *(complex) reflection* is a linear transformation of V that has finite order and whose fix space $\mathrm{Fix}(w) := \ker(\mathbb{1}-w)$ is a hyperplane in V. Such a hyperplane is called *reflection hyperplane*. A *complex* (or *unitary*) *reflection group* W is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. We denote the set of reflections in W by $\mathcal{R}$ and the set of corresponding reflection hyperplanes by $\mathcal{R}^*$. The space V and the action of W on V are called *reflection* (or *natural*) *representation* of W. Throughout this paper, we assume that W is irreducible. This means that W cannot be decomposed as a product of two reflection subgroups acting on proper subspaces of V. In this case, the dimension n of V is called *rank* of W. G.C. Shephard and J.A. Todd classified such groups in [ST54]. They moreover used this classification to show that when W acts on the symmetric algebra $S = \mathrm{Sym}(V^*) \cong \mathbb{C}[x_1, \ldots, x_n]$, its ring of invariants S^W is again a polynomial algebra, generated by homogeneous polynomials $f_1, \ldots, f_n$ of uniquely determined degrees $d_1 \leq \cdots \leq d_n$. These are called *degrees* of W. This result was as well proven by C. Chevalley in [Che55].

It turns out that there is an equivalent way to define the degrees using the *coinvariant algebra* $S/\langle \mathbf{f} \rangle$, where $\langle \mathbf{f} \rangle = \langle f_1, \ldots, f_n \rangle = S_+^W$ is the ideal in S generated by all invariants without constant term. Both Shephard and Todd [ST54] and Chevalley [Che55] showed that $S/\langle \mathbf{f} \rangle$ carries the regular representation of W. Thus, $S/\langle \mathbf{f} \rangle$ contains exactly k copies of any irreducible W-representation U of dimension k. In particular $S/\langle \mathbf{f} \rangle$ contains n copies of V. The *U-exponents* $e_1(U), \ldots, e_k(U)$ are the degrees of the homogeneous components of $S/\langle \mathbf{f} \rangle$ in which these k copies of U occur. It is known, see e.g. [LT09, Section 4.1], that the degrees of W are uniquely determined by saying that the V-exponents are equal to $d_1-1, \ldots, d_n-1$. This characterization has the advantage that one can as well define the *codegrees* $d_1^* \geq \ldots \geq d_n^*$ by saying that the V^*-exponents are given by $d_1^*+1, \ldots, d_n^*+1$. It is well known that the degrees and the codegrees determine the number of reflections and the number of reflecting hyperplanes in W,

$$|\mathcal{R}| = (d_1 - 1) + \ldots + (d_n - 1) \qquad |\mathcal{R}^*| = (d_1^* + 1) + \ldots + (d_n^* + 1), \qquad (4)$$

see [LT09, Theorem 4.14(ii) and Appendix C, Section 1.2]. As in [Bes07], we say that W is *well-generated* if the following two equivalent properties hold,

(i) W is generated by n reflections,
(ii) the degrees and the codegrees satisfy $d_i + d_i^* = d_n$.

The equivalence of these two properties was observed in [OS80], using the Shephard-Todd classification. For well-generated complex reflection groups, the *Coxeter number* h is defined to be the largest degree d_n of W. Observe that (ii) together with

the counting formulas in (4) yields

$$|R| + |R^*| = nh. \tag{5}$$

An element c in a well-generated complex reflection group W is called *regular* if it has an eigenvector lying in the complement of the reflecting hyperplanes for W and furthermore ζ-*regular* if this eigenvector may be taken to have eigenvalue ζ. In this case, the multiplicative order d of ζ is called a *regular number* for W. An integer d is a regular number if and only if it divides as many degrees as codegrees, see e.g. [LT09, Theorem 11.28]. Since for all i on has $d_i + d_i^* = h$, this implies that h is a regular number. Therefore for any primitive h-th root of unity ζ there exists a regular element $c_\zeta \in W$ with eigenvalue ζ, see [LT09, Remark 11.23]. Any such element is called *Coxeter element*. For any ζ, all ζ-regular elements are W-conjugate, see [LT09, Corollary 11.25]. Thus, the class C_ζ of Coxeter elements for a particular primitive h-th root of unity ζ is closed under conjugation. Nonetheless, observe that according to our definition, there are in general more than a single conjugacy class of Coxeter elements, since the conjugacy classes for the various primitive h-th roots of unity may differ. For comparison, we notice that the papers [Bes07] define Coxeter elements as $e^{\frac{2i\pi}{h}}$-regular elements, i.e. elements of the class C_ζ with $\zeta = e^{\frac{2i\pi}{h}}$. Therefore all Coxeter elements in the sense of [Bes07] are also Coxeter elements with our definition, but the converse is not true in general.

To check that an element of W is a Coxeter element, we will use several times the following necessary and sufficient condition.

Proposition 2.1. *An element $w \in W$ is a Coxeter element if and only if it has an eigenvalue that is a primitive h-th root of unity.*

This result is well known to specialists, but we have not found it explicitly in the literature in the most general case. For real groups, a proof can be found for example in [Kan01, Theorem 32-2C]. Here, for completeness, we provide a proof of the general case that follows exactly the same lines. The arguments are very similar to the ones used in the proof of [LT09, Theorem 11.15].

Proof. Since W is irreducible and well-generated, we have

$$h = d_n > d_{n-1}, \ldots, d_2, d_1. \tag{6}$$

Fix now ζ to be a primitive h-th root of unity. We have already seen that there exists a regular element $c_\zeta \in W$ with eigenvalue ζ, and that its conjugacy class C_ζ is the conjugacy class of all ζ-regular elements in W. It follows from [LT09, Theorem 11.24(iii)] and (6) that

$$|\mathcal{C}_\zeta| = |W|/|Z(c_\zeta)| = d_1 \cdots d_{n-1},$$

where $Z(c_\zeta)$ denotes the centralizer of c_ζ, and where we used the fact that $|W| = d_1 \cdots d_n$. On the other hand, we have that the Pianzola-Weiss polynomial for ζ is

$$P(T) := \sum_{w \in W} T^{\dim V(w,\zeta)} = d_1 \cdots d_{n-1}(T + h - 1),$$

where $V(w, \zeta) = \{v \in V : w(v) = \zeta \cdot v\}$ is the eigenspace for w with eigenvalue ζ, see [LT09, Corollary 10.39]. The coefficient $d_1 \cdots d_{n-1}$ of T in $P(T)$ counts the number of elements in W that have an eigenvalue ζ. Since all elements in $\mathcal{C}_\zeta$ have ζ as an eigenvalue, we conclude that all elements in W that have ζ as an eigenvalue are already contained in $\mathcal{C}_\zeta$, and are thus Coxeter elements. The statement follows. $\square$

rank	Well-generated exceptional classification types
2	$G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}$
3	$G_{23} = H_3, G_{24}, G_{25}, G_{26}, G_{27}$
4	$G_{28} = F_4, G_{29}, G_{30} = H_4, G_{32}$
5	G_{33}
6	$G_{34}, G_{35} = E_6$
7	$G_{36} = E_7$
8	$G_{37} = E_8$

TABLE 1. All well-generated exceptional classification types.

2.1. **The classification of well-generated complex reflection groups.** In the remainder of this section, we recall the classification of irreducible well-generated complex reflection groups from [ST54]. A *monomial* matrix is a square matrix with exactly one non-zero entry in each row and column. Let $G(r, p, n)$ with p dividing r be the group of all monomial $(n \times n)$-matrices with entries being r-th roots of unity, such that the product of the non-zero entries is an (r/p)-th root of unity. It has order $n!r^n/p$.

Theorem 2.2 (Shephard, Todd)**.** *Let W be an irreducible complex reflection group. Then W is either isomorphic to $G(r, p, n)$ for some positive integers r, p, n, or to one of 34 exceptional groups.*

Following [ST54], we denote the 34 exceptional *Shephard-Todd classification types* by G_4 through G_{37}. The degrees and codegrees of all irreducible complex reflection groups are known, see [ST54]. For the groups $G(r, p, n)$, the degrees are given by

$$r, 2r, \ldots, (n - 1)r, rn/p, \tag{7}$$

while the codegrees are given by

$$0, r, \ldots, (n - 1)r \text{ if } p < r,$$
$$0, r, \ldots, (n - 2)r, (n - 1)r - n \text{ if } p = r. \tag{8}$$

The complete list of well-generated Shephard-Todd exceptional classification types is given in Table 1, their degrees and codegrees can be found in Table 2 in Appendix B. This implies that the two infinite families of irreducible well-generated complex reflection groups are given by $G(r, 1, n)$ and $G(r, r, n)$. Observe that the infinite families of real reflection groups are the particular cases

- $G(m, m, 2) = I_2(m)$,
- $G(1, 1, n) = A_{n-1}$,
- $G(2, 1, n) = B_n$,
- $G(2, 2, n) = D_n$.

We will use this classification to prove Theorem 1.1. In particular, see Appendix B for the representation theoretic data for the exceptional types.

3. DIRECT PROOF OF THE MAIN THEOREM FOR THE DIHEDRAL GROUP $I_2(m)$

As a warmup exercise, we prove the main theorem for the dihedral group $I_2(m)$. Recall that the group $I_2(m)$ can be realized as the group of orthogonal symmetries

of the regular m-gon. We think of its vertices as the complex m-th roots of unity $\{\zeta^\ell : 0 \leq \ell < m\}$ in $\mathbb{R}^2 \cong \mathbb{C}$ where ζ is the primitive m-th root of unity $e^{2\pi i/m}$. It consists of the m rotations $\{r_i : 0 \leq i < m\}$ where $r_i(\zeta^\ell) = \zeta^{i+\ell}$ is the rotation of angle $\frac{2i\pi}{m}$, and the m reflections $\{s_i : 0 \leq i < m\}$ where $s_i(\zeta^\ell) = \zeta^{i-\ell}$ is the orthogonal reflection through the line of angle $\frac{i\pi}{m}$ with the x-axis. $I_2(m)$ is generated by the reflections s_0, s_1, and thus, $c = s_1 s_0 = r_1$ is a Coxeter element. Note that $s_i s_j = c$ if and only if $i = j + 1 \pmod{m}$. In particular, since any product of $2g + 1$ reflections is again a reflection, there is a unique other reflection such that the complete product of all $2g + 2$ reflections equals c. Therefore the number of factorizations of c into $2g + 2$ reflections is equal to m^{2g+1}, and we obtain

$$\mathrm{FAC}_{I_2(m)}(t) = \sum_{g \geq 0} \frac{t^{2g+2}}{(2g+2)!} m^{2g+1} = \frac{e^{mt} + e^{-mt} - 2}{2m} = \frac{1}{2m}\left(e^{mt/2} - e^{-mt/2}\right)^2.$$

Since for the dihedral group we have $n = 2$ and $h = m$, this yields Formula (2) in this case.

4. A REPRESENTATION THEORETIC APPROACH TO THE MAIN THEOREM

In this section, we recall a classical approach to the enumeration of factorizations in groups via representation theory. This approach is widely used in the literature to enumerate factorizations in the symmetric group, see e.g. [LZ04, Appendix A] and the references therein. It is based on the Frobenius formula given in Formula (10) below. For completeness, we recall its proof here, see as well e.g. [LZ04, Appendix A].

Let W be a complex reflection group. The group algebra $\mathbb{C}[W]$ is decomposed into irreducible W-modules as

$$\mathbb{C}[W] = \bigoplus_{\lambda \in \Lambda} \dim(\lambda) V^\lambda, \tag{9}$$

where V^λ, $\lambda \in \Lambda$ is a complete list of irreducible representations of W. We let $\mathcal{R} \subset W$ be the union of the conjugacy classes of all reflections. We use the same letter, in non-calligraphic style, to denote the corresponding sum in the group algebra,

$$R := \sum_{\tau \in \mathcal{R}} \tau.$$

Moreover, we extend class functions linearly from W to $\mathbb{C}[W]$. In particular if $\chi : W \to \mathbb{C}$ is a class function, we use the notation

$$\chi(R) := \sum_{\tau \in \mathcal{R}} \chi(\tau).$$

We consider the action of W on $\mathbb{C}[W]$ by left multiplication. Since an element $w \in W$ acts on $\mathbb{C}[W]$ by permuting the canonical basis, the trace of this action is equal to the number of fixed points under this action, which is $|W|$ if $w = 1$ and 0 otherwise. Since R is in the center of $\mathbb{C}[W]$, it acts as a scalar on every irreducible

representation, so the decomposition (9) gives the Frobenius formula

$$
\#\Big\{(\tau_1, \tau_2, \ldots, \tau_\ell) \in \mathcal{R}^\ell \ : \ \tau_1\tau_2\ldots\tau_\ell = c\Big\}
$$
$$
= \frac{1}{|W|}\mathrm{Tr}_{\mathbb{C}[W]}\Big(R^\ell c^{-1}\Big)
$$
$$
= \frac{1}{|W|}\sum_{\lambda\in\Lambda}\dim(\lambda)\mathrm{Tr}_{V^\lambda}\Big(R^\ell c^{-1}\Big)
$$
$$
= \frac{1}{|W|}\sum_{\lambda\in\Lambda}\dim(\lambda)^{1-\ell}\chi_\lambda(R)^\ell\chi_\lambda(c^{-1}), \quad (10)
$$

where χ_λ is the character of the representation λ. Using (10), we can now deduce that $\mathrm{FAC}_W(t)$ is given by the finite sum

$$
\mathrm{FAC}_W(t) = \frac{1}{|W|}\sum_{\lambda\in\Lambda}\dim(\lambda)\chi_\lambda(c^{-1})\exp\left(t\cdot\frac{\chi_\lambda(R)}{\dim(\lambda)}\right). \quad (11)
$$

We will use this formula, together with the classification of complex reflection groups and the knowledge of their irreducible representations to prove Theorem 1.1. In order to simplify notation, we will often make use of the *normalized character* defined as the character divided by the dimension of the corresponding representation, $\chi_\lambda(\mathbb{1}) = \dim(\lambda)$. By convention, a normalized character will be denoted as

$$
\widetilde{\chi}_\lambda := \frac{1}{\dim(\lambda)}\cdot\chi_\lambda.
$$

5. Proof of the main theorem for the families $G(r,1,n)$ and $G(r,r,n)$

In this section, we treat the case of the two infinite families of well-generated complex reflection groups, namely $G(r,1,n)$ and $G(r,r,n)$. As a preliminary, and to setup some notation, we treat the case of the symmetric group $G(1,1,n) = \mathcal{S}_n$, following an approach similar to [SSV97] or [LZ04, Appendix A.2.4].

5.1. **The proof for the symmetric group $\mathcal{S}_n$.** The symmetric group $\mathcal{S}_n$ is the group of all permutation matrices of size n. Its irreducible representations are classically indexed by the set of partitions of n, see e.g. [Sag01]. In this section we are going to use extensively two classical results, the *hook-length formula* and the *Murnaghan-Nakayama rule*. These formulas give the dimensions of these representations and the evaluation of their characters, respectively. We refer the reader to the original papers [FRT54, Mur37, Nak41] or to [Sag01, Sections 3.10 and 4.10] for a detailed treatment of these results. We will use them several times without again recalling these references.

A *partition* of n is a non-increasing sequence $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ of positive integers summing to n. The integer n is called the *size* of λ, denoted by $|\lambda| = n$, and we write $\lambda \vdash n$. The integers λ_i are called the *parts* of λ. To denote a partition, we often use superscript notation, we write for example $\lambda = [4^2, 3^1, 1^3]$ for the partition $\lambda = [4, 4, 3, 1, 1, 1]$ of 14. By convention, the *empty partition* $\varnothing$ is the unique partition of 0.

If not otherwise stated, we denote by χ_λ the character of the representation of $\mathcal{S}_n$ indexed by the partition $\lambda \vdash n$.

$\mathcal{S}_n$ is generated by the transpositions $s_i = (i, i+1)$ for $1 \leq i < n$. Since the Coxeter element $c = s_1 s_2 \ldots s_{n-1} = (1, 2, \ldots, n)$ is a long cycle, as is its inverse c^{-1}, the Murnaghan-Nakayama rule implies that $\chi_\lambda(c^{-1}) = 0$ unless λ is a *hook*, i.e., unless λ is of the form $\mathfrak{h}_k^n := [n-k, 1^k]$ for some $0 \leq k < n$. The following lemma is a straightforward consequence of the hook-length formula and of the Murnaghan-Nakayama rule, and will be useful in several places.

Lemma 5.1 (**Hook characters: dimension and evaluation on a long cycle**). *Let $n \geq 1$ and $0 \leq k < n$. Then*

$$\dim(\mathfrak{h}_k^n) = \binom{n-1}{k} \quad and \quad \chi_{\mathfrak{h}_k^n}\big((1, 2, \ldots, n)\big) = (-1)^k.$$

In view of evaluating (11), we will also need the value of hook characters on transpositions. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r] \vdash n$, we define the multiset of its contents as

$$\mathrm{Content}(\lambda) := \{\!\{ j - i : 1 \leq i \leq r, 1 \leq j \leq \lambda_i \}\!\},$$

where we indicate the multiset notation by double braces. Notice that $\mathrm{Content}(\lambda)$ has cardinality n. We have the following classical lemma relating character evaluations and contents.

Lemma 5.2 (**Normalized character evaluated on a transposition**). *Let $\lambda \vdash n$ be a partition and $\tau \in \mathcal{S}_n$ be a transposition. Then we have*

$$\widetilde{\chi}_\lambda(\tau) = \frac{2}{n(n-1)} \sum_{x \in \mathrm{Content}(\lambda)} x.$$

Proof. As in Section 4, we let $R = \sum_{\tau \in \mathcal{R}} \tau$ be the sum of all transpositions in the group algebra $\mathbb{C}[\mathcal{S}_n]$. Write $R = \sum_{i=1}^n J_i$ where $J_i = \sum_{i<j}(i, j)$ is the j-th Jucys-Murphy element. It is well known – see e.g. [VO04] – that given any symmetric function f in n variables, the element $f(J_1, J_2, \ldots, J_n)$ acts on the irreducible module $V^\lambda \subset \mathbb{C}[\mathcal{S}_n]$ as the scalar $f(\mathrm{Content}(\lambda))$. The statement follows by taking $f(x_1, x_2, \ldots x_n) = x_1 + x_2 + \cdots + x_n$, and observing that $\chi_\lambda(R) = \frac{n(n-1)}{2}\chi_\lambda(\tau)$. $\square$

In the particular case of hooks, we obtain

$$\widetilde{\chi}_{\mathfrak{h}_k^n}(\tau) = \frac{n - 2k - 1}{n - 1}. \tag{12}$$

We now recalled all ingredients needed to provide a proof of Theorem 1.1 for the symmetric group.

Proof of Theorem 1.1 for the symmetric group $W = \mathcal{S}_n$. Using Equation (12) together with Lemma 5.1, Formula (11) can be rewritten as

$$\begin{aligned}
\mathrm{FAC}_W(t) &= \frac{1}{|W|} \sum_{k=0}^{n-1} \binom{n-1}{k}(-1)^k e^{t\frac{n(n-2k-1)}{2}} \\
&= \frac{1}{|W|} e^{t\binom{n}{2}} \left(1 - e^{-tn}\right)^{n-1} \\
&= \frac{1}{|W|} \left(e^{t\frac{n}{2}} - e^{-t\frac{n}{2}}\right)^{n-1}.
\end{aligned}$$

Since the symmetric group $\mathcal{S}_n$ has rank $n-1$, and since for $\mathcal{S}_n$ one has $|\mathcal{R}| = |\mathcal{R}^*| = \binom{n}{2}$, this coincides with (2) in this case. $\square$

5.2. **The structure of the proof.** In the cases of $G(r,1,n)$ and $G(r,r,n)$, the structure of the proof will be similar to what we just did for the symmetric group. It can be decomposed into the following four steps.

step 1 Provide a list of all irreducible characters not vanishing on the inverse of the Coxeter element (for $\mathcal{S}_n$, these were exactly the hook characters).

step 2 Compute the dimension of these characters and their evaluation on the inverse of the Coxeter element (for $\mathcal{S}_n$, this was given by Lemma 5.1).

step 3 Compute the evaluation of these characters on the conjugacy classes of reflections (for $\mathcal{S}_n$, this was given by Equation (12)).

step 4 Use Formula (11) and simplify the sum (for $\mathcal{S}_n$, we just applied Newton's binomial formula) to prove Formula (2).

5.3. **The proof for the group $G(r,1,n)$ with $r > 1$.** The Coxeter number for $G(r,1,n)$ is $h = nr$. Let ζ be a primitive h-th root of unity, and let $\xi = \zeta^n$, which is a primitive r-th root of unity. Recall that $G(r,1,n)$ is the group of all monomial matrices whose non-zero entries are powers of ξ. Given $\sigma \in \mathcal{S}_n$ and $(i_1, i_2, \ldots, i_n)$ with $0 \leq i_1, \ldots, i_n < r$, we denote by $w = \sigma \wr (i_1, i_2, \ldots, i_n)$ the element of $G(r,1,n)$ with entry ξ^{i_ℓ} at position (ℓ, σ_ℓ) for $1 \leq \ell \leq n$. Observe that our presentation depends on the choice of ξ and thus on the choice of ζ. We moreover denote by $|w| := \sigma$ the projection onto $\mathcal{S}_n$, and by $\|w\| := i_1 + \ldots + i_n \pmod r$.

Consider the element $c = c_0 \wr (0, 0, \ldots, 0, 1) \in G(r,1,n)$ where $c_0 \in \mathcal{S}_n$ is given by the long cycle $(1, 2, \ldots, n)$. We claim that c is a Coxeter element of $G(r,1,n)$. To see that, according to the discussion preceding Section 2.1, it is enough to check that c has an eigenvalue equal to ζ. This is indeed the case, since the fact that $\xi = \zeta^n$ implies that the element c has n eigenvalues given by $\{\zeta, \zeta^{r+1}, \zeta^{2r+1}, \ldots, \zeta^{(n-1)r+1}\}$.

Remark 5.3. In the rest of this section, we will prove Theorem 1.1 for the group $G(r,1,n)$, for the particular choice of Coxeter element $c = c_0 \wr (0, 0, \ldots, 0, 1)$. Since c belongs to the conjugacy class $\mathcal{C}_\zeta$ of all Coxeter elements which are ζ-regular, and since the number of factorizations into reflections is the same for elements in the same conjugacy class, this will also prove the theorem for all elements $c' \in \mathcal{C}_\zeta$. Finally, since we fixed ζ as an *arbitrary* primitive h-th root of unity, this will imply the result for *all* Coxeter elements of the group $G(r,1,n)$.

The following proposition is well known and describes the irreducible characters of $G(r,1,n)$. We refer to Appendix A.2 for its proof and references.

Proposition 5.4. *The complete and unambiguous list of irreducible characters of $G(r,1,n)$ is obtained as follows. Let $\vec{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ be an r-tuple of partitions of total size n, and let $k_\ell = |\lambda^{(\ell)}|$. One may view*

$$B := G(r,1,k_0) \times G(r,1,k_1) \times \cdots \times G(r,1,k_{r-1})$$

as a subgroup of $G(r,1,n)$, formed by block-diagonal matrices. The character $\chi_{\vec{\lambda}}$ of $G(r,1,n)$ is then given by

$$\chi_{\vec{\lambda}}(w) = \frac{1}{|B|} \sum_{\substack{s \in G(r,1,n) \\ s^{-1}ws \in B}} \prod_{\ell=0}^{r-1} \chi_{\lambda_\ell}(|w_\ell|) \cdot \xi^{\ell \cdot \|w_\ell\|}, \tag{13}$$

where $s^{-1}ws \in B$ in the sum is denoted by $(w_0, \ldots, w_{r-1})$, and where $\chi_{\lambda^{(\ell)}}$ denotes the $\mathcal{S}_{k_\ell}$-character indexed by $\lambda^{(\ell)}$.

Having the list of irreducible characters at hand, we can now proceed with the four steps in the proof as explained in Section 5.2.

Step 1. The following lemma describes those characters not vanishing on the inverse of the Coxeter element.

Lemma 5.5. *The character $\chi_{\vec{\lambda}}$ defined in* (5.4) *vanishes on c^{-1} unless*

$$\vec{\lambda} = {}_q\mathfrak{h}_k^n := (0, \ldots, 0, \mathfrak{h}_k^n, 0, \ldots 0) \text{ for } 0 \leq q < r \text{ and } 0 \leq k < n,$$

where as before $\mathfrak{h}_k^n$ is a hook of size n, and where the hook appears at position q.

Proof. Let us consider the evaluation of Formula (13) with $w = c^{-1}$. Observe that $|c^{-1}|$ is a long cycle. Thus, the cyclic group it generates acts transitively on $\{1, 2, \ldots, n\}$, which implies that the sum in (13) is empty unless exactly one of the numbers $k_0, \ldots, k_{r-1}$ is equal to n, say k_q. Moreover, the Murnaghan-Nakayama rule implies that the evaluation $\chi_{\lambda_q}(c_0)$ is zero unless $\lambda^{(k_q)}$ is a hook. We are therefore left with all the partition vectors listed in the lemma. $\square$

Step 2. By the Murnaghan-Nakayama rule and the fact that $\|c^{-1}\| = r - 1$, the evaluation of (13) gives

$$\chi_{q\mathfrak{h}_k^n}(c^{-1}) = (-1)^k \xi^{-q}.$$

Moreover, their degrees are easily determined by evaluating (13) on the identity element,

$$\dim({}_q\mathfrak{h}_k^n) = \chi_{\mathfrak{h}_k^n}(1) = \binom{n-1}{k}.$$

Step 3. The set of reflections of $G(r, 1, n)$ is divided into r conjugacy classes, $\mathcal{R} = \bigcup_{\ell=0}^{r-1} \mathcal{R}_\ell$, where

- for $1 \leq \ell < r$, $\mathcal{R}_\ell$ is the set of matrices obtained from the identity matrix by replacing one of its entries by ξ^ℓ. Hence $\mathcal{R}_\ell$ has cardinality n, for each ℓ.
- $\mathcal{R}_0$ is the set of matrices $M_{i,j}^{(k)}$, for $1 \leq i < j \leq n$, $1 \leq k \leq r$, where $M_{i,j}^{(k)}$ is obtained from the matrix of the transposition (i, j) by replacing its entries in columns i and j by ξ^k and ξ^{-k}, respectively. Hence, $\mathcal{R}_0$ has cardinality $r\binom{n}{2}$.

For $0 \leq \ell < r$ we let $R_\ell = \sum_{\tau \in \mathcal{R}_\ell} \tau$ be the corresponding element of the group algebra. We need to evaluate the character $\chi_{q\mathfrak{h}_k^n}$ on each of the elements R_ℓ. If $\tau \in \mathcal{R}_\ell$ with $\ell > 0$, then $\|\tau\| = \ell$ and $|\tau| = \mathbb{1}$. Therefore, evaluating (13) yields $\chi_{q\mathfrak{h}_k^n}(\tau) = \dim(\mathfrak{h}_k^n)\xi^{q\ell}$, and thus

$$\chi_{q\mathfrak{h}_k^n}(R_\ell) = n \dim(\mathfrak{h}_k^n)\xi^{q\ell} = n \dim({}_q\mathfrak{h}_k^n)\xi^{q\ell}.$$

If $\tau \in \mathcal{R}_0$, then $\|\tau\| = 0$ and $|\tau|$ is a transposition, so by (12) the character (13) evaluates as

$$\chi_{q\mathfrak{h}_k^n}(\tau) = \chi_{\mathfrak{h}_k^n}(|\tau|) = \dim(\mathfrak{h}_k^n)\frac{n - 2k - 1}{n - 1} = \dim({}_q\mathfrak{h}_k^n)\frac{n - 2k - 1}{n - 1}.$$

Adding all contributions and letting R be the sum of all reflections as in Section 4, we obtain

$$\widetilde{\chi}_{q\mathfrak{h}_k^n}(R) = \sum_{\ell=1}^{r-1} n\xi^{q\ell} + \frac{nr(n - 2k - 1)}{2}.$$

Step 4. From Steps 1–3, Formula (11) for $W = G(r, 1, n)$ finally rewrites as

$$\text{FAC}_W(t) = \frac{1}{|W|} \sum_{q=0}^{r-1} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \xi^{-q} \exp\left(t \sum_{\ell=1}^{r-1} n\xi^{q\ell} + tr\frac{n(n-2k-1)}{2}\right)$$

$$= \frac{1}{|W|} e^{tr\binom{n}{2}} \left(\sum_{q=0}^{r-1} \xi^{-q} e^{t \cdot \sum_{l=1}^{r-1} n\xi^{ql}}\right) \left(\sum_{k=0}^{n-1} \binom{n-1}{k}(-1)^k e^{-trnk}\right)$$

$$= \frac{1}{|W|} e^{tr\binom{n}{2}} \left(1 - e^{-trn}\right)^{n-1} \left(\sum_{q=0}^{r-1} \xi^{-q} e^{t \cdot \sum_{\ell=1}^{r-1} n\xi^{q\ell}}\right).$$

To finish the proof of Theorem 1.1 for the group $G(r, 1, n)$, we use the following lemma.

Lemma 5.6. *Let ξ be a primitive r-th root of unity. Then*

$$\sum_{q=0}^{r-1} \xi^{-q} e^{t \cdot \sum_{\ell=1}^{r-1} n\xi^{q\ell}} = e^{t(r-1)n} - e^{-tn}$$

Proof. If $1 < q < r - 1$ then $\xi^{-q} \neq 1$ and $\sum_{\ell=1}^{r-1} \xi^{q\ell} = -1$ by summing the geometric progression (notice that here we use that $r \geq 2$). We therefore obtain

$$\sum_{q=0}^{r-1} \xi^{-q} e^{t \cdot \sum_{\ell=1}^{r-1} n\xi^{q\ell}} = \xi^0 e^{n(r-1)t} + \sum_{q=1}^{r-1} \xi^{-q} e^{-nt} = e^{(n-1)rt} - e^{-nt}.$$

$\square$

Proof of Theorem 1.1 for the group $W = G(r, 1, n)$ with $r > 1$. Using Lemma 5.6, our last expression of $\text{FAC}_W(t)$ rewrites as

$$\text{FAC}_W(t) = \frac{1}{|W|} e^{tr\binom{n}{2}} \left(1 - e^{-trn}\right)^{n-1} \left(e^{t(r-1)n} - e^{-tn}\right)$$

$$= \frac{1}{|W|} e^{tr\binom{n}{2}+t(r-1)n} \left(1 - e^{-trn}\right)^n$$

$$= \frac{1}{|W|} \left(e^{t\left(r\frac{n+1}{2}-1\right)} - e^{-t\left(r\frac{n-1}{2}+1\right)}\right)^n.$$

Since we have for $W = G(r, 1, n)$ that $|\mathcal{R}| = r\binom{n+1}{2} - n$ and $|\mathcal{R}^*| = r\binom{n}{2} + n$, this coincides with (2) in this case. $\square$

5.4. The proof for the group $G(r, r, n)$ with $r > 1$ and $n > 2$. The group $G(r, r, 2)$ is the dihedral group $I_2(r)$, which we have already treated in Section 3. We thus assume $n > 2$. Now, the Coxeter number for $G(r, r, n)$ is $h = (n-1)r$. Let ζ be a primitive h-th root of unity, and let $\xi = \zeta^{n-1}$, which is a primitive r-th root of unity. The group $G(r, r, n)$ is a subgroup of $G(r, 1, n)$, and we reuse the notation defined in the previous section, still using the letter ξ for the primitive r-th root of unity underlying the presentation.

Consider the element $c = c_0 \wr (0, 0, \ldots, 0, 1, r - 1) \in G(r, r, n)$, where $c_0 \in \mathcal{S}_n$ is given by the cycle $(1, 2, \ldots, n - 1)(n)$. We claim that c is a Coxeter element of $G(r, r, n)$. To see that, from the discussion preceding Section 2.1, it is enough to prove that c has an eigenvalue equal to ζ. This is true, since the

fact that $\xi = \zeta^{n-1}$ implies that the element c has $n-1$ eigenvalues given by $\{\zeta, \zeta^{r+1}, \zeta^{2r+1}, \ldots, \zeta^{(n-2)r+1}\}$, while the n-th eigenvalue is given by $\zeta^{(n-1)(r-1)}$.

Remark 5.7. In the rest of the proof for the group $G(r,r,n)$ we will work with the particular Coxeter element $c = c_0 \wr (0, 0, \ldots, 0, 1, r-1)$. This is enough to prove Theorem 1.1 for *all* Coxeter elements of the group $G(r,r,n)$, for the same reasons as described in Remark 5.3.

Step 1. The description of irreducible characters is now more complicated than for the group $G(r,1,n)$. Therefore we consider here only the irreducible characters that are of interest to us in the proof of Theorem 1.1. We refer to Appendix A for a description of all irreducible characters, and for a proof of the following lemma.

Lemma 5.8. *Let χ be an irreducible character of $G(r,r,n)$ not vanishing on c^{-1}. Then χ is equal to the restriction of the character $\chi_{\vec{\lambda}}$ of $G(r,1,n)$ defined by (13), where $\vec{\lambda}$ is one of the following partition vectors.*

- *$\vec{\lambda} = ([n], \ldots)$ or $\vec{\lambda} = ([1^n], \ldots)$, where the dots denote a list of $(r-1)$ empty partitions,*
- *$\vec{\lambda} = (\mathfrak{qh}_k^n, \ldots)$ for some $1 \le k \le n-3$, where $\mathfrak{qh}_k^n$ denotes the quasi-hook $\mathfrak{qh}_k^n := [n-k-1, 2, 1^{k-1}]$,*
- *$\vec{\lambda} = (\mathfrak{h}_k^{n-1}, \ldots, 1, \ldots)$ for some $1 \le j < r$ and $0 \le k \le n-2$. Here $\mathfrak{h}_k^{n-1}$ is a hook of size $n-1$ and the partition of size 1 appears in position j. As before the dots denote lists of empty partitions, respectively of lengths $j-1$ and $r-j-1$.*

Step 2. This step is encapsulated in the following lemma.

Lemma 5.9. *The dimension and evaluation on the Coxeter element c of the characters listed in Lemma 5.8 are given as follows.*

$$\dim([n], \ldots) = 1, \quad \chi_{[n],\ldots}(c^{-1}) = 1,$$
$$\dim([1^n], \ldots) = 1, \quad \chi_{[1^n],\ldots}(c^{-1}) = (-1)^n,$$
$$\dim(\mathfrak{qh}_k^n, \ldots) = \frac{(n-2-k)k}{n-1}\binom{n}{k+1}, \quad \chi_{\mathfrak{qh}_k^n,\ldots}(c^{-1}) = (-1)^k,$$
$$\dim(\mathfrak{h}_k^{n-1}, \ldots, 1, \ldots) = n \cdot \binom{n-2}{k}, \quad \chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}(c^{-1}) = (-1)^k \cdot \xi^{-j}.$$

Proof. If λ is a partition of n, observe that in the sum (13) defining $\chi_{\lambda,\ldots}$, all the elements $s \in G(r,1,n)$ contribute. Therefore one has $\chi_{\lambda,\ldots}(w) = \chi_\lambda(|w|)$ for all $w \in G(r,1,n)$. This settles the first three cases considered in the lemma, using the Murnaghan-Nakayama rule and the hook-length formula (the only non immediate fact here is that $\dim(\mathfrak{qh}_k^n) = \frac{(n-2-k)k}{n-1}\binom{n}{k+1}$; it follows by a simple computation using the hook-length formula, that we leave to the reader).

In the fourth case we have to be more careful about the elements s contributing to the sum (13). When evaluating $\chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}$ on c^{-1}, an element $s \in G(r,1,n)$ contributes to the sum if and only if s belongs to the group B of diagonal matrices defined in Proposition 5.4. From this we obtain $\chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}(c^{-1}) = \chi_{\mathfrak{h}_k^{n-1}}((1, 2, \ldots, n-$

1)) $\cdot \xi^{-j}$, which equals $(-1)^k \cdot \xi^{-j}$ by Lemma 5.1. When evaluating the same character on the identity, then all the elements $s \in G(r, 1, n)$ contribute, so we obtain

$$\chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}(\mathbb{1}) = \frac{|G(r, 1, n)|}{|B|}\chi_{\mathfrak{h}_k^{n-1}}(\mathbb{1}) = n\dim(\mathfrak{h}_k^{n-1}) = n\binom{n-2}{k},$$

where we again used Lemma 5.1. $\qquad\square$

Step 3. The set of reflections of $G(r, r, n)$ is formed of one conjugacy class, namely the class $\mathcal{R}_0$ defined in Section 5.3. The following lemma gives the evaluation of the normalized irreducible characters at the corresponding group algebra element $R_0 := \sum_{\tau \in \mathcal{R}_0} \tau$.

Lemma 5.10. *The normalized character evaluations on the conjugacy class of reflections for the group $G(r, r, n)$ is given by*

$$\widetilde{\chi}_{[n],\ldots}(R_0) = r\binom{n}{2}, \tag{14}$$

$$\widetilde{\chi}_{[1^n],\ldots}(R_0) = -r\binom{n}{2}, \tag{15}$$

$$\widetilde{\chi}_{\mathfrak{q}\mathfrak{h}_k^n,\ldots}(R_0) = \frac{r(n-1)(n-2-2k)}{2}, \tag{16}$$

$$\widetilde{\chi}_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}(R_0) = \frac{r(n-1)(n-2k-2)}{2}. \tag{17}$$

Proof. As already observed in the proof of Lemma 5.9, for λ a partition of n we have $\chi_{\lambda,\ldots}(w) = \chi_\lambda(|w|)$ for all w. Therefore we have $\chi_{\lambda,\ldots}(R_0) = r\binom{n}{2}\chi_\lambda(|\tau|)$ where τ is any element of $\mathcal{R}_0$. Since $|\tau|$ is a transposition, the three first cases (14), (15), (16) therefore follow from Lemma 5.2 (we leave to the reader the details of the computation of the sum of contents of the quasi-hook).

Now let us consider the evaluation $\chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}(\tau)$ where $\tau \in \mathcal{R}_0$. Note that $|\tau|$ is a transposition, say $|\tau| = (i, j)$. In (13), an element $s \in G(r, 1, n)$ is such that $s^{-1}\tau s \in B$ if and only if $|s^{-1}(n)| \notin \{i, j\}$. There are $r^n(n-2)(n-1)!$ such elements in $G(r, 1, n)$. Therefore (13) evaluates as

$$\begin{aligned}\chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots}(\tau) &= \frac{r^n(n-2)(n-1)!}{r^n(n-1)!}\chi_{\mathfrak{h}_k^{n-1}}(|\tau|)\\ &= (n-2)\chi_{\mathfrak{h}_k^{n-1}}(|\tau|)\\ &= (n-2k-2)\dim(\mathfrak{h}_k^{n-1}),\end{aligned}$$

where we used (12) in the last equality. (17) then follows from the equality $\dim(\mathfrak{h}_k^{n-1}) = n\dim(\chi_{\mathfrak{h}_k^{n-1},\ldots,1,\ldots})$. $\qquad\square$

Step 4. From Steps 1–3, Formula (11) rewrites as

$$\begin{aligned}|W| \cdot \mathrm{FAC}_W(t) = {}&e^{tr\binom{n}{2}} + (-1)^n e^{-tr\binom{n}{2}}\\ &+ \sum_{k=1}^{n-3}\frac{(n-2-k)k}{n-1}\binom{n}{k+1}(-1)^k e^{tr\frac{(n-1)(n-2-2k)}{2}}\\ &+ \sum_{k=0}^{n-2}n\binom{n-2}{k}(-1)^k e^{tr\frac{(n-1)(n-2-2k)}{2}}\left(\sum_{j=1}^{r-1}\xi^{-j}\right).\end{aligned}$$

To simplify this expression, first note that the range of summation in the first sum may be extended to $k \in [0, n-2]$ since boundary values vanish. Then, note that $\sum_{j=1}^{r-1} \xi^{-j} = -1$. Finally, observe that we have

$$n \binom{n-2}{k} - \frac{(n-2-k)k}{n-1} \binom{n}{k+1} = \binom{n}{k+1}.$$

Therefore the two sums can be merged together and we obtain

$$\mathrm{FAC}_W(t) = \frac{1}{|W|} \left(e^{tr\binom{n}{2}} + (-1)^n e^{-tr\binom{n}{2}} \right.$$

$$\left. + \sum_{k=0}^{n-2} \binom{n}{k+1} (-1)^{k+1} e^{tr \frac{(n-1)(n-2k-2)}{2}} \right).$$

We can now finish the proof of Theorem 1.1 for the group $G(r, r, n)$.

Proof of Theorem 1.1 for the group $W = G(r, r, n)$ with $r > 1$ and $n > 2$. Notice that the two isolated terms in the last equality may be incorporated to the sum by extending the summation range to $k \in [-1, n-1]$. Making the change of index $k \to k+1$ we obtain

$$\mathrm{FAC}_W(t) = \frac{1}{|W|} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{tr \frac{(n-1)(n-2k)}{2}}$$

$$= \frac{1}{|W|} e^{tr \frac{(n-1)}{2}} \left(1 - e^{-tr(n-1)} \right)^n$$

$$= \frac{1}{|W|} \left(e^{tr \frac{(n-1)}{2}} - e^{-tr \frac{(n-1)}{2}} \right)^n.$$

Since we have for $W = G(r, r, n)$ that $|\mathcal{R}| = |\mathcal{R}^*| = r\binom{n}{2}$, this coincides with (2) in this case. $\qquad\square$

6. Proof of the main theorem for the exceptional groups

We now discuss the verification of Theorem 1.1 for the well-generated exceptional Shephard-Todd classification types. Those were listed in Table 1. In Table 2 in Appendix B, we summarize more detailed information concerning reflections, degrees, codegrees, and number of irreducible representations. The order, degrees, and codegrees are taken from [BMR95], while the number of irreducible representations were computed using `Chevie` [GHL+96]. This software package does not only provide the number of irreducible representations but also the complete character tables[2]. The irreducible characters are uniquely determined (and indexed in `Chevie`) by their *degrees* given by the dimension of the corresponding irreducible representation, $\deg(\chi) = \chi(\mathbb{1})$, together with the smallest integer k for which the irreducible representation V_χ occurs within the k-th symmetric power of the reflection representation V,

$$\mathrm{occ}(\chi) := \min \left\{ k : V_\chi \text{ is a summand of } \mathrm{Sym}^k(V) \right\}.$$

Moreover, in `Chevie`, for each character value at a conjugacy class, a class representative within W is given as a permutation of the roots for W. Its matrix in

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[2]see http://www.math.jussieu.fr/~jmichel/gap3/htm/chap073.htm for examples and a detailed description.

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(c^{-1})$	$\chi(R)$
1	0	1	1	8
1	4	ζ_3^2	ζ_3	-4
1	8	ζ_3	ζ_3^2	-4
2	1	ζ_3	ζ_3^2	4
2	3	ζ_3^2	ζ_3	4
2	5	1	1	-8
3	2	0	0	0

FIGURE 1. Character values for G_4

$\mathrm{GL}(V)$ thus has rows given by the images of the simple roots. Therefore, one can simply find the class representatives of the class $\mathcal{C}_\zeta$ given by all Coxeter elements which are ζ-regular, for each primitive h-th root of unity ζ (see the discussion in the paragraph preceding Section 2.1). Similarly, one can find the class representatives of the conjugacy classes of reflections. For the later, we computed as well the class sizes. To be more precise, we used the Sage [S$^+$11] interface to GAP3 and Chevie, together with the Sage patch on reflection groups[3] to determine the conjugacy classes of Coxeter elements, and the classes of reflections, and to compute the class sizes and to evaluate the irreducible characters at these classes.

As a backup check, we rechecked this data for the irreducible characters using the Sage interface to GAP4 together with the GAP4 implementation of irreducible characters of permutation groups. As mentioned above, we used here the presentation of the irreducible well-generated complex reflection groups as permutation groups acting on roots.

In Figure 1, the computed data for the exceptional complex reflection group G_4 is shown as an example. The first two columns provide $\deg(\chi)$ and $\mathrm{occ}(\chi)$ which uniquely determine the given irreducible representation χ of G_4. We can read from Table 2 that the Coxeter number for G_4 is 6. We thus have two primitive 6-th roots of unity, namely $\zeta = e^{2\pi i/6}$ and $\zeta^{-1} = e^{10\pi i/6}$. Columns 3 and 4 in Figure 1 show the evaluation $\chi(c)$ and $\chi(c^{-1})$ of the irreducible characters at representatives of the two conjugacy classes $\mathcal{C}_\zeta$ and $\mathcal{C}_{\zeta^{-1}}$. The last column finally gives the evaluation $\chi(R) = \sum_{\tau \in \mathcal{R}} \chi(\tau)$ of the irreducible characters at the sum of the reflections. Evaluating the finite sum (11) with this data, one easily checks that Theorem 1.1 holds for the group $W = G_4$.

For the interested reader, the complete and long list of the evaluations for all exceptional well-generated complex reflection groups can be found in Tables 3–28 in Appendix B. To save space, we only show the evaluation of the irreducible characters at a representative c of the conjugacy class $\mathcal{C}_\zeta$ for $\zeta = e^{2\pi i/h}$, and omit the other primitive h-th roots of unity. Nonetheless, we checked the other classes as well.

This procedure allowed us to compute (11) for all well-generated exceptional Shephard-Todd classification types, and thus to verify the following theorem which completes the proof of Theorem 1.1.

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[3]see http://trac.sagemath.org/sage_trac/ticket/11187 for further information.

Theorem 6.1. *Theorem 1.1 holds for all well-generated exceptional Shephard-Todd classification types.*

APPENDIX A. IRREDUCIBLE CHARACTERS OF $G(r,1,n)$ AND $G(r,r,n)$.

In this appendix we give the proofs of Proposition 5.4 and Lemma 5.8. We do not claim much originality. The proof of Proposition 5.4 is easily found in the literature, see e.g. [Mac95, Appendix B]. However, for the group $G(r,r,n)$, we were unable to provide references where the characters are described explicitly enough so that Lemma 5.8 could be given as an immediate corollary[4]. Therefore we include its proof here, and since it follows from the same general theory and helps setting up notation, we also include the proof of Proposition 5.4. Both proofs are exercises in elementary representation theory, and only use the description of $G(r,p,n)$ as the semi-direct product of $\mathcal{S}_n$ by an abelian group, see below.

A.1. **Irreducible characters of semidirect products with abelian groups.** The semidirect product with an abelian group is a standard situation in representation theory. Here for convenience we recall briefly the general theory, taken verbatim from [Ser77, Section 8.2], to which we refer for the proofs. We follow the notation in this reference, expect that we denote by K the group denoted by H in [Ser77], and by κ, instead of h, elements of this group.

We let G, A, K be finite groups such that A is abelian, $G = A \cdot K$, and $A \cap K = \{1\}$. We let $X = \mathrm{Hom}(A, \mathbb{C}^*)$ be the group of complex irreducible characters of A. The group G operates on X by $gx(a) := x(g^{-1}ag)$. We fix $(x_\iota)_{\iota \in X/K}$ be a complete system of representative of the orbits of K in X, and for $\iota \in X/K$, we let K_ι be the subgroup of K formed by elements κ such that $\kappa x_\iota = x_\iota$. We let $G_\iota = A \cdot K_\iota$ be the corresponding subgroup of G. We extend x_ι to G_ι by defining $x_\iota(a \cdot \kappa) = x_\iota(a)$. this defines a one-dimensional character of G_ι. Now, let ρ be an irreducible representation of K_ι. By composing ρ with the projection $G_\iota \to K_\iota$, we obtain an irreducible representation $\tilde{\rho}$ of G_ι. By considering the tensor product $x_\iota \otimes \tilde{\rho}$ we obtain an irreducible representation of G_ι. We let $\theta_{\iota,\rho} := \mathrm{Ind}_{G_\iota}^{G}(x_\iota \otimes \tilde{\rho})$ be the corresponding induced representation of G.

Proposition A.1 ([Ser77, Proposition 25]). *The representations $\theta_{\iota,\rho}$ form a complete list of irreducible representations of G. Moreover two such representations $\theta_{\iota,\rho}$ and $\theta_{\iota',\rho'}$ are isomorphic if and only if $\iota = \iota'$ and ρ is isomorphic to ρ'.*

Note that what precedes applies to the group $G = G(r,p,n)$, upon taking $K = \mathcal{S}_n$, and letting A be the group of diagonal matrices of size n, whose entries are r-th roots of unity $(\xi_1, \xi_2, \ldots, \xi_n)$ such that the product $\xi_1\xi_2\ldots\xi_n$ is an (r/p)-th root of unity. In the remainder of this section we apply the general theory to the two cases that are of interest to us, namely $p = 1$ and $p = r$. In what follows ξ is a fixed primitive r-th root of unity.

A.2. **Irreducible characters for the group $G(r,1,n)$.** In the case of $G(r,1,n)$ we have $A \cong (\mathbb{Z}/r\mathbb{Z})^n$, and $X = \mathrm{Hom}(A, \mathbb{C}^*)$ is formed of the homomorphisms $\{x_\ell, \ell \in A\}$ given by

$$x_\ell(a) = \xi^{\ell_1 a_1 + \ell_2 a_2 + \cdots + \ell_n a_n}$$

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[4][Ste89] gives a description of the irreducible representations of $G(r,r,n)$ in a way that is essentially equivalent to ours.

for $a \in A$. The orbits of X under the action of $K = \mathcal{S}_n$ are indexed by vectors $\iota = (i_0, i_1, \ldots, i_{r-1})$ of sum n, think of i_m as the number of coordinates of ℓ equal to m. For a given vector ι there is a natural representative $x_\iota = x_{(0^{i_0}, 1^{i_1}, \ldots)}$, and we obtain this way a complete system of representatives of X/K. The subgroup K_ι of $\mathcal{S}_n$ formed by elements κ such that $\kappa x_\iota = x_\iota$ is, clearly, the group of diagonal matrices

$$K_\iota = \mathcal{S}_{i_0} \times \mathcal{S}_{i_1} \times \cdots \times \mathcal{S}_{i_{r-1}}.$$

The irreducible representations of this group are given by $\rho = \rho_{\lambda_0} \otimes \rho_{\lambda_1} \otimes \cdots \otimes \rho_{\lambda_{r-1}}$ where $\vec{\lambda}$ is a vector of partitions as in Proposition 5.4 and ρ_{λ_m} is the representation of $\mathcal{S}_{i_m}$ indexed by λ_m, for $0 \le m < r$. Note that the group $G_\iota = A.K_\iota$ is nothing but the group B of block-diagonal matrices defined in Proposition 5.4. With the notation of Paragraph A.1, the tensor product $\tilde{\rho} \otimes x_\iota$ is a representation of G_ι of character

$$\chi_{\tilde{\rho} \otimes x_\iota}(w_0, w_1 \ldots, w_{r-1}) = \prod_{\ell=0}^{r-1} \chi_{\lambda_\ell}(|w_\ell|)\xi^{\ell \cdot \|w_\ell\|}.$$

The representation $\theta_{\iota,\rho}$ is, by definition, the induced representation $\mathrm{Ind}_{G_\iota}^G(\tilde{\rho} \otimes x_\iota)$, whose character is given by (13) (for general background on characters of induced representations, see e.g. [Ser77, Theorem 12]). Therefore, Proposition A.1 becomes Proposition 5.4 is this setting.

A.3. **Irreducible characters for the group** $G(r, r, n)$**.** We are going to use the general theory of Paragraph A.1 to describe the irreducible representations of $G(r, r, n)$ as induced representations, but we will not go as far as in the case of $G(r, 1, n)$. More precisely, in the notation of A.1, we will describe explicitly the groups G_ι, but we will not describe explicitly their irreducible representations in general. Instead we will keep in mind that, to prove Lemma 5.8, we are only interested in characters that do not vanish on the inverse c^{-1} of the Coxeter element, so we will disregard as soon as possible any group G_ι that contains no conjugate of c^{-1}. As a side remark, let us mention finally that we could have proved Lemma 5.8 directly using Proposition 5.4 and classical criteria regarding restrictions of induced representations ([Ser77, Chapter 7]). However the approach presented here is not longer, and it requires no more background than the one presented in Appendix A.1.

Recall that in the case of $G(r, r, n)$ the Coxeter element is $c = c_0 \wr (0, \ldots, 0, 1, r-1)$ where $c_0 = (1, 2, \ldots, n-1)(n)$ has cycle type $[n-1, 1]$. As in Section 5.4 we assume that $n > 2$.

In the case of $G(r, r, n)$ we have $A \cong \mathrm{Ker}\,\pi$ where $\pi : (\mathbb{Z}/r\mathbb{Z})^n \to \mathbb{Z}/r\mathbb{Z}$ is the sum-of-coordinate mapping. Therefore $X = \mathrm{Hom}(A, \mathbb{C}^*)$ is formed of the homomorphisms

$$x_\ell : a \in A \longmapsto \xi^{\ell_1 a_1 + \ell_2 a_2 + \ldots \ell_n a_n}$$

for $\ell \in (\mathbb{Z}/r\mathbb{Z})^n/\approx$, where $\ell \approx \ell'$ if and only if $\ell - \ell'$ is of the form $(z, z, \ldots, z)$. The orbits of X under the action of $K = \mathcal{S}_n$ are indexed by vectors $\iota = (i_0, i_1, \ldots, i_{r-1})$ of total sum n considered up to circular permutation (think of i_m as the number of coordinates of ℓ equal to m, and observe that a translation on ℓ acts as a circular permutation on ι). For each such orbit we fix a representative $x_\iota = x_{0^{i_0} 1^{i_1} \ldots}$, and we obtain this way a complete system of representatives of X/K.

Fix $\iota = (i_0, i_1, \ldots, i_{r-1})$ and let d be the smallest positive integer such that $i_{\ell+d} = i_\ell$ for all ℓ, indices being taken modulo r. Note that d is a divisor of r, say

$r = md$, so that ι consists of m copies of the list

$$\iota^\circ := i_0, i_1, \ldots, i_{d-1}.$$

We now determine the group K_ι, in the notation of A.1. By definition, an element $\kappa \in \mathfrak{S}_n$ is such that $\kappa x_\iota = x_\iota$ if and only if

$$\kappa(0^{i_0}, 1^{i_1}, \ldots, (r-1)^{i_{r-1}}) = (0^{i_0}, 1^{i_1}, \ldots, (r-1)^{i_{r-1}}) \text{ modulo } \approx,$$

where $\approx$ is the equivalence relation defined above, i.e. iff there exists $k \in \{1, \ldots, r\}$ such that

$$\kappa(0^{i_0}, 1^{i_1}, \ldots, (r-1)^{i_{r-1}}) = (k^{i_0}, (k+1)^{i_1}, \ldots, (k+r-1)^{i_{r-1}}). \qquad (18)$$

Counting the number of coordinates equal to ℓ, this implies in particular that $i_\ell = i_{k+\ell}$ for all ℓ, so that k is a multiple of d, say $k = ad$. More precisely, Equation (18) says that κ realizes a bijection between the set of coordinates containing the integer ℓ to those containing the integer $\ell + k$ for all ℓ. In other words, K_ι is the group of permutation matrices having the "cycle-block form" as shown in Figure 2 for some $0 \le a < m$, and for some permutation matrices $\pi_0, \pi_1, \ldots, \pi_{r-1}$ of respective sizes $i_0, i_1, \ldots, i_{r-1}$. The group G_ι is the group of matrices having the same form, but where entries equal to 1 can be replaced by r-th roots of unity in such a way that the product of all entries is 1.

Lemma A.2. *Unless one of the entries of ι is equal to n or $n-1$, the inverse c^{-1} of the Coxeter element has no conjugates in the group G_ι.*

Proof. First note that c^{-1} has a conjugate in G_ι if and only if $c_0^{-1} = |c^{-1}|$ has a conjugate in K_ι. Since $c_0^{-1} = (n-1, n-2, \ldots, 1)(n)$ has a fixed point, if such a conjugate exists it is such that $a = 0$, keeping previous notation. In other words, the conjugate belongs to the group of block-diagonal matrices $\mathcal{S}_{i_0} \times \mathcal{S}_{i_1} \times \cdots \times \mathcal{S}_{i_{r-1}}$. But since c_0^{-1} has cycle type $[n-1, 1]$, one of the blocks must be of size $n-1$ or n, in order to fit the cycle of length $(n-1)$. □

Recalling the classical expression of induced characters as a sum over conjugates, see e.g. [Ser77, Theorem 12], this implies the following lemma.

Lemma A.3. *Unless one of the entries of ι is equal to n or $n-1$, the character of the representation $\theta_{\iota,\rho}$ defined in A.1 vanishes on the Coxeter element, for any irreducible representation ρ of K_ι.*

Since we only want to prove Lemma 5.8, we may thus focus only on the case when ι has a coordinate equal to n or $n-1$. We treat both cases separately.

In the first case, we have that $\iota = (n, 0, \ldots, 0)$ up to circular permutation. The group K_ι coincides with $\mathcal{S}_n$, and G_ι coincides with $G(r, r, n)$. For any irreducible representation $\rho = \rho_\lambda$ of $\mathcal{S}_n$, the character of the irreducible representation $\theta_{\iota,\rho}$ on an element $w \in G(r, r, n)$ is given, simply, by the $\mathcal{S}_n$-character $\chi_\lambda(|w|)$.

It remains to determine which partitions λ of n are such that the character χ_λ does not vanish on the permutation c_0^{-1} of type $[n-1, 1]$. Clearly, the Murnaghan-Nakayama rule implies than λ is a hook or a quasi-hook, since these are the only partitions containing a strip of length $n-1$. A closer look at the case of hooks shows, moreover, that the only hooks for which the character does not vanish are $[n]$ and $[1^n]$. Finally, it is clear that the characters thus defined coincide with the restrictions to $G(r, r, n)$ of the representations $\chi_{[n], \ldots}$, $\chi_{[1^n], \ldots}$ and $\chi_{\mathfrak{qh}_k^n, \ldots}$ as stated in Lemma 5.8.
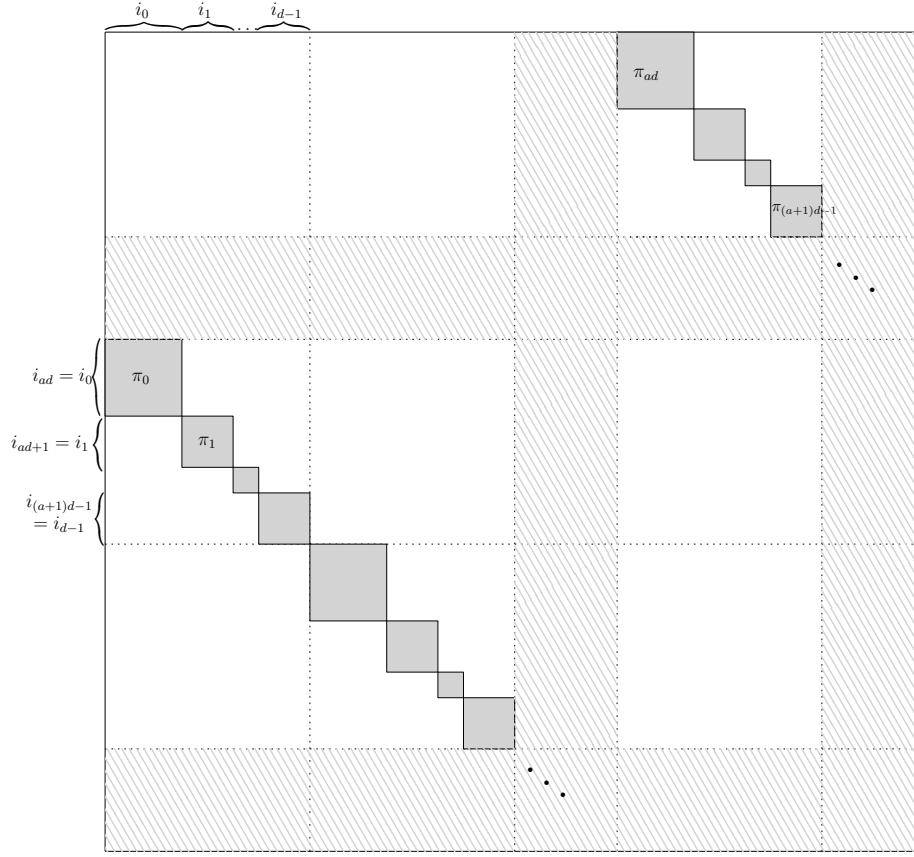
FIGURE 2. A permutation matrix in "cycle-block form". All entries in the white regions are equal to zero.

In the second case, we have that, up to a circular permutation, $\iota = (n - 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where the "1" appears in position j, for $1 \leq j < r$. Notice that since $n \neq 2$ we have $n - 1 \neq 1$ so ι is fixed by no non-trivial circular permutation. In previous notation, this means that $d = r$, so that the groups K_ι is the group of block-diagonal matrices

$$K_\iota = \mathcal{S}_{n-1} \times \mathcal{S}_1.$$

The group G_ι is the subgroup of $G(r, r, n)$ formed by matrices w such that $|w| \in K_\iota$. Irreducible representations of K_ι are indexed by partitions λ of $n - 1$, and for each such partition we may thus construct the representation $\theta_{\iota,\rho}$ of $G(r, r, n)$ as in A.1. By definition, its character is

$$\chi(w) = \sum_{\substack{s \in G(r,r,n) \\ s^{-1}ws \in G_\iota}} \chi_\lambda(|w_1|)\xi^{j\|w_2\|}$$

where $s^{-1}ws \in B$ in the sum is denoted by (w_1, w_2) with w_1, w_2 matrices of respective sizes $n-1$ and 1. Since $|c^{-1}| = c_0^{-1}$ has cycle type $[n-1, 1]$, the only conjugates $s^{-1}c^{-1}s$ of the Coxeter element that are in G_ι are such that $|w_1|$ is an $(n-1)$-cycle.

Therefore $\chi(c^{-1}) = 0$ unless λ is a hook of size $n-1$. This concludes the proof of Lemma 5.8, by observing that the character we just described coincides with the restriction of the $G(r, 1, n)$-character $\chi_{\lambda,\dots,1,\dots}$ as in the statement of Lemma 5.8.

APPENDIX B. CHARACTER EVALUATIONS FOR THE EXCEPTIONAL GROUPS

type	order	degrees	codegrees	# irr.rep.
G_4	24	$4, 6$	$0, 2$	7
G_5	72	$6, 12$	$0, 6$	21
G_6	48	$4, 12$	$0, 8$	14
G_8	96	$8, 12$	$0, 4$	16
G_9	192	$8, 24$	$0, 16$	32
G_{10}	288	$12, 24$	$0, 12$	48
G_{14}	144	$6, 24$	$0, 18$	24
G_{16}	600	$20, 30$	$0, 10$	45
G_{17}	1200	$20, 60$	$0, 40$	90
G_{18}	1800	$30, 60$	$0, 30$	135
G_{20}	360	$12, 30$	$0, 18$	27
G_{21}	720	$12, 60$	$0, 48$	54
G_{23}	120	$2, 6, 10$	$0, 4, 8$	10
G_{24}	336	$4, 6, 14$	$0, 8, 10$	12
G_{25}	648	$6, 9, 12$	$0, 3, 6$	24
G_{26}	1296	$6, 12, 18$	$0, 6, 12$	48
G_{27}	2160	$6, 12, 30$	$0, 18, 24$	34
G_{28}	1152	$2, 6, 8, 12$	$0, 4, 6, 10$	25
G_{29}	7680	$4, 8, 12, 20$	$0, 8, 12, 16$	37
G_{30}	14400	$2, 12, 20, 30$	$0, 10, 18, 28$	34
G_{32}	155520	$12, 18, 24, 30$	$0, 6, 12, 18$	102
G_{33}	51840	$4, 6, 10, 12, 18$	$0, 6, 8, 12, 14$	40
G_{34}	39191040	$6, 12, 18, 24, 30, 42$	$0, 12, 18, 24, 30, 36$	169
G_{35}	51840	$2, 5, 6, 8, 9, 12$	$0, 3, 4, 6, 7, 10$	25
G_{36}	2903040	$2, 6, 8, 10, 12, 14, 18$	$0, 4, 6, 8, 10, 12, 16$	60
G_{37}	696729600	$2, 8, 12, 14, 18, 20, 24, 30$	$0, 6, 10, 12, 16, 18, 22, 28$	112

TABLE 2. The order, degrees and codegrees of all well-generated exceptional Shephard-Todd classification types.

TABLE 3. Character values for G_4

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	8
1	4	ζ_3^2	-4
1	8	ζ_3	-4
2	1	ζ_3	4
2	3	ζ_3^2	4
2	5	1	-8
3	2	0	0

TABLE 4. Character values for G_5

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	16
1	4	ζ_3	4
1	4	ζ_3	4
1	8	ζ_3^2	4
1	8	ζ_3^2	-8
1	8	ζ_3^2	4
1	12	1	-8
1	12	1	-8
1	16	ζ_3	-8
2	1	0	8
2	3	0	8
2	3	0	8
2	5	0	-4
2	5	0	8
2	5	0	-4
2	7	0	-4
2	7	0	-4
2	9	0	-16
3	2	$-\zeta_3^2$	0
3	4	$-\zeta_3$	0
3	6	-1	0

TABLE 5. Character values for G_6

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	14
1	4	ζ_3	2
1	6	-1	2
1	8	ζ_3^2	2
1	10	$-\zeta_3$	-10
1	14	$-\zeta_3^2$	-10
2	1	ζ_{12}^{11}	4
2	3	$-\zeta_{12}^{11}$	4
2	3	$-\zeta_{12}^{7}$	4
2	5	ζ_{12}^{7}	4
2	5	ζ_4	-8
2	7	$-\zeta_4$	-8
3	2	0	6
3	4	0	-6

TABLE 6. Character values for G_8

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	18
1	6	-1	-6
1	12	1	-6
1	18	-1	-6
2	1	ζ_4	12
2	4	-1	12
2	7	$-\zeta_4$	12
2	7	$-\zeta_4$	-12
2	10	1	-12
2	13	ζ_4	-12
3	2	0	6
3	4	0	6
3	6	0	6
3	8	0	-18
4	3	ζ_4	0
4	5	$-\zeta_4$	0

TABLE 7. Character values for G_9

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	30
1	6	ζ_4	6
1	12	-1	6
1	12	-1	6
1	18	$-\zeta_4$	-18
1	18	$-\zeta_4$	6
1	24	1	-18
1	30	ζ_4	-18
2	1	$-\zeta_8^3$	12
2	4	1	12
2	5	ζ_8^3	12
2	7	ζ_8	-12
2	7	ζ_8	12
2	8	-1	12
2	10	ζ_4	-12
2	11	$-\zeta_8$	-12
2	11	$-\zeta_8$	12
2	13	ζ_8^3	-12
2	14	$-\zeta_4$	-12
2	17	$-\zeta_8^3$	-12
3	2	0	18
3	4	0	18
3	6	0	-6
3	6	0	18
3	8	0	-6
3	8	0	-6

3	10	0	-6
3	12	0	-30
4	3	ζ_8	0
4	5	$-\zeta_8^3$	0
4	7	$-\zeta_8$	0
4	9	ζ_8^3	0

4	3	0	16
4	5	0	-8
4	7	0	-8
4	9	0	16
4	11	0	-8
4	13	0	-8

TABLE 8. Character values for G_{10}

$\deg(\chi)$	$\operatorname{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	34
1	6	ζ_4	10
1	8	ζ_3	10
1	12	-1	10
1	14	ζ_{12}^7	-14
1	16	ζ_3^2	10
1	18	$-\zeta_4$	10
1	20	$-\zeta_3$	-14
1	22	ζ_{12}^{11}	-14
1	26	$-\zeta_{12}^7$	-14
1	28	$-\zeta_3^2$	-14
1	34	$-\zeta_{12}^{11}$	-14
2	1	0	20
2	4	0	20
2	5	0	20
2	7	0	20
2	7	0	-4
2	8	0	20
2	9	0	-4
2	10	0	-4
2	11	0	20
2	11	0	-4
2	12	0	-4
2	13	0	-4
2	14	0	-4
2	15	0	-4
2	15	0	-28
2	17	0	-4
2	18	0	-28
2	21	0	-28
3	2	$-\zeta_{12}^7$	6
3	4	$-\zeta_3^2$	6
3	6	$-\zeta_4$	6
3	6	ζ_4	6
3	8	$-\zeta_3$	6
3	8	ζ_3	-18
3	10	ζ_{12}^{11}	6
3	10	$-\zeta_{12}^{11}$	6
3	12	1	-18
3	12	-1	6
3	14	ζ_{12}^7	6
3	16	ζ_3^2	-18

TABLE 9. Character values for G_{14}

$\deg(\chi)$	$\operatorname{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	28
1	8	ζ_3	4
1	12	-1	4
1	16	ζ_3^2	4
1	20	$-\zeta_3$	-20
1	28	$-\zeta_3^2$	-20
2	1	$\zeta_{24} + \zeta_{24}^{19}$	8
2	4	0	8
2	5	$-\zeta_{24}^{11} - \zeta_{24}^{17}$	8
2	7	$-\zeta_{24} - \zeta_{24}^{19}$	8
2	8	0	8
2	9	$\zeta_8 + \zeta_8^3$	-16
2	11	$\zeta_{24}^{11} + \zeta_{24}^{17}$	8
2	12	0	-16
2	15	$-\zeta_8 - \zeta_8^3$	-16
3	2	$-\zeta_3$	12
3	4	$-\zeta_3^2$	12
3	6	-1	12
3	6	1	-12
3	8	ζ_3	-12
3	10	ζ_3^2	-12
4	3	0	16
4	5	0	-8
4	7	0	-8

TABLE 10. Character values for G_{16}

$\deg(\chi)$	$\operatorname{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	48
1	12	ζ_5^2	-12
1	24	ζ_5^4	-12
1	36	ζ_5	-12
1	48	ζ_5^3	-12
2	1	ζ_5	36
2	7	ζ_5^2	36
2	13	ζ_5^3	36
2	13	ζ_5^3	-24
2	19	ζ_5^4	36
2	19	ζ_5^4	-24
2	25	1	-24
2	25	1	-24
2	31	ζ_5	-24
2	37	ζ_5^2	-24

deg(χ)	occ(χ)	$\chi(c)$	$\chi(R)$
3	2	0	24
3	6	0	24
3	10	0	24
3	10	0	24
3	14	0	24
3	14	0	−36
3	18	0	24
3	18	0	−36
3	22	0	−36
3	26	0	−36
4	3	$-\zeta_5^3$	12
4	6	ζ_5	12
4	8	ζ_5^3	12
4	9	$-\zeta_5^4$	12
4	11	$-\zeta_5$	12
4	12	ζ_5^2	12
4	14	ζ_5^4	12
4	15	-1	−48
4	17	$-\zeta_5^2$	12
4	20	1	−48
5	4	$-\zeta_5^4$	0
5	8	$-\zeta_5^3$	0
5	10	-1	0
5	12	$-\zeta_5^2$	0
5	16	$-\zeta_5$	0
6	5	0	48
6	7	0	−12
6	9	0	−12
6	11	0	−12
6	13	0	−12

TABLE 11. Character values for G_{17}

deg(χ)	occ(χ)	$\chi(c)$	$\chi(R)$
1	0	1	78
1	12	ζ_5	18
1	24	ζ_5^2	18
1	30	-1	18
1	36	ζ_5^3	18
1	42	$-\zeta_5$	−42
1	48	ζ_5^4	18
1	54	$-\zeta_5^2$	−42
1	66	$-\zeta_5^3$	−42
1	78	$-\zeta_5^4$	−42
2	1	ζ_{20}^{17}	36
2	7	$-\zeta_{20}^9$	36
2	11	$-\zeta_{20}^{17}$	36
2	13	ζ_{20}	36
2	13	ζ_{20}	−24
2	17	ζ_{20}^9	36
2	19	$-\zeta_{20}^{13}$	−24
2	19	$-\zeta_{20}^{13}$	36
2	23	$-\zeta_{20}$	36

deg(χ)	occ(χ)	$\chi(c)$	$\chi(R)$
2	23	$-\zeta_{20}$	−24
2	25	ζ_4	−24
2	25	ζ_4	−24
2	29	ζ_{20}^{13}	36
2	29	ζ_{20}^{13}	−24
2	31	$-\zeta_{20}^{17}$	−24
2	35	$-\zeta_4$	−24
2	35	$-\zeta_4$	−24
2	37	ζ_{20}^9	−24
2	41	ζ_{20}^{17}	−24
2	47	$-\zeta_{20}^9$	−24
3	2	0	54
3	6	0	54
3	10	0	54
3	10	0	54
3	12	0	−6
3	14	0	54
3	14	0	−6
3	16	0	−6
3	18	0	54
3	18	0	−6
3	20	0	−6
3	20	0	−6
3	22	0	−6
3	24	0	−6
3	24	0	−66
3	26	0	−6
3	28	0	−6
3	28	0	−66
3	32	0	−66
3	36	0	−66
4	3	ζ_{20}	12
4	6	$-\zeta_5^3$	12
4	8	ζ_5^4	12
4	9	$-\zeta_{20}^{13}$	12
4	11	ζ_{20}^{17}	12
4	12	ζ_5	12
4	13	$-\zeta_{20}$	12
4	14	$-\zeta_5^2$	12
4	15	ζ_4	−48
4	16	ζ_5^3	12
4	17	$-\zeta_{20}^9$	12
4	18	$-\zeta_5^4$	12
4	19	ζ_{20}^{13}	12
4	20	1	−48
4	21	$-\zeta_{20}^{17}$	12
4	22	$-\zeta_5$	12
4	24	ζ_5^2	12
4	25	$-\zeta_4$	−48
4	27	ζ_{20}^9	12
4	30	-1	−48

5	4	$-\zeta_5^2$	30
5	8	$-\zeta_5^4$	30
5	10	1	−30
5	12	$-\zeta_5$	30
5	14	ζ_5^2	−30
5	16	$-\zeta_5^3$	30
5	18	ζ_5^4	−30
5	20	−1	30
5	22	ζ_5	−30
5	26	ζ_5^3	−30
6	5	0	48
6	7	0	−12
6	9	0	−12
6	11	0	−12
6	13	0	−12
6	15	0	48
6	17	0	−12
6	19	0	−12
6	21	0	−12
6	23	0	−12

TABLE 12. Character values for G_{18}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	88
1	12	ζ_5	28
1	20	ζ_3	28
1	24	ζ_5^2	28
1	32	ζ_{15}^8	−32
1	36	ζ_5^3	28
1	40	ζ_3^2	28
1	44	ζ_{15}^{11}	−32
1	48	ζ_5^4	28
1	52	ζ_{15}^{13}	−32
1	56	ζ_{15}^{14}	−32
1	64	ζ_{15}	−32
1	68	ζ_{15}^2	−32
1	76	ζ_{15}^4	−32
1	88	ζ_{15}^7	−32
2	1	0	56
2	7	0	56
2	11	0	56
2	13	0	56
2	13	0	−4
2	17	0	56
2	19	0	56
2	19	0	−4
2	21	0	−4
2	23	0	56
2	23	0	−4
2	25	0	−4
2	25	0	−4
2	27	0	−4
2	29	0	56
2	29	0	−4
2	31	0	−4
2	33	0	−4
2	33	0	−64
2	35	0	−4
2	35	0	−4
2	37	0	−4
2	39	0	−4
2	39	0	−64
2	41	0	−4
2	45	0	−64
2	45	0	−64
2	47	0	−4
2	51	0	−64
2	57	0	−64
3	2	$-\zeta_{15}^8$	24
3	6	$-\zeta_5^3$	24
3	10	$-\zeta_3^2$	24
3	10	$-\zeta_3^2$	24
3	12	$-\zeta_5$	24
3	14	$-\zeta_{15}^{11}$	24
3	14	$-\zeta_{15}^{11}$	−36
3	16	$-\zeta_{15}^4$	24
3	18	$-\zeta_5^4$	24
3	18	$-\zeta_5^4$	−36
3	20	$-\zeta_3$	24
3	20	$-\zeta_3$	24
3	22	$-\zeta_{15}^{13}$	24
3	22	$-\zeta_{15}^{13}$	−36
3	24	$-\zeta_5^2$	24
3	24	$-\zeta_5^2$	−36
3	26	$-\zeta_{15}^{14}$	24
3	26	$-\zeta_{15}^{14}$	−36
3	28	$-\zeta_{15}^7$	24
3	28	$-\zeta_{15}^7$	−36
3	30	−1	24
3	30	−1	24
3	32	$-\zeta_{15}^8$	−36
3	34	$-\zeta_{15}$	24
3	34	$-\zeta_{15}$	−36
3	36	$-\zeta_5^3$	−36
3	38	$-\zeta_{15}^2$	24
3	38	$-\zeta_{15}^2$	−36
3	42	$-\zeta_5$	−36
3	46	$-\zeta_{15}^4$	−36
4	3	0	52
4	6	0	52
4	8	0	−8
4	9	0	52

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
4	11	0	-8
4	12	0	52
4	13	0	-8
4	14	0	-8
4	15	0	-8
4	16	0	-8
4	17	0	-8
4	18	0	52
4	19	0	-8
4	20	0	-68
4	21	0	52
4	22	0	-8
4	23	0	-8
4	24	0	52
4	25	0	-68
4	26	0	-8
4	27	0	52
4	28	0	-8
4	29	0	-8
4	30	0	-8
4	31	0	-8
4	32	0	-8
4	34	0	-8
4	35	0	-68
4	37	0	-8
4	40	0	-68
5	4	ζ_{15}	20
5	8	ζ_{15}^{2}	20
5	10	ζ_{3}^{2}	20
5	12	ζ_{5}	-40
5	14	ζ_{15}^{11}	20
5	16	ζ_{15}^{4}	20
5	18	ζ_{5}^{4}	-40
5	20	ζ_{3}	20
5	22	ζ_{15}^{13}	20
5	24	ζ_{5}^{2}	-40
5	26	ζ_{15}^{14}	20
5	28	ζ_{15}^{7}	20
5	30	1	-40
5	32	ζ_{15}^{8}	20
5	36	ζ_{5}^{3}	-40
6	5	0	48
6	7	0	-12
6	9	0	-12
6	11	0	-12
6	13	0	-12
6	15	0	48
6	17	0	-12
6	19	0	-12
6	21	0	-12
6	23	0	-12

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
6	25	0	48
6	27	0	-12
6	29	0	-12
6	31	0	-12
6	33	0	-12

TABLE 13. Character values for G_{20}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	40
1	20	ζ_{3}^{2}	-20
1	40	ζ_{3}	-20
2	1	$-\zeta_{15}^{2} - \zeta_{15}^{8}$	20
2	7	$-\zeta_{15}^{11} - \zeta_{15}^{14}$	20
2	11	$-\zeta_{15}^{7} - \zeta_{15}^{13}$	20
2	17	$-\zeta_{15} - \zeta_{15}^{4}$	20
2	21	$-\zeta_{5} - \zeta_{5}^{4}$	-40
2	27	$-\zeta_{5}^{2} - \zeta_{5}^{3}$	-40
3	2	$-\zeta_{15}^{7} - \zeta_{15}^{13}$	0
3	6	$-\zeta_{5}^{2} - \zeta_{5}^{3}$	0
3	10	$-\zeta_{15}^{11} - \zeta_{15}^{14}$	0
3	10	$-\zeta_{15}^{2} - \zeta_{15}^{8}$	0
3	12	$-\zeta_{5} - \zeta_{5}^{4}$	0
3	14	$-\zeta_{15} - \zeta_{15}^{4}$	0
4	3	1	40
4	6	-1	40
4	8	$-\zeta_{3}^{2}$	-20
4	11	ζ_{3}^{2}	-20
4	13	ζ_{3}	-20
4	16	$-\zeta_{3}$	-20
5	4	0	20
5	8	0	20
5	12	0	-40
6	5	$-\zeta_{3}^{2}$	0
6	7	$-\zeta_{3}$	0
6	9	-1	0

TABLE 14. Character values for G_{21}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	70
1	20	ζ_{3}	10
1	30	-1	10
1	40	ζ_{3}^{2}	10
1	50	$-\zeta_{3}$	-50
1	70	$-\zeta_{3}^{2}$	-50
2	1	$-\zeta_{60}^{19} - \zeta_{60}^{31}$	20
2	7	$\zeta_{60}^{7} + \zeta_{60}^{43}$	20
2	11	$\zeta_{60}^{11} + \zeta_{60}^{59}$	20
2	13	$-\zeta_{60}^{7} - \zeta_{60}^{43}$	20
2	17	$-\zeta_{60}^{23} - \zeta_{60}^{47}$	20
2	19	$\zeta_{60}^{19} + \zeta_{60}^{31}$	20
2	21	$-\zeta_{20}^{13} - \zeta_{20}^{17}$	-40

2	23	$\zeta_{60}^{23} + \zeta_{60}^{47}$	20
2	27	$\zeta_{20} + \zeta_{20}^{9}$	-40
2	29	$-\zeta_{60}^{11} - \zeta_{60}^{59}$	20
2	33	$-\zeta_{20} - \zeta_{20}^{9}$	-40
2	39	$\zeta_{20}^{13} + \zeta_{20}^{17}$	-40
3	2	$\zeta_{15}^{11} + \zeta_{15}^{14}$	30
3	6	$\zeta_{5} + \zeta_{5}^{4}$	30
3	10	$\zeta_{15}^{7} + \zeta_{15}^{13}$	30
3	10	$\zeta_{15} + \zeta_{15}^{4}$	30
3	12	$-\zeta_{5}^{2} - \zeta_{5}^{3}$	-30
3	14	$\zeta_{15}^{2} + \zeta_{15}^{8}$	30
3	16	$-\zeta_{15}^{7} - \zeta_{15}^{13}$	-30
3	18	$\zeta_{5}^{2} + \zeta_{5}^{3}$	30
3	20	$-\zeta_{15}^{11} - \zeta_{15}^{14}$	-30
3	20	$-\zeta_{15}^{2} - \zeta_{15}^{8}$	-30
3	24	$-\zeta_{5} - \zeta_{5}^{4}$	-30
3	28	$-\zeta_{15} - \zeta_{15}^{4}$	-30
4	3	$-\zeta_{4}$	40
4	6	1	40
4	8	$-\zeta_{3}$	-20
4	9	ζ_{4}	40
4	11	$-\zeta_{12}^{7}$	-20
4	12	-1	40
4	13	ζ_{12}^{11}	-20
4	14	ζ_{3}	-20
4	16	$-\zeta_{3}^{2}$	-20
4	17	ζ_{12}^{7}	-20
4	19	$-\zeta_{12}^{11}$	-20
4	22	ζ_{3}^{2}	-20
5	4	0	50
5	8	0	50
5	10	0	-10
5	12	0	-10
5	14	0	-10
5	18	0	-70
6	5	$-\zeta_{12}^{7}$	0
6	7	ζ_{12}^{11}	0
6	9	$-\zeta_{4}$	0
6	11	ζ_{12}^{7}	0
6	13	$-\zeta_{12}^{11}$	0
6	15	ζ_{4}	0

TABLE 15. Character values for H_3

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	15
1	15	-1	-15
3	1	$\zeta_{5} + \zeta_{5}^{4}$	15
3	3	$\zeta_{5}^{2} + \zeta_{5}^{3}$	15
3	6	$-\zeta_{5} - \zeta_{5}^{4}$	-15
3	8	$-\zeta_{5}^{2} - \zeta_{5}^{3}$	-15
4	3	1	0
4	4	-1	0

5	2	0	15
5	5	0	-15

TABLE 16. Character values for G_{24}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	21
1	21	-1	-21
3	1	$-\zeta_{7} - \zeta_{7}^{2} - \zeta_{7}^{4}$	21
3	3	$-\zeta_{7}^{3} - \zeta_{7}^{5} - \zeta_{7}^{6}$	21
3	8	$\zeta_{7} + \zeta_{7}^{2} + \zeta_{7}^{4}$	-21
3	10	$\zeta_{7}^{3} + \zeta_{7}^{5} + \zeta_{7}^{6}$	-21
6	2	-1	42
6	9	1	-42
7	3	0	21
7	6	0	-21
8	4	1	0
8	5	-1	0

TABLE 17. Character values for G_{25}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	24
1	12	1	-12
1	24	1	-12
2	3	0	12
2	9	0	12
2	15	0	-24
3	1	ζ_{3}	36
3	5	ζ_{3}^{2}	36
3	5	ζ_{3}^{2}	0
3	6	-1	0
3	13	ζ_{3}	-36
3	13	ζ_{3}	0
3	17	ζ_{3}^{2}	-36
6	2	0	36
6	4	0	36
6	4	0	0
6	8	0	-36
6	8	0	0
6	10	0	-36
8	3	0	48
8	6	0	-24
8	9	0	-24
9	5	$-\zeta_{3}^{2}$	0
9	7	$-\zeta_{3}$	0

TABLE 18. Character values for G_{26}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	33
1	9	-1	15
1	12	ζ_{3}^{2}	-3

1	21	$-\zeta_3^2$	-21
1	24	ζ_3	-3
1	33	$-\zeta_3$	-21
2	3	ζ_3	30
2	9	ζ_3^2	30
2	12	$-\zeta_3$	-6
2	15	1	-6
2	18	$-\zeta_3^2$	-6
2	24	-1	-42
3	1	0	45
3	4	0	27
3	5	0	45
3	5	0	9
3	6	0	27
3	8	0	27
3	8	0	-9
3	13	0	-27
3	13	0	9
3	15	0	-27
3	16	0	-45
3	16	0	-9
3	17	0	-27
3	20	0	-45
6	2	0	54
6	4	0	54
6	4	0	18
6	5	0	18
6	7	0	18
6	7	0	-18
6	8	0	-18
6	8	0	18
6	10	0	-18
6	11	0	-54
6	11	0	-18
6	13	0	-54
8	3	1	48
8	6	-1	48
8	6	$-\zeta_3^2$	-24
8	9	ζ_3^2	-24
8	9	ζ_3	-24
8	12	$-\zeta_3$	-24
9	5	0	27
9	7	0	27
9	8	0	-27
9	10	0	-27

3	7	$\zeta_{15}^2 + \zeta_{15}^8$	45
3	16	$-\zeta_{15}^{11} - \zeta_{15}^{14}$	-45
3	20	$-\zeta_{15}^7 - \zeta_{15}^{13}$	-45
3	20	$-\zeta_{15} - \zeta_{15}^4$	-45
3	22	$-\zeta_{15}^2 - \zeta_{15}^8$	-45
5	6	0	45
5	6	0	45
5	15	0	-45
5	15	0	-45
6	2	ζ_3^2	90
6	4	ζ_3	90
6	17	$-\zeta_3^2$	-90
6	19	$-\zeta_3$	-90
8	6	$-\zeta_5^2 - \zeta_5^3$	0
8	9	$\zeta_5^2 + \zeta_5^3$	0
8	9	$\zeta_5 + \zeta_5^4$	0
8	12	$-\zeta_5 - \zeta_5^4$	0
9	4	$-\zeta_3$	45
9	6	-1	45
9	8	$-\zeta_3^2$	45
9	9	1	-45
9	11	ζ_3^2	-45
9	13	ζ_3	-45
10	3	0	90
10	12	0	-90
15	5	0	45
15	7	0	45
15	8	0	-45
15	10	0	-45

TABLE 20. Character values for F_4

$\deg(\chi)$	$\text{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	24
1	12	1	0
1	12	1	0
1	24	1	-24
2	4	-1	24
2	4	-1	24
2	16	-1	-24
2	16	-1	-24
4	1	0	48
4	7	0	0
4	7	0	0
4	8	1	0
4	13	0	-48
6	6	-1	0
6	6	-1	0
8	3	0	48
8	3	0	48
8	9	0	-48
8	9	0	-48
9	2	0	72

TABLE 19. Character values for G_{27}

$\deg(\chi)$	$\text{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	45
1	45	-1	-45
3	1	$\zeta_{15}^{11} + \zeta_{15}^{14}$	45
3	5	$\zeta_{15}^7 + \zeta_{15}^{13}$	45
3	5	$\zeta_{15} + \zeta_{15}^4$	45

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9	6	0	0
9	6	0	0
9	10	0	-72
12	4	1	0
16	5	0	0

TABLE 21. Character values for G_{29}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	40
1	40	1	-40
4	1	$-\zeta_4$	80
4	3	ζ_4	80
4	4	-1	80
4	21	$-\zeta_4$	-80
4	23	ζ_4	-80
4	24	-1	-80
5	8	0	40
5	16	0	-40
6	10	-1	0
6	10	-1	0
6	10	-1	0
6	10	-1	0
6	12	1	0
10	2	0	160
10	6	0	80
10	14	0	-80
10	18	0	-160
15	4	0	120
15	4	0	120
15	12	0	-120
15	12	0	-120
16	3	$-\zeta_4$	160
16	5	ζ_4	160
16	13	ζ_4	-160
16	15	$-\zeta_4$	-160
20	5	0	80
20	6	0	80
20	7	0	80
20	9	0	-80
20	10	0	-80
20	11	0	-80
24	6	1	0
24	7	ζ_4	0
24	9	$-\zeta_4$	0
30	8	0	0

TABLE 22. Character values for H_4

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	60
1	60	1	-60
4	1	$\zeta_5 + \zeta_5^4$	120
4	7	$\zeta_5^2 + \zeta_5^3$	120
4	31	$\zeta_5 + \zeta_5^4$	-120
4	37	$\zeta_5^2 + \zeta_5^3$	-120
6	12	$-\zeta_5 - \zeta_5^4$	0
6	20	$-\zeta_5^2 - \zeta_5^3$	0
8	12	0	0
8	13	-1	0
9	2	0	180
9	6	0	180
9	22	0	-180
9	26	0	-180
10	12	-1	0
16	3	1	240
16	6	-1	240
16	11	$\zeta_5^2 + \zeta_5^3$	0
16	13	$\zeta_5 + \zeta_5^4$	0
16	18	-1	-240
16	21	1	-240
18	10	0	0
24	6	$-\zeta_5^2 - \zeta_5^3$	0
24	7	1	0
24	11	1	0
24	12	$-\zeta_5 - \zeta_5^4$	0
25	4	0	300
25	16	0	-300
30	10	$\zeta_5 + \zeta_5^4$	0
30	10	$\zeta_5^2 + \zeta_5^3$	0
36	5	0	360
36	15	0	-360
40	8	1	0
48	9	-1	0

TABLE 23. Character values for G_{32}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	80
1	40	ζ_3	-40
1	80	ζ_3^2	-40
4	1	ζ_3	200
4	11	ζ_3^2	200
4	21	1	-40
4	41	ζ_3^2	-160
4	51	1	-40
4	61	ζ_3	-160
5	4	0	160
5	12	0	40
5	20	0	160
5	36	0	40
5	44	0	-200
5	52	0	-200
6	8	ζ_3^2	120
6	28	ζ_3	120
6	48	1	-240

10	2	0	320
10	10	0	320
10	14	0	−40
10	30	0	−280
10	34	0	−40
10	42	0	−280
15	6	0	480
15	8	0	120
15	16	0	120
15	22	0	−240
15	24	0	−240
15	38	0	−240
20	3	0	520
20	5	0	280
20	7	0	280
20	9	0	520
20	9	0	160
20	12	0	160
20	13	0	−80
20	16	0	−80
20	17	0	280
20	19	0	280
20	20	0	−80
20	21	0	160
20	25	0	−440
20	29	0	−80
20	29	0	−440
20	31	0	−440
20	33	0	−560
20	35	0	−440
24	6	−1	480
24	16	$-\zeta_3$	−240
24	26	$-\zeta_3^2$	−240
30	4	0	600
30	8	0	600
30	12	0	−120
30	12	0	240
30	16	0	−120
30	20	0	−120
30	20	0	−480
30	24	0	−120
30	28	0	−480
36	5	$-\zeta_3^2$	720
36	7	$-\zeta_3$	720
36	15	−1	−360
36	17	$-\zeta_3^2$	−360
36	25	$-\zeta_3$	−360
36	27	−1	−360
40	8	0	560
40	10	0	560
40	14	0	−160

40	18	0	−400
40	22	0	−160
40	24	0	−400
45	6	0	360
45	10	0	360
45	14	0	360
45	18	0	360
45	22	0	−720
45	26	0	−720
60	7	0	480
60	9	0	120
60	11	0	480
60	11	0	120
60	12	0	480
60	13	0	120
60	15	0	120
60	15	0	−240
60	16	0	−240
60	17	0	−600
60	19	0	−600
60	20	0	−240
64	8	$-\zeta_3^2$	320
64	11	ζ_3^2	320
64	13	ζ_3	320
64	16	$-\zeta_3$	320
64	18	−1	−640
64	21	1	−640
80	9	0	640
80	13	0	−320
80	17	0	−320
81	10	ζ_3	0
81	12	1	0
81	14	ζ_3^2	0

TABLE 24. Character values for G_{33}

$\deg(\chi)$	$\operatorname{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	45
1	45	−1	−45
5	1	ζ_3^2	135
5	3	ζ_3	135
5	28	$-\zeta_3^2$	−135
5	30	$-\zeta_3$	−135
6	5	0	90
6	20	0	−90
10	8	ζ_3^2	90
10	8	ζ_3	90
10	17	$-\zeta_3^2$	−90
10	17	$-\zeta_3$	−90
15	2	0	315
15	9	0	45
15	12	0	−45
15	23	0	−315

20	6	-1	180
20	15	1	-180
24	4	0	360
24	19	0	-360
30	3	0	450
30	4	0	270
30	6	0	270
30	13	0	-270
30	15	0	-270
30	18	0	-450
40	5	$-\zeta_3^2$	360
40	5	$-\zeta_3$	360
40	14	ζ_3^2	-360
40	14	ζ_3	-360
45	7	0	135
45	9	0	135
45	10	0	-135
45	12	0	-135
60	7	0	180
60	10	0	-180
64	8	1	0
64	9	-1	0
81	6	0	405
81	11	0	-405

TABLE 25. Character values for G_{34}

$\deg(\chi)$	$\mathrm{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	126
1	126	1	-126
6	1	ζ_3^2	504
6	5	ζ_3	504
6	85	ζ_3^2	-504
6	89	ζ_3	-504
15	14	ζ_3	630
15	16	ζ_3^2	630
15	56	ζ_3	-630
15	58	ζ_3^2	-630
20	33	1	0
20	33	1	0
21	2	0	1386
21	4	0	1386
21	6	0	1134
21	60	0	-1134
21	68	0	-1386
21	70	0	-1386
35	6	0	1890
35	18	0	630
35	36	0	-630
35	60	0	-1890
56	3	0	3024
56	9	0	2016
56	45	0	-2016
56	57	0	-3024
70	9	0	2520
70	9	0	2520
70	9	0	2520
70	45	0	-2520
70	45	0	-2520
70	45	0	-2520
84	13	0	2016
84	17	0	2016
84	37	0	-2016
84	41	0	-2016
90	6	-1	3780
90	48	-1	-3780
105	4	0	4410
105	8	0	4410
105	8	0	3150
105	10	0	3150
105	20	0	630
105	22	0	630
105	26	0	-630
105	28	0	-630
105	38	0	-3150
105	40	0	-3150
105	46	0	-4410
105	50	0	-4410
120	5	$-\zeta_3$	5040
120	7	$-\zeta_3^2$	5040
120	21	-1	0
120	21	-1	0
120	47	$-\zeta_3$	-5040
120	49	$-\zeta_3^2$	-5040
126	5	0	4536
126	7	0	4536
126	41	0	-4536
126	43	0	-4536
140	12	0	2520
140	21	0	0
140	30	0	-2520
189	18	0	1134
189	24	0	-1134
210	8	0	6300
210	10	0	6300
210	12	0	3780
210	13	0	2520
210	17	0	2520
210	25	0	-2520
210	29	0	-2520
210	30	0	-3780
210	38	0	-6300
210	40	0	-6300
280	12	0	5040

$\deg(\chi)$	$\operatorname{occ}(\chi)$	$\chi(c)$	$\chi(R)$
280	12	0	5040
280	30	0	−5040
280	30	0	−5040
315	6	0	9450
315	10	0	5670
315	14	0	5670
315	18	0	1890
315	24	0	−1890
315	28	0	−5670
315	32	0	−5670
315	36	0	−9450
336	8	0	8064
336	10	0	8064
336	17	0	2016
336	19	0	2016
336	23	0	−2016
336	25	0	−2016
336	32	0	−8064
336	34	0	−8064
384	8	$-\zeta_3$	8064
384	10	$-\zeta_3^2$	8064
384	11	ζ_3	8064
384	13	ζ_3^2	8064
384	29	ζ_3	−8064
384	31	ζ_3^2	−8064
384	32	$-\zeta_3$	−8064
384	34	$-\zeta_3^2$	−8064
420	7	0	10080
420	11	0	10080
420	12	0	7560
420	14	0	2520
420	16	0	2520
420	20	0	−2520
420	21	0	0
420	22	0	−2520
420	30	0	−7560
420	31	0	−10080
420	35	0	−10080
504	9	0	12096
504	15	0	3024
504	21	0	−3024
504	33	0	−12096
540	17	$-\zeta_3$	0
540	19	$-\zeta_3^2$	0
540	21	−1	0
540	21	−1	0
560	9	0	10080
560	18	0	0
560	18	0	0
560	18	0	0
560	27	0	−10080
630	11	0	7560
630	13	0	7560
630	14	0	3780
630	15	0	7560
630	16	0	3780
630	20	0	−3780
630	22	0	−3780
630	23	0	−7560
630	25	0	−7560
630	27	0	−7560
720	16	$-\zeta_3^2$	0
720	20	$-\zeta_3$	0
729	10	ζ_3^2	10206
729	12	1	10206
729	14	ζ_3	10206
729	24	1	−10206
729	26	ζ_3	−10206
729	28	ζ_3^2	−10206
756	14	0	4536
756	16	0	4536
756	20	0	−4536
756	22	0	−4536
840	11	0	10080
840	13	0	10080
840	13	0	5040
840	17	0	5040
840	19	0	−5040
840	23	0	−5040
840	23	0	−10080
840	25	0	−10080
896	12	0	8064
896	15	0	8064
896	21	0	−8064
896	24	0	−8064
945	14	0	5670
945	16	0	5670
945	20	0	−5670
945	22	0	−5670
1260	17	0	0
1260	19	0	0
1280	15	1	0
1280	18	−1	0

TABLE 26. Character values for $E6$

$\deg(\chi)$	$\operatorname{occ}(\chi)$	$\chi(c)$	$\chi(R)$
1	0	1	36
1	36	1	−36
6	1	−1	144
6	25	−1	−144
10	9	−1	0
15	4	−1	180
15	5	0	180

deg(χ)	occ(χ)	$\chi(c)$	$\chi(R)$
15	16	−1	−180
15	17	0	−180
20	2	0	360
20	10	1	0
20	20	0	−360
24	6	0	144
24	12	0	−144
30	3	1	360
30	15	1	−360
60	5	0	360
60	8	1	0
60	11	0	−360
64	4	0	576
64	13	0	−576
80	7	0	0
81	6	0	324
81	10	0	−324
90	8	−1	0

TABLE 27. Character values for $E7$

deg(χ)	occ(χ)	$\chi(c)$	$\chi(R)$
1	0	1	63
1	63	−1	−63
7	1	−1	315
7	46	1	−315
15	7	0	315
15	28	0	−315
21	3	0	693
21	6	0	567
21	33	0	−567
21	36	0	−693
27	2	0	945
27	37	0	−945
35	4	−1	945
35	13	1	315
35	22	−1	−315
35	31	1	−945
56	3	1	1512
56	30	−1	−1512
70	9	−1	630
70	18	1	−630
84	12	0	252
84	15	0	−252
105	5	0	2205
105	6	0	1575
105	12	0	315
105	15	0	−315
105	21	0	−1575
105	26	0	−2205
120	4	0	2520

120	25	0	−2520
168	6	0	2520
168	21	0	−2520
189	5	0	3213
189	7	0	2457
189	10	0	1323
189	17	0	−1323
189	20	0	−2457
189	22	0	−3213
210	6	0	3150
210	10	0	630
210	13	0	−630
210	21	0	−3150
216	9	0	1512
216	16	0	−1512
280	8	1	2520
280	9	−1	2520
280	17	−1	−2520
280	18	1	−2520
315	7	0	2835
315	16	0	−2835
336	11	0	1008
336	14	0	−1008
378	9	0	1890
378	14	0	−1890
405	8	0	2835
405	15	0	−2835
420	10	0	1260
420	13	0	−1260
512	11	1	0
512	12	−1	0

TABLE 28. Character values for $E8$

deg(χ)	occ(χ)	$\chi(c)$	$\chi(R)$
1	0	1	120
1	120	1	−120
8	1	−1	720
8	91	−1	−720
28	8	0	1680
28	68	0	−1680
35	2	0	2520
35	74	0	−2520
50	8	0	2400
50	56	0	−2400
56	19	1	1680
56	49	1	−1680
70	32	−1	0
84	4	−1	5040
84	64	−1	−5040
112	3	1	6720
112	63	1	−6720

160	7		0	7680	1575	34	0	−37800
160	55		0	−7680	1680	22	1	0
168	24		0	0	2016	19	1	0
175	12		0	4200	2100	16	0	25200
175	36		0	−4200	2100	20	0	0
210	4		0	10080	2100	28	0	−25200
210	52		0	−10080	2240	10	0	40320
300	8		0	10800	2240	28	0	−40320
300	44		0	−10800	2268	10	0	45360
350	14		0	8400	2268	30	0	−45360
350	38		0	−8400	2400	17	0	14400
400	7		0	14400	2400	23	0	−14400
400	43		0	−14400	2688	20	0	0
420	20		−1	0	2800	13	0	33600
448	9		−1	13440	2800	25	0	−33600
448	25		−1	0	2835	14	0	22680
448	39		−1	−13440	2835	22	0	−22680
525	12		0	12600	3150	18	0	0
525	36		0	−12600	3200	16	0	19200
560	5		0	23520	3200	22	0	−19200
560	47		0	−23520	3240	9	0	71280
567	6		0	22680	3240	31	0	−71280
567	46		0	−22680	3360	13	0	40320
700	6		0	25200	3360	25	0	−40320
700	16		0	8400	4096	11	1	61440
700	28		0	−8400	4096	12	−1	61440
700	42		0	−25200	4096	26	−1	−61440
840	13		0	15120	4096	27	1	−61440
840	14		0	10080	4200	12	0	50400
840	26		0	−10080	4200	15	0	25200
840	31		0	−15120	4200	18	0	0
972	12		0	19440	4200	21	0	−25200
972	32		0	−19440	4200	24	0	−50400
1008	9		−1	30240	4480	16	1	0
1008	39		−1	−30240	4536	13	0	45360
1050	10		0	25200	4536	18	1	0
1050	34		0	−25200	4536	23	0	−45360
1134	20		1	0	5600	15	0	33600
1296	13		0	25920	5600	19	0	0
1296	33		0	−25920	5600	21	0	−33600
1344	8		1	40320	5670	18	−1	0
1344	19		0	0	6075	14	0	48600
1344	38		1	−40320	6075	22	0	−48600
1400	7		0	42000	7168	17	−1	0
1400	8		0	33600				
1400	11		0	25200				
1400	20		0	0				
1400	29		0	−25200				
1400	32		0	−33600				
1400	37		0	−42000				
1575	10		0	37800				

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