PARTIAL DUALITY OF HYPERMAPS

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Abstract. We introduce a collection of new operations on hypermaps, partial duality, which include the classical Euler-Poincaré dualities as particular cases. These operations generalize the partial duality for maps, or ribbon graphs, recently discovered in a connection with knot theory. Partial duality is different from previous studied operations of S. Wilson, G. Jones, L. James, and A. Vince.

Combinatorially hypermaps may be described in one of three ways: as three involutions on the set of flags (τ-model), or as three permutations on the set of half-edges (σ-model in orientable case), or as edge 3-colored graphs. We express partial duality in each of these models.

1. Introduction

Maps can be thought of as graphs embedded into surfaces. *Hypermaps* are hypergraphs embedded into surfaces. In other words in hypermaps a (hyper) edge is allowed to connect more then two vertices, so having more than two *half-edges*, or just a single half-edge (see Figure 10).

One way of combinatorially study oriented hypermaps, the σ-model, is to consider permutation of its half-edges, also know as *darts*, around each vertex, around each hyperedge, and around each face according to the orientation. This model has been carefully worked out by R. Cory [Co75], however it can be traced back to L. Heffter [He891]. It became popular after the work of J. R. Edmonds [Ed60]. It is very important for the Grothendieck dessins d'Enfants theory, see [LZ04], where the σ-model is called 3-*constellation*. We review the σ-model in Subsection 2.3.

Another combinatorial description of hypermaps, the τ-model, goes through three involutions acting on the set of *local flags*, also know as *blades*, represented by triples (vertex, edge, face). The motivation for this model was the study of symmetry of regular polyhedra which is a group generated by reflections (involutions). As such it may be traced back to Ancient Greeks. It was used systematically by F. Klein in [Kl884] and later by Coxeter and Moser in [CM80]. More recently this model was used in the context of maps and hypermaps in [Wi, JT83, Ja, JP10]. We review the τ-model in Subsection 2.2.

In 1975 T. Walsh noted [Wal75] that, if we consider a small regular neighborhood of vertices and hyperedges, then we can regard hypermaps as cell decomposition of a compact closed surface into disks of three types, vertices, hyperedges, and faces, such that the disks of the same type do not intersect and the disks of different

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types may intersect only on arcs of their boundaries. These arcs form a 3-regular graph whose edges are colored in 3 colors depending on the types of cells they are adjacent to. The arcs of intersection of hyperedge-disks with face-disks bear the color 0. The color 1 stands for the arcs of intersection of vertex-disks with face-disks. And the arcs of intersection of vertex-disks with hyperedge-disks are colored by 2. Thus we came to the concept of [2]-colored graphs, where [2] stands for the set of three colors $[2] := \{0, 1, 2\}$. It turns out that such a [2]-colored graph carries all the information about the original hypermap. This gives another combinatorial model for description of hypermaps. We review this model in Subsection 2.4.

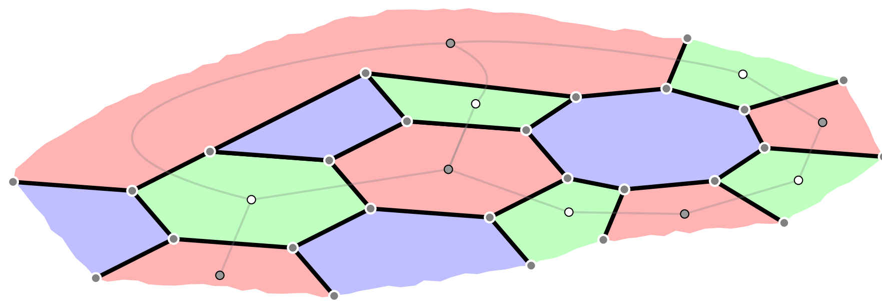

Figure 1. Local view of a hypermap with its Walsh map superimposed. Vertices are red, edges are green.

About the same time this concept was generalized to higher dimensions. Namely, in the 1970's M. Pezzana [Pez74, Pez75] discovered a way of coding a piecewise-linear (PL) manifold by a properly edge-colored graph. The idea goes as follows: choose a triangulation K of this given manifold M. Consider then its first barycentric subdivision K_1. The 1-skeleton of K_1^* is a properly edge-colorable graph. It turns out that the coloring of the graph is sufficient to reconstruct M completely. The discovery of M. Pezzana allows to bring combinatorial and graph theoretical methods into PL topology. This correspondence between PL manifolds and colored graphs has been further developed by M. Ferri, C. Gagliardi and their group [FGG86]. It has also been independently rediscovered by A. Vince [Vin83], S Lins and A. Mandel [LM85], and, to a certain extent, by R. Gurau [Gur11].

Originally the partial duality relative to a subset of edges was defined for ribbon graphs in [Ch09] under the name of generalized duality. The motivation came from an idea to unify various versions of the Thistlethwaite theorems in knot theory relating the Jones polynomial of knots with the Tutte-like polynomial of (ribbon) graphs. Then it was thoughtfully studied and developed in papers [VT09, Mo10, Mo11, EMM12, BBC12, Mo13, HM13]. We refer to [EMM13] for an excellent account on this development.

The main result of this paper is a generalization of partial duality to hypermaps in Section 3. There we define the partial duality in Subsection 3.1 and then describe it in each of the three combinatorial models in subsequent subsections. Independently this generalization was found by Ben Smith [Sm14], but his motivation and the direction of development were completely different. The operation of partial duality usually is different from the operations of [JP10, JT83] and from the operation of [Vin95]. Typically it changes the genus of a hypermap.

We finish the paper with general remarks about future directions of research on partial duality in higher dimensions.

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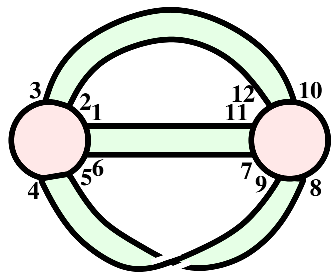
2. Hypermaps

2.1. Geometrical model.

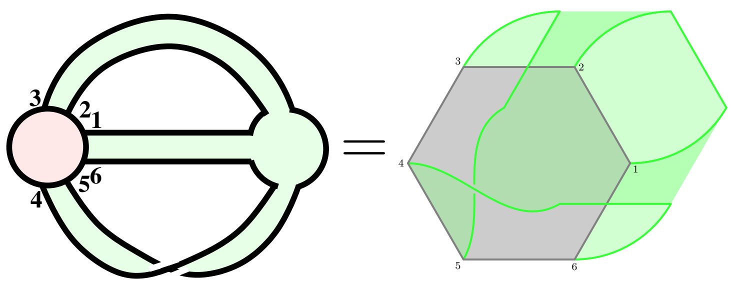
A *map* is a cellularly embedded graph in a (not necessarily orientable) compact closed surface. The edges of a graph are represented by smooth arcs on the surface connecting two (not necessarily distinct) vertices. A small regular neighborhood of such a graph on the surface is a surface with boundary, called *ribbon graph*, equipped with a decomposition into a union of topological disks of two types, the neighborhoods of vertices and the neighborhoods of edges. The last one can be regarded as a narrow quadrilateral along the edges attached to the corresponding vertex discs at the two opposite sides. Attaching disks called *faces* to the boundary components of a ribbon graph restores the original closed surface. Thus a map may be regarded as a cell decomposition of a compact closed surface into disks of three types, vertices, edges, and faces, such that the disks of the same type do not intersect and the disks of different types may intersect only on arcs of their boundaries and the edge-disks intersect with at most two vertex-disks and at most two face-disks.

Hypermaps differ from maps that the edges are allowed to be *hyperedges* and may connect several vertices. Hypermaps may be considered as cellularly embedded hypergraphs where hyperedges are embedded as one dimensional star-shapes. Consequently hypermaps can be defined as cell decomposition of a compact closed surface into disks of three types, vertices, hyperedges, and faces, such that the disks of the same type do not intersect and the disks of different types may intersect only on arcs of their boundaries. The restriction on edge-disks to be quadrilaterals is released here comparable to the definition of a map. So the definition of a hypermap is completely symmetrical with respect to the types of the cells.

Figure 2 shows a non orientable map $\mathfrak{m}_0$ and a hypermap $\mathfrak{hm}_0$ obtained from $\mathfrak{m}_0$ by uniting the right vertex with the three edges into a single hyperedge.



a. A non orientable map $\mathfrak{m}_0$ on a projective plane with two vertices and three edges.



b. A non orientable hypermap $\mathfrak{hm}_0$ on a projective plane with one vertex and one hyperedge.

Figure 2. Maps or hypermaps

2.2. Permutational τ-model.

A hypermap $\mathfrak{hm}$ can be described in a pure combinatorial way as three fixed point free involutions, τ_0, τ_1, and τ_2, acting on a set X of *local flags* of $\mathfrak{hm}$. A (local) flag is a triple (v, e, f) consisting of a vertex v, the intersection e of a hyperedge incident to v with a small neighborhood of v, the intersection f of a face adjacent to v and e with the same neighborhood of v. Another way of defining a local flag is to consider a triangle in the barycentric subdivision of faces of $\mathfrak{hm}$ considered as an embedded hypergraph. We will depict a flag as a small black copy of such a triangle attached to the vertex v. When a hypermap is understood

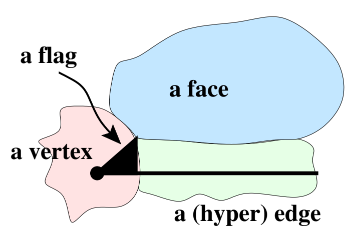


Figure 3. A local flag

as a [2]-colored cell decomposition of a surface, the local flags correspond to the points where all three types of cells meet together. Three lines of cell intersections emanate from each such point, the 2-line of intersection of the vertex-disk with the (hyper) edge-disk, the 1-line of intersection of the vertex-disk with the face-disk, and the 0-line of intersection of the edge-disk with the face-disk. These lines yield three partitions of the set X of local flags into pairs of flags whose corresponding points are connected by 0-, 1-, or 2-lines. The permutation τ_i swaps the flags in the pairs connected by the i-lines.

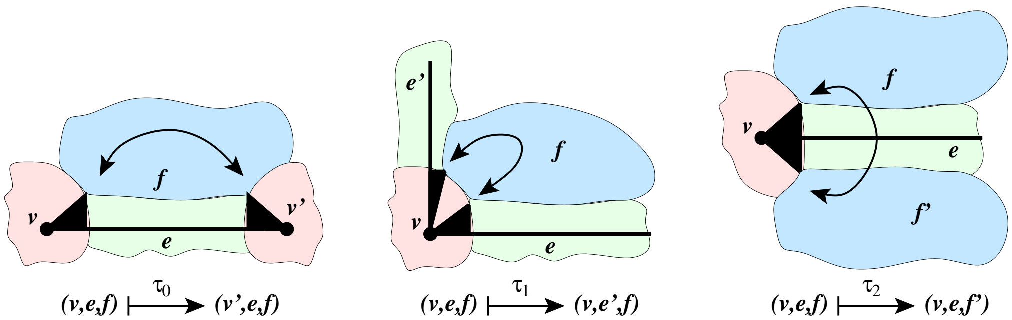


Figure 4. Involutions τ_0, τ_1, τ_2

In Figure 2 the local flags are labeled by numbers. For these hypermaps the permutations τ_i are the following. For the map $\mathfrak{m}_0$, $\tau_0 = (1,11)(2,12)(3,10)(4,9)(5,8)(6,7)$, $\tau_1 = (1,2)(3,4)(5,6)(7,9)(8,10)(11,12)$, $\quad \tau_2 = (1,6)(2,3)(4,5)(7,11)(8,9)(10,12)$. For $\mathfrak{hm}_0$, $\tau_0 = (1,2)(3,5)(4,6)$, $\quad \tau_1 = (1,2)(3,4)(5,6)$, $\quad \tau_2 = (1,6)(2,3)(4,5)$.

Any three fixed-point free involutions on a set X yield a hypermap. Its vertices correspond to orbits of the subgroup $\langle\tau_1,\tau_2\rangle$ generated by τ_1 and τ_2, edges to the orbits of $\langle\tau_0,\tau_2\rangle$, and faces to the orbits of $\langle\tau_0,\tau_1\rangle$. A hyperedge is a genuine edge if the corresponding orbit consists of four elements. Thus a hypermap is a map if and only if $\tau_0\tau_2$ is also an involution.

Remark 2.1. W. Tutte [Tut84] introduced less symmetrical description of combinatorial maps in terms of three permutations θ, ϕ, and P. They can be expressed in terms of τ_0, τ_1, and τ_2 as follows.

$$\theta = \tau_2, \quad \phi = \tau_0, \quad P = \tau_1\tau_2.$$

2.3. **Permutational $\boldsymbol{\sigma}$-model.**

This model gives a presentation of an oriented hypermap in terms of three permutations σ_V, σ_E, and σ_F of its *half-edges* H satisfying the relation $\sigma_F\sigma_E\sigma_V = 1$. A *half-edge* is a small part of a hyperedge near a vertex incident to this hyperedge. We may think of half-edges as non complete local flags (v,e) consisting of a vertex v and the intersection e of a hyperedge incident to v with a small neighborhood of v. So a genuine edge has two half-edges, but a hyperedge may have more than two half-edges or even a single half-edge. If we think of a hyperedge as a one-dimensional star embedded to the surface, then the rays of the star are the half-edges of the hyperedge. We place an empty square at the center of the star in order to distinguish it from a vertex.

The permutation σ_V is a cyclic permutation of half-edges incident to a vertex according to the orientation of the hypermap. The permutation σ_E acts as the cyclic permutation of rays in each star according to the orientation. For the permutation σ_F we need to direct the half-edges with arrows pointing away from the vertices to which they are attached. These arrows point toward the centers of the stars of the hyperedges. The permutation σ_F cyclically permutes those half-edges in each face which are directed along the orientation of the face. One

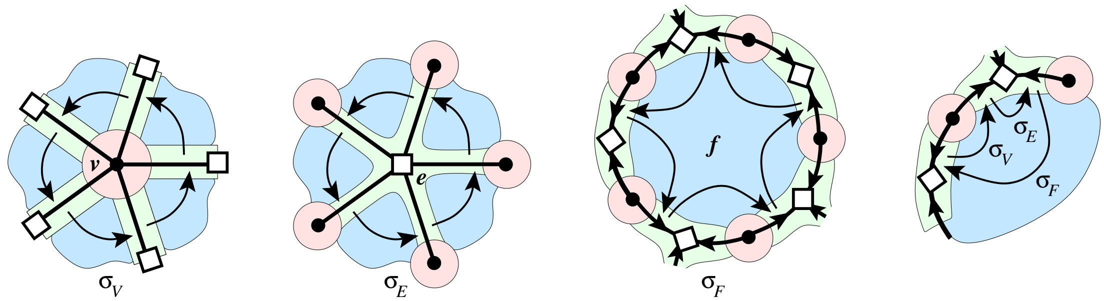


Figure 5. Permutations σ_V, σ_E, σ_F, and the identity $\sigma_F\sigma_E\sigma_V = 1$

can easily check that $\sigma_F\sigma_E\sigma_V = 1$, see Figure 5. The cycles of σ_V correspond to the vertices of the hypermap, the cycles of σ_E correspond to the hyperedges, and the cycles of σ_F correspond to the faces of the hypermap. Consequently any three permutations σ_V, σ_E, and σ_F of a set H satisfying the relation $\sigma_F\sigma_E\sigma_V = 1$ uniquely determines an oriented hypermap.

Now let us describe the relation with the τ-model of Subection 2.2. Each half-edge has two local flags in which it participates, if $x \in X$ is one of them, then $\tau_2(x)$ is the another one. Therefore the cardinality of H is twice smaller that the cardinality of X.

Suppose an oriented hypermap $\mathfrak{hm}$ is given by its σ-model on the set of half-edges $H = \{1, \dots, m\}$. We set X to be a double of H, $X := \{1_-, 1_+, 2_-, 2_+, \dots, m_-, m_+\}$, and the involution τ_2 to swap i_- and i_+. Define the permutation τ_0 to be $\tau_0(i_-) := (\sigma_E(i))_+$ and $\tau_0(i_+) := (\sigma_E^{-1}(i))_-$. Finally, define τ_1 as $\tau_1(i_-) := (\sigma_V^{-1}(i))_+$ and $\tau_1(i_+) := (\sigma_V(i))_-$. Obviously they are involutions and the hypermap they define is $\mathfrak{hm}$.

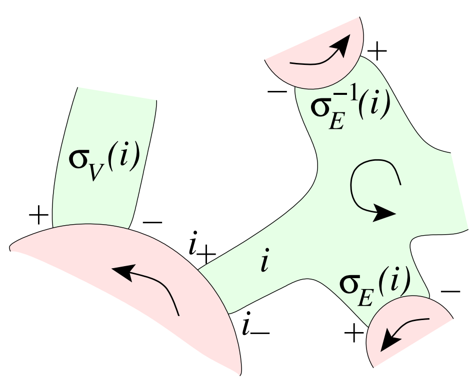
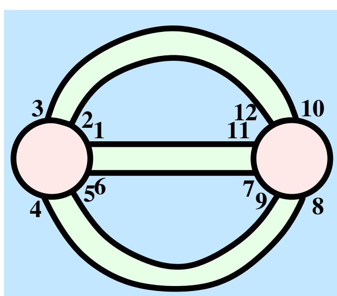


Figure 6. σ and τ permutations

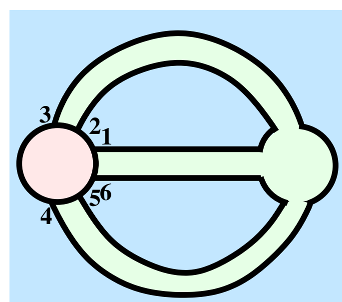
In the opposite way, suppose a hypermap $\mathfrak{hm}$ is given by its τ-model on the set of local flags $X = \{1, \dots, n\}$. Also suppose that $\mathfrak{hm}$ is *connected*. That means the group generated by τ_0, τ_1, and τ_2 acts transitively on X. For orientable hypermap we can consistently arrange X in pairs with subscripts $+$ and $-$ as in Figure 6. For non orientable hypermap such an arrangement is impossible. One may observe that τ's always change the subscript to the opposite one. This means that the subgroup G of words of even length in τ's preserve the subscript. The group G is generated by $\tau_2\tau_1$, $\tau_0\tau_2$, and $\tau_1\tau_0$. For non orientable hypermap, the subgroup G also acts transitively on X. In the orientable case, X splits into two orbits of G, one with the subscript $+$ and another one with the subscript $-$. Let H be the one with subscript $+$. Then we set σ_V (resp. σ_E and σ_F) be the restriction of $\tau_2\tau_1$ (resp. $\tau_0\tau_2$, and $\tau_1\tau_0$) on the orbit H. Obviously these restrictions satisfy the relation $\sigma_F\sigma_E\sigma_V = (\tau_1\tau_0)(\tau_0\tau_2)(\tau_2\tau_1) = 1$. It is clear from Figure 6 that the σ-model constructed in this way give the original orientable hypermap $\mathfrak{hm}$. The restrictions to the "$-$"-orbit gives the same hypermap with the opposite orientation.

Example 2.2. For hypermaps on Figure 2 the subgroup G is generated by the following permutations. For $\mathfrak{m}_0$, $\tau_2\tau_1 = (1,3,5)(2,6,4)(7,8,12)(9,11,10)$, $\tau_0\tau_2 = (1,7)(2,10)(3,12)(4,8)(5,9)(6,11)$, $\tau_1\tau_0 = (1,12)(2,11)(3,8,6,9)(4,7,5,10)$. For $\mathfrak{hm}_0$, $\tau_2\tau_1 = (1,3,5)(2,6,4)$, $\tau_0\tau_2 = (1,4,3)(2,5,6)$, $\tau_1\tau_0 = (1,2)(3,6)(4,510)$. In both cases the group G acts transitively on flags. This is a combinatorial expression of the fact that these two hypermaps are non orientable.

On the contrary, consider the two oriented hypermaps of Figure 7. The permuta-



a. An orientable map $\mathfrak{m}_1$ on a sphere with two vertices and three edges.



b. An orientable hypermap $\mathfrak{hm}_1$ on a sphere with one vertex and one hyperedge.

Figure 7. Maps or hypermaps, second example.

tions τ_i are the following. For the map $\mathfrak{m}_1$, $\tau_0 = (1,11)(2,12)(3,10)(4,8)(5,9)(6,7)$, $\tau_1 = (1,2)(3,4)(5,6)(7,9)(8,10)(11,12)$, $\tau_2 = (1,6)(2,3)(4,5)(7,11)(8,9)(10,12)$. For $\mathfrak{hm}_1$, $\tau_0 = (1,2)(3,4)(5,6)$, $\tau_1 = (1,2)(3,4)(5,6)$, $\tau_2 = (1,6)(2,3)(4,5)$.

The generators of the subgroup G for $\mathfrak{m}_1$ are: $\tau_2\tau_1 = (1,3,5)(2,6,4)(7,8,12)(9,11,10)$, $\tau_0\tau_2 = (1,7)(2,10)(3,12)(4,9)(5,8)(6,11)$, $\tau_1\tau_0 = (1,12)(2,11)(3,8)(4,10)(5,7)(6,9)$.

For $\mathfrak{hm}_1$: $\tau_2\tau_1 = (1,3,5)(2,6,4)$, $\tau_0\tau_2 = (1,5,3)(2,4,6)$, $\tau_1\tau_0 = 1$. One can see that the group G has two orbits on the set of flags. The "$+$"-orbits are: for $\mathfrak{m}_1$, $H = \{1,3,5,7,8,12\}$; for $\mathfrak{hm}_1$, $H = \{1,3,5\}$. The restriction of the generators on this orbit gives the σ-models.

For $\mathfrak{m}_1$: $\sigma_V = (1,3,5)(7,8,12), \sigma_E = (1,7)(3,12)(5,8), \sigma_F = (1,12)(3,8)(5,7)$.
For $\mathfrak{hm}_1$: $\sigma_V = (1,3,5),\ \sigma_E = (1,5,3),\ \sigma_F = 1$.

There is an elegant formula for the Euler characteristic of a hypermap in terms of its σ-model.

Lemma 2.3. [LZ04, Proposition 1.5.3] *Let $\mathfrak{hm} = (\sigma_V, \sigma_E, \sigma_F)$ be an oriented hypermap given by its σ model on the set X of n half-edges, $n := \#X$. Let c_V (resp. c_E and c_F) denote the number of cycles of σ_V (resp. σ_E and σ_F). Then the Euler characteristic $\chi(\mathfrak{hm}) = 2 - 2g$ (g is the genus) of the surface of $\mathfrak{hm}$ is computable by*

$$\chi(\mathfrak{hm}) = c_V + c_E + c_F - n\ .$$

Proof. Let $\mathcal{T}$ be the cell decomposition (tesselation) given by the hypermap $\mathfrak{hm}$. We first triangulate $\mathcal{T}$ by stellar subdividing it, see Figure 8. The result is easily seen to be a simplicial complex $\mathcal{S}$. Clearly $\chi(\mathfrak{hm}) = \chi(\mathcal{T}) = \chi(\mathcal{S})$.

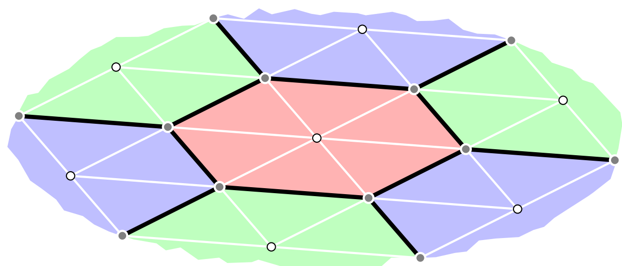


Figure 8. Local view of the stellar subdivision of a hypermap.

Note that

- $2n$ is the number of vertices of $\mathcal{T}$,
- the number of polygons in $\mathcal{T}$ is $c_V + c_E + c_F$,
- the number of edges of $\mathcal{T}$ is $3n$,
- the number of faces (triangles) of $\mathcal{S}$ is twice the number of edges of $\mathcal{T}$.

Then the numbers of vertices v, of edges e and of faces f of $\mathcal{S}$ are given by:

$$\mathsf{v} = 2n + c_V + c_E + c_F, \quad \mathsf{e} = 3n + 6n, \quad \mathsf{f} = 6n.$$

And $\chi = \mathsf{v} - \mathsf{e} + \mathsf{f}$ gives the claimed formula. □

2.4. Edge Colored Graphs.

As indicated in Subsection 2.2 the boundaries of cells of a hypermap form a 3-regular graph embedded into the surface of the hypermap. It carries a natural edge coloring: the arcs of intersection of hyperedges and faces are colored by 0, the arcs of intersection of vertices and faces are colored by 1, and the arcs of intersection of vertices and hyperedges are colored by 2. In this subsection we show that the entire hypermap can be reconstructed from this information.

Definition 2.4. Let κ be a finite set. A *κ-colored graph* is a connected graph such that each edge carries a “color” in κ and each vertex is incident to exactly one edge of each color.

Note that a κ-colored graph is necessarily $\#\kappa$-regular and has no loops (but may contain multiple edges). In the following, for all $I \subset \kappa$, we will denote $\kappa \setminus I$ by $\overline{I}$.

Let $\kappa = \{1, 2, \ldots, \#\kappa\}$. For a κ-colored graph Γ we can define a permutational τ-model as a set of involutions $\tau_1, \tau_2, \ldots, \tau_{\#\kappa}$ acting on the set X of vertices of Γ as follows. τ_i interchange the vertices connected by an edge of color i. For the colored graphs coming from hypermaps these permutations coincide with the τ-model from Subsection 2.2.

Each colored graph Γ contains some special colored subgraphs called *bubbles* in [Gur11] and *residues* in [Vin83].

Definition 2.5. Let Γ be a κ-colored graph and $I \subset \kappa$. An *$\#I$-bubble* of colors I in Γ is a connected component of the I-colored subgraph of Γ induced by the edges of Γ with colors in I.

In particular 0-bubbles, corresponding to $I = \emptyset$, are the vertices of Γ.

The set of bubbles in Γ of colors $I \subset \kappa$ is denoted by $\mathcal{B}^I(\Gamma)$ or $\mathcal{B}^I$ if there is no ambiguity. B^I is its cardinality $\#\mathcal{B}^I$. We also define $\mathcal{B}_n(\Gamma)$, $0 \leqslant n \leqslant \#\kappa - 1$, to be the set of all n-bubbles in Γ: $\mathcal{B}_n \coloneqq \bigcup_{I \subset \kappa, \#I = n} \mathcal{B}^I$; $B_n \coloneqq \#\mathcal{B}_n$. Finally the whole set of bubbles of Γ, $\bigcup_{0 \leqslant n \leqslant \#\kappa - 1} \mathcal{B}_n$, is written as $\mathcal{B}(\Gamma)$. The subgraph inclusion relation provides $\mathcal{B}(\Gamma)$ with a poset structure.

2.4.1. ***Topology of edge colored graphs.*** To each colored graph Γ, one can associate two cell complexes, $\boldsymbol{\Delta^*(\Gamma)}$ and its dual $\boldsymbol{\Delta(\Gamma)}$, as follows.

For each $D \in \mathbb{N}$, let $[D]$ be the set $\{0, 1, \ldots, D\}$.

The dual complex $\boldsymbol{\Delta^*(\Gamma)}$. Let Γ be a $[D]$-colored graph. To each D-bubble $b \in \mathcal{B}^{\bar{i}}(\Gamma)$, one associates a 0-simplex $\mathfrak{s}(b)$ colored i. To each $(D-1)$-bubble $b \in \mathcal{B}^{\overline{\{i,j\}}}$, one associates an edge $\mathfrak{s}(b)$ the endpoints of which are respectively colored i and j. In general, to each k-bubble $b \in \mathcal{B}^{\overline{\{i_1, \cdots, i_k\}}}$, one associates an abstract $(D-k)$-simplex $\mathfrak{s}(b)$ colored $[D] \setminus \{i_1, \cdots, i_k\}$. Now, the poset structure of $\mathcal{B}(\Gamma)$ provides gluing data for those simplices. Indeed, let us consider two $(D-k)$-simplices $\mathfrak{s}(b)$ and $\mathfrak{s}(b')$. If the corresponding k-bubbles b and b' are contained in a common $(k+1)$-bubble b'', identify $\mathfrak{s}(b)$ and $\mathfrak{s}(b')$ along their common facet $\mathfrak{s}(b'')$. This gluing respects the coloring structure of the simplices. It can be shown that such a complex is a trisp (for triangulated space) [Koz08]. A. Vince [Vin83, p.4] called the topological space of this simplicial complex the *underlying topological space of the combinatorial map* Γ.

But there is also another complex associated to Γ, dual to Δ^*.

The direct complex $\boldsymbol{\Delta(\Gamma)}$. It is constructed inductively, like a CW complex. To each k-bubble, $0 \leqslant k \leqslant D$, one will associate a k-dimensional topological space. To each 0-bubble b, i.e. to each vertex of Γ, corresponds a point $|b|$. Each edge e of Γ, i.e. each 1-bubble, contains two vertices u and v. Define $|e|$ as the cone over $|u| \times |v|$. The realization $|e|$ of e is thus a segment. Now consider a 2-bubble b. It is a bicolored cycle in Γ. b contains a bunch of edges whose realization form a circle. $|b|$ is defined as a cone over this circle hence a 2-disk. In general, let b

be k-bubble. It contains a set $\mathcal{B}_{k-1}(b)$ of $(k-1)$-bubbles. The realization of any $b' \in \mathcal{B}_{k-1}(b)$ has been defined at the previous induction step. The realizations $|b_1|$ and $|b_2|$ for $b_1, b_2 \in \mathcal{B}_{k-1}$ are identified along $|b_1 \cap b_2|$ (which is a union of lower dimensional bubbles). Then the whole set $\mathcal{B}_{k-1}(b)$ has a (connected) realization that we note $\partial|b|$. Finally $|b|$ is defined as the cone over $\partial|b|$ (hence the name).

The realization $|\Gamma|$ of Γ corresponds to the gluing of the D-cells of $\Delta(\Gamma)$. $\Delta(\Gamma)$ is a complex whose cells are topological spaces glued along their common boundaries. But in general, its cells are not homeomorphic to balls. And indeed $|\Gamma|$ is generally not a manifold but a normal pseudo-manifold [Gur10].

2.4.2. ***Hypermaps as edge-colored graphs.*** It was mentioned at the beginning of this subsection that a hypermap $\mathfrak{hm}$ determines a [2]-colored graph $\Gamma_{\mathfrak{hm}}$. Its vertices corresponds to (local) flags of $\mathfrak{hm}$ and its edges of color i correspond to the orbits of the involution τ_i.

Here is an inverse construction. Let us consider a [2]-colored graph Γ. The 2-cells of its direct complex $\Delta(\Gamma)$ are polygons and $|\Gamma|$ is thus the result of the gluing of polygons along common boundaries. Whereas not true in general, the gluing of polygons, *dictated by a colored graph*, is always a manifold and thus a closed compact (not necessarily orientable) surface. Moreover those polygons are of three types: they are bounded by either 01-, 02- or 12-cycles (2-bubbles). Said differently, $\Delta(\Gamma)$ is, in dimension 2, a polygonal tessellation of a closed compact (not necessarily orientable) surface with polygons of three different types, i.e. a hypermap $\mathfrak{hm}$. Thus [2]-colored graphs provide another description of hypermaps.

Example 2.6. Here are the examples of [2]-colored graph for hypermaps $\mathfrak{hm}_0$ from Figure 2 and $\mathfrak{hm}_1$ from Figure 7. A reader may enjoy constructing the direct

Figure 9. [2]-colored graphs $\Gamma_{\mathfrak{hm}_0}$ and $\Gamma_{\mathfrak{hm}_1}$

complexes $\Delta(\Gamma_{\mathfrak{hm}_0})$ and $\Delta(\Gamma_{\mathfrak{hm}_1})$, and checking that they are indeed isomorphic to the hypermaps from Figures 2 and 7.

Lemma 2.7. *A hypermap corresponding to a* [2]*-colored graph* Γ *is orientable if and only if* Γ *is bipartite.*

Proof. According to Subsection 2.3 a hypermap is orientable if and only if the vertices of Γ can be split into two parts with subscripts + and − as on Figure 6. □

Remark 2.8. In [Vin83] A. Vince proposed a way to associate a $[D]$-colored graph Γ to any cell decomposition K of a closed D-manifold. Γ is defined as the 1-skeleton of the complex dual to the first barycentric subdivision of K. Whereas his method works for any cell complex associated to closed manifolds, it does not define a one-to-one correspondence between hypermaps and [2]-colored graphs (not all colored graphs have a dual complex which is the barycentric subdivision of another cell complex). Moreover the colored graph thus associated to K is of higher order than ours.

3. Partial duality

3.1. **Definition.**

Assume that a hypermap $\mathfrak{hm}$ is connected. Otherwise we will need to do partial duality for each connected component separately and then take the disjoint union. Let S be a subset of cells of $\mathfrak{hm}$ of the same type, either vertex-cells, or hyperedge-cells, or face-cells. We will define the *partial dual hypermap* $\mathfrak{hm}^S$ relative to S. If S is the set of all cells of the given type, the partial duality relative to S is the total duality which swaps the two types of the remaining cells without changing the cells themselves and reverses the orientation in an oriented case.

For example, if $\mathfrak{hm}$ is a graph cellularly embedded into a surface then the total duality relative to the whole set of edges is a classical duality of graphs on surfaces which interchanges vertices and faces. Since the concept of hypermap is completely symmetrical we can make the total duality relative to the set of vertices for example. Then the edges and faces will be interchanged. The hypermap $\mathfrak{hm}_1$ from Figure 7 has one vertex one hyperedge and 3 faces. So we have three total duals relative to the vertex, relative to the hyperedge, and relative to all three faces, which differ only by the color of the corresponding cells. On Figure 10 the three duals are shown as cell decomposed spheres together with the corresponding embeddings of the hypergraphs; the hyperedges are embedded as one-dimensional stars when needed. The left picture represents $\mathfrak{hm}_1^{\{v\}}$ and has 3 hyperedges, each

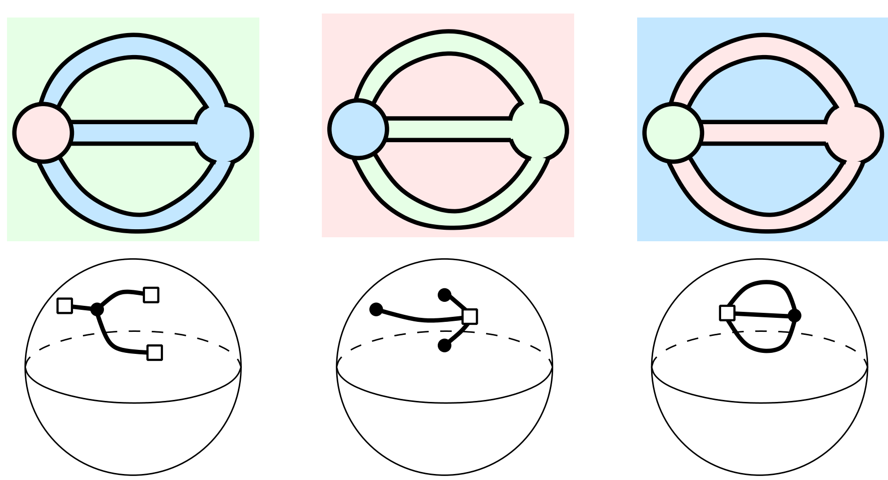


Figure 10. Total duals of the hypermap $\mathfrak{hm}_1$ from Figure 7

consisting of a single half-edge. The middle picture represents $\mathfrak{hm}_1^{\{e\}}$. The right picture is $\mathfrak{hm}_1^{\{f_1,f_2.f_3\}}$. It is isomorphic to the original hypermap $\mathfrak{hm}_1$.

Definition 3.1. Without loss of generality we may assume that S is a subset of the set vertex-cells. Choose a different type of cells, say hyperedges. Later in

Lemma 3.3 we show that the resulting hypermap does not depend on this choice; we could choose faces instead of hyperegdes if we want to.

Step 1. Consider the boundary ∂F of the surface F which is the union of the cells from S and all cells of the chosen type, hyperedges in our case.

Step 2. Glue a disk to each connected component of ∂F. These will be the *hyperedge-cells* for $\mathfrak{hm}^S$. Not that we do not include the interior of F into the hyperedges. Although if ∂F has only one component, gluing a disk to it results in the surface F itself, and then we may consider F as the single hyperedge of $\mathfrak{hm}^S$. See Figure 11.

Step 3. Take a copy of every vertex. These disks will be the *vertex-cells* for $\mathfrak{hm}^S$. Their attachment to the hyperedges is as follows. Every vertex disk of the original hypermap $\mathfrak{hm}$ contributes one or several intervals to ∂F. Indeed, if a vertex belong to S, then it contributes to F itself and a part of its boundary contributes to ∂F. If a vertex is not in S, then it has some hyperededges attached to it because $\mathfrak{hm}$ is assumed to be connected. So such a vertex-disk has a common boundary intervals with F and therefore contributes these intervals to ∂F. The new copies of the vertex-disks, as vertices of $\mathfrak{hm}^S$, are attached to hyperedges exactly along the same intervals as the old ones. See Figure 12.

Step 4. At the previous steps we constructed the vertex and hyperedge cells for the partial dual $\mathfrak{hm}^S$. Their union forms a surface with boundary. Glue a disk to each of its boundary components. These are going to be the *faces* of $\mathfrak{hm}^S$. See Figure 13.

This finishes the construction of the partial dual hypermap $\mathfrak{hm}^S$.

Example 3.2. We examplify the construction of the partial dual $\mathfrak{m}_1^{\{v\}}$ for the map $\mathfrak{m}_1$ from Figure 7 relative to its left vertex v.

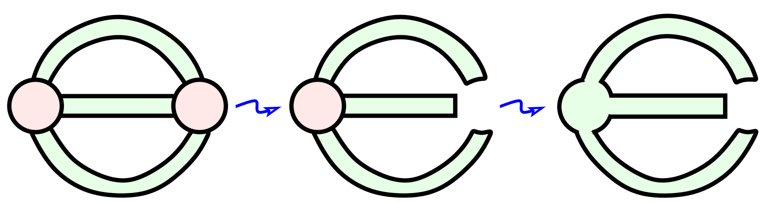


Figure 11. Steps 1 & 2: forming hyperedges of $\mathfrak{m}_1^{\{v\}}$

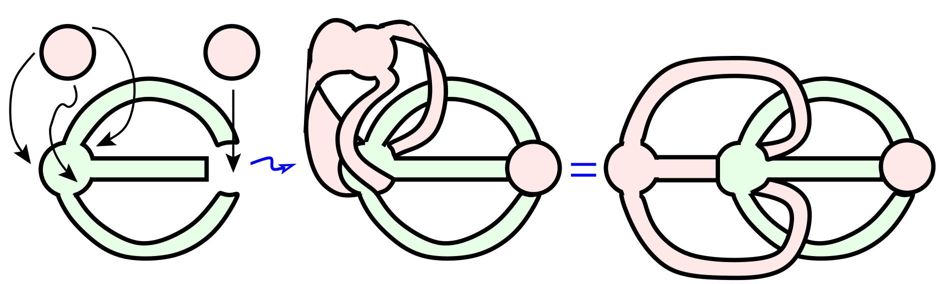


Figure 12. Step 3: copying vertices and gluing them to hyperedges

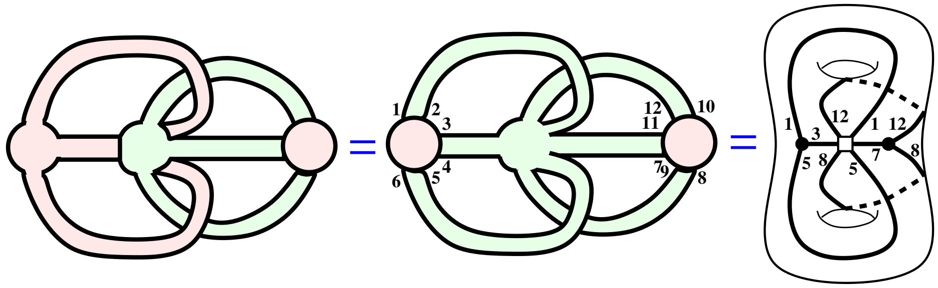


Figure 13. Step 4: gluing faces and the resulting hypermap $\mathfrak{m}_1^{\{v\}}$.

Similarly one may find the partial dual $\mathfrak{m}_0^{\{v\}}$ for a non orientable map $\mathfrak{m}_0$ from Figure 2. The resulting surface after the step 3 will be similar to the one above, only one half-edge will be twisted. It still has one boundary component, and therefore a single face. So its Euler characteristic is still -2, only now the resulting hypermap will be non orientable. It represents a surface homeomorphic to a connected sum of 4 copies of the projective plane.

Lemma 3.3. *The resulting hypermap does not depend on the choice of type at the beginning of Definition 3.1.*

Proof. Decompose the boundary circles of faces on a hypermap $\mathfrak{hm}$ into the union of three sets of arcs intersecting only at the end points of the arcs, $D_0(\mathfrak{hm}) \cup D_{1,S}(\mathfrak{hm}) \cup D_{1.\not S}(\mathfrak{hm})$. The set $D_0(\mathfrak{hm})$ consist of arc of intersection of faces with hyperedges, $D_{1,S}(\mathfrak{hm})$ — of faces with vertices from the set S, and $D_{1.\not S}(\mathfrak{hm})$ — of faces with vertices not from S. Analyzing the result of Step 3 of the construction one can easily note that $D_0(\mathfrak{hm}) = D_0(\mathfrak{hm}^S)$ and $D_{1.\not S}(\mathfrak{hm}) = D_{1.\not S}(\mathfrak{hm}^S)$. Moreover, $D_{1,S}(\mathfrak{hm}^S)$ consists of the complementary arcs of the boundary circles of vertices from S to the arcs $D_{1,S}(\mathfrak{hm})$; formally the complementary arcs on the second copies of the vertices of S. This means that the boundary circles of faces of $\mathfrak{hm}^S$ are exactly the boundary circles of the surface obtained by the union of vertices of S and the all the faces. In other word on Step 1 we may take faces instead of hyperedges and we will get the same boundary circles as for $\mathfrak{hm}^S$. Then, by symmetry the hyperedges will also be the same. □

Analogously to [Ch09, Sec.1.8] the following lemma describes simple properties of the partial duality for hypermaps. Its proof is obvious.

Lemma 3.4.

(a) $\left(\mathfrak{hm}^S\right)^S = \mathfrak{hm}$.

(b) *There is a bijection between the cells of type S in $\mathfrak{hm}$ and the cells of the same type in $\mathfrak{hm}^S$. This bijection preserves the valency of cells. The number of cells of other types may change.*

(c) *If $s \notin S$ but has the same type as the cells of S, then $\mathfrak{hm}^{S\cup\{s\}} = \left(\mathfrak{hm}^S\right)^{\{s\}}$.*

(d) $\left(\mathfrak{hm}^{S'}\right)^{S''} = \mathfrak{hm}^{\Delta(S',S'')}$, *where $\Delta(S',S'') := (S' \cup S'') \setminus (S' \cap S'')$ is the symmetric difference of sets.*

(e) *Partial duality preserves orientability of hypermaps.*

3.2. Partial duality in $\boldsymbol{\sigma}$-model.

For an oriented hypermap $\mathfrak{hm}$ represented in the σ-model of Subection 2.3 we shell write

$$\mathfrak{hm} = (\sigma_V, \sigma_E, \sigma_F)\ .$$

Theorem 3.5. *Let S be a subsets $S := V'$ of vertices (resp. subset of hyperedges $S := E'$ and subset of faces $S := F'$) of a hypermap $\mathfrak{hm}$. Then its partial dual is given by the permutations*

$$\begin{aligned}
\mathfrak{hm}^{V'} &= (\sigma_{\overline{V'}}\sigma_{V'}^{-1},\ \sigma_E\sigma_{V'},\ \sigma_{V'}\sigma_F) \\
\mathfrak{hm}^{E'} &= (\sigma_{E'}\sigma_V,\ \sigma_{\overline{E'}}\sigma_{E'}^{-1},\ \sigma_F\sigma_{E'}) \\
\mathfrak{hm}^{F'} &= (\sigma_V\sigma_{F'},\ \sigma_{F'}\sigma_E,\ \sigma_{\overline{F'}}\sigma_{F'}^{-1})\ ,
\end{aligned}$$

where $\sigma_{V'}$, $\sigma_{E'}$, $\sigma_{F'}$ denote the permutations consisting of the cycles corresponding to the elements of V', E', F' respectively, and overline means the complementary set of cycles.

Proof. Because of the symmetry it is sufficient to prove the theorem only for the case $S = V'$.

From Definition 3.1 it follows that the number of half-edges is preserved by the partial duality. We need to make a bijection between half-edges of $\mathfrak{hm}$ and those of $\mathfrak{hm}^S$ such that the corresponding permutations are related as in the first equation of the theorem.

Half-edges are attached to vertices. If a vertex does not belong to S then the attachment of half-edges to it does not change with partial duality (Step 2). So for those half-edges the required bijection is the identity.

Half-edges attached to vertices of S change, so we need to indicate the bijection for them. Consider a vertex-disk from the set S of the original hypermap $\mathfrak{hm}$. It can be represented as a $2k$-gon because the arcs of its boundary circle intersecting with hyperedges and faces alternate. In $\mathfrak{hm}$ it has k half-edges attached along every other side. We call them *old half-edges*. These half-edges together with the vertex-disk form a piece of the surface F on Step 1 near the vertex. The orientation of F induces an orientation on its boundary ∂F. In the partial dual $\mathfrak{hm}^S$ the hyperedges, the *new hyperedges*, are attached to every connected component of ∂F (Step 2). The orientation of ∂F induces the orientation on new hyperedges. They are attached to a new vertex (Step 3) along the other sides of the $2k$-gon, which form *new half-edges*. Set the label of a new half-edge to be the same as the label of the old one preceding the new half-edge in the direction of the orientation of the old vertex. This gives the bijection of half-edges around vertices of S. The orientation on the new vertex, as well as on the entire hypermap $\mathfrak{hm}^S$, is induced from the new hyperedges. Figure 14 shows that the labels of the new half-edges appear around new vertices in the order opposite to the one around old ones. This means that the cycle in the permutation σ_V corresponding to a vertex in S of the initial hypermap $\mathfrak{hm}$ should be inverted to get the cycle for $\mathfrak{hm}^S$. This proves that the first term of the first formula of the theorem $\sigma_V(\mathfrak{hm}^S) = \sigma_{V'}^{-1}(\mathfrak{hm})\sigma_{\overline{V'}}(\mathfrak{hm})$.

For the second term we need to analyze the cyclic order of new half-edges around new hyperedge according to its orientation. It may be read off from labels of the

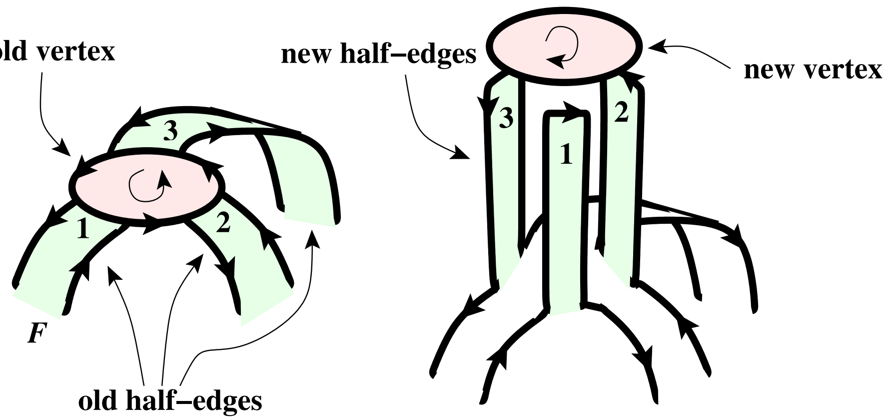


Figure 14. Permutation $\sigma_V(\mathfrak{hm}^S)$

half-edges met when traveling along the boundary of the hyperedge in the direction of its orientation. Such a boundary for $\mathfrak{hm}^S$ is exactly a connected component of ∂F with the orientation induced from $\mathfrak{hm}$. The last is given precisely by the product of permutations $\sigma_E(\mathfrak{hm})\sigma_{V'}(\mathfrak{hm})$. Indeed, consider Figure 15 and suppose that $\sigma_E(\mathfrak{hm}) : 2 \mapsto i$ for some i. Then the new half-edge labels appear at ∂F in the order $\ldots, 1, i, \ldots$. So $\sigma_E(\mathfrak{hm}^S) : 1 \mapsto i$, which is equal to $\sigma_E(\mathfrak{hm})\sigma_{V'}(\mathfrak{hm})$:

$$1 \xmapsto{\sigma_{V'}(\mathfrak{hm})} 2 \xmapsto{\sigma_E(\mathfrak{hm})} i \quad .$$

This proves the second term.

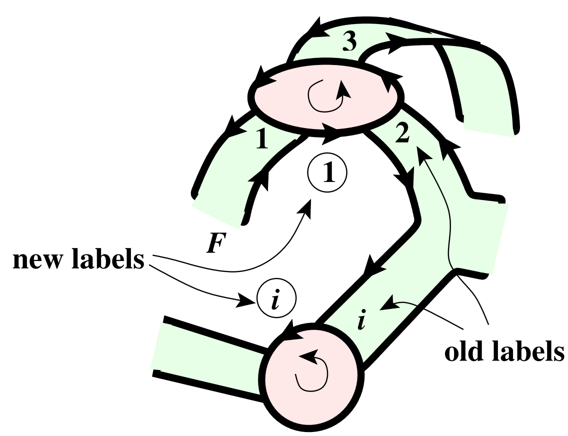


Figure 15. Permutation $\sigma_E(\mathfrak{hm}^S)$

The third term follows from the relation $\sigma_F\sigma_E\sigma_V = 1$. □

Example 3.6. This is a continuation of Example 2.2. We found that for the map $\mathfrak{m}_1$ on Figure 7 the permutations σ's act on the set of half-edges $H = \{1, 3, 5, 7, 8, 12\}$ as

$$\sigma_V = (1, 3, 5)(7, 8, 12), \sigma_E = (1, 7)(3, 12)(5, 8), \sigma_F = (1, 12)(3, 8)(5, 7).$$

The cycle $(1, 3, 5)$ of σ_V corresponds to the the left vertex v. For the σ-model of the partial dual $\mathfrak{m}_1^{\{v\}}$, we set $V' = \{v\}$. Then $\sigma_{V'} = (1, 3, 5)$ and $\sigma_{\overline{V'}} = (7, 8, 12)$. According to the theorem

$$\begin{aligned}
\sigma_V(\mathfrak{m}_1^{\{v\}}) &= \sigma_{\overline{V'}}\sigma_{V'}^{-1} = (1,5,3)(7,8,12)\ , \\
\sigma_E(\mathfrak{m}_1^{\{v\}}) &= \sigma_E\sigma_{V'} = (1,7)(3,12)(5,8)(1,3,5) = (1,12,3,8,5,7)\ , \\
\sigma_F(\mathfrak{m}_1^{\{v\}}) &= \sigma_{V'}\sigma_F = (1,3,5)(1,12)(3,8)(5,7) = (1,12,3,8,5,7)\ .
\end{aligned}$$

One may check that these permutations agree with the last picture on Figure 13.

Corollary 3.7. *The total duality with respect to $S \coloneqq V$ (resp. $S \coloneqq E$ and $S \coloneqq F$) is reduced to the classical Euler-Poincaré duality which swaps the names of two remaining types of cells and reverse the orientation.*

In σ-model it is given by the formulae

$$\begin{aligned}
\mathfrak{hm}^V &= (\sigma_V^{-1},\ \sigma_E\sigma_V,\ \sigma_V\sigma_F) = (\sigma_V^{-1},\ \sigma_F^{-1},\ \sigma_E^{-1}) \\
\mathfrak{hm}^E &= (\sigma_E\sigma_V,\ \sigma_E^{-1},\ \sigma_F\sigma_E) = (\sigma_F^{-1},\ \sigma_E^{-1},\ \sigma_V^{-1}) \\
\mathfrak{hm}^F &= (\sigma_V\sigma_F,\ \sigma_F\sigma_E,\ \sigma_F^{-1}) = (\sigma_E^{-1},\ \sigma_V^{-1},\ \sigma_F^{-1})\ .
\end{aligned}$$

The inverse of these permutations are responsible for the change of orientation of the hypermap.

3.3. Partial duality in $\boldsymbol{\tau}$-model.

Theorem 3.8. *Consider the τ-model of a hypermap $\mathfrak{hm}$ given by the permutations $\tau_0(\mathfrak{hm}) : (v, e, f) \mapsto (v', e, f)$, $\tau_1(\mathfrak{hm}) : (v, e, f) \mapsto (v, e', f)$, $\tau_2(\mathfrak{hm}) : (v, e, f) \mapsto (v, e, f')$ of its local flags. Let V' be a subset of its vertices, $\tau_1^{V'}$ be the product of all transpositions in τ_1 for $v \in V'$, and $\tau_2^{V'}$ be the product of all transpositions in τ_2 for $v \in V'$. Then its partial dual $\mathfrak{hm}^{V'}$ is given by the permutations*

$$\tau_0(\mathfrak{hm}^{V'}) = \tau_0, \qquad \tau_1(\mathfrak{hm}^{V'}) = \tau_1\tau_1^{V'}\tau_2^{V'}, \qquad \tau_2(\mathfrak{hm}^{V'}) = \tau_1\tau_1^{V'}\tau_2^{V'}\ .$$

In other words the permutations τ_1 and τ_2 swap their transpositions of local flags around the vertices in V'. Similar statements hold for partial dualities relative to the subsets of hyperedges E' and of faces F'.

Proof. From the Definition 3.1 one may see that if a vertex does not participate in the partial duality, $v \notin V'$, then nothing changes with local flags around it. But if $v \in V'$, then the roles of edges and faces in its local flags are interchanged. This may be seen on Step 3 and also on Figures 14 and 15 when the second copy of the vertex is attached to the new hyperedges. So, if two such local flags were transposed by τ_1 of the original hypermap, then they will be transposed by τ_2 of the partial dual and vise versa. □

Example 3.9. In Example 2.2, we found the τ-model for the map $\mathfrak{m}_1$ of Figure 7: $\tau_0 = (1, 11)(2, 12)(3, 10)(4, 8)(5, 9)(6, 7)$, $\tau_1 = (1, 2)(3, 4)(5, 6)(7, 9)(8, 10)(11, 12)$, $\tau_2 = (1, 6)(2, 3)(4, 5)(7, 11)(8, 9)(10, 12)$.

There are six flags around the left vertex v labeled by $1, \ldots, 6$. The corresponding transpositions around this vertex are

$$\tau_1^{\{v\}} = (1,2)(3,4)(5,6), \qquad \tau_2^{\{v\}} = (1,6)(2,3)(4,5)\ .$$

Swapping them between τ_1 and τ_2 we get the τ-model of the partial dual $\tau_0(\mathfrak{hm}^{\{v\}}) = (1,11)(2,12)(3,10)(4,8)(5,9)(6,7)$, $\tau_1(\mathfrak{hm}^{\{v\}}) = (1,6)(2,3)(4,5)(7,9)(8,10)(11,12)$, $\tau_2(\mathfrak{hm}^{\{v\}}) = (1,2)(3,4)(5,6)(7,11)(8,9)(10,12)$. This agrees with the labeling of flags on Figure 13.

Corollary 3.10. *The τ-model of the total dual $\mathfrak{hm}^V$ of a hypermap $\mathfrak{hm}$ is given by the involutions*

$$\tau_0(\mathfrak{hm}^V) = \tau_0(\mathfrak{hm}), \qquad \tau_1(\mathfrak{hm}^V) = \tau_2(\mathfrak{hm}), \qquad \tau_2(\mathfrak{hm}^V) = \tau_1(\mathfrak{hm}) .$$

One may check that this agrees with Corollary 3.7 in the case of oriented hypermaps.

3.4. **Partial duality for colored graphs.**

Let $\Gamma_{\mathfrak{hm}}$ be the [2]-colored graph corresponding to a hypermap $\mathfrak{hm}$. Let I be a subset of two out of three colors, for example $I = \{1,2\}$, and let S be a subset of 2-bubbles in $\mathcal{B}^I$ which corresponds to a subset of vertices of $\mathfrak{hm}$.

Theorem 3.11. *The* [2]*-colored graph $\Gamma_{\mathfrak{hm}^S}$ of the partial dual hypermap $\mathfrak{hm}^S$ is obtained from $\Gamma_{\mathfrak{hm}}$ by swapping the colors 1 and 2 for all edges in the 2-bubbles of S. In particular, the underlying graphs of $\Gamma_{\mathfrak{hm}^S}$ and $\Gamma_{\mathfrak{hm}}$ are the same.*

Proof. The edges of $\Gamma_{\mathfrak{hm}}$ of color 1 (resp. 2) correspond to 2-element orbits of τ_1 (resp. τ_2). According to Theorem 3.8 the partial dual hypermap is obtained by swapping the corresponding transpositions of τ_1 and τ_2. This corresponds to swapping the colors 1 and 2 in the bubbles of S. □

Example 3.12. Here are the [2]-colored graphs $\Gamma_{\mathfrak{hm}_1^V}$, $\Gamma_{\mathfrak{hm}_1^E}$ and $\Gamma_{\mathfrak{hm}_1^F}$ for the total duals of the hypermap $\mathfrak{hm}_1$ from Figure 7 relative to the set of all vertices V, all edges E, and all faces F. These dual hypermaps are shown on Figure 10.

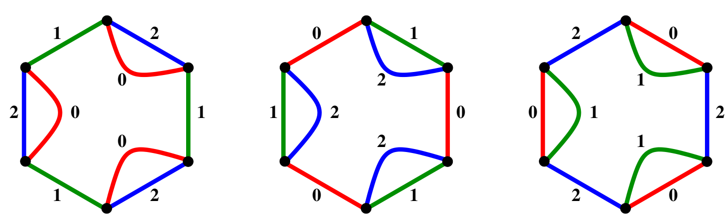


Figure 16. [2]-colored graphs of total duals $\Gamma_{\mathfrak{hm}_1^V}$, $\Gamma_{\mathfrak{hm}_1^E}$, $\Gamma_{\mathfrak{hm}_1^F}$

Remark 3.13. Higher dimensional partial duality. Such an easy interpretation of partial duality for [2]-colored graphs easily allows to make a higher dimensional generalization for [D]-colored graphs Γ. Namely, fix a set I of D colors out of the total number of $D+1$ colors, and let S be a subset of D-bubbles in $\mathcal{B}^I$. The *partial dual* Γ^S *relative to* S is a [D]-colored graph obtained from Γ by a permutation of the colors of the edges in S.

In this case the word "duality" is appropriate. It is rather an action of a symmetric group S_D on colors of edges of bubbles of S. In the hypermap case, $D = 2$.

This group is isomorphic to $\mathbb{Z}_2$, so the partial duality corresponds to the only non-trivial element of order 2. But for higher D the group S_D contains higher order elements so they will not be "dualities" anymore.

This concept of higher dimensional partial duality is completely unexplored up to now. It would be very interesting to study it. In particular, is it true that if the realization $|\Gamma|$ of Γ through its direct complex $\Delta(\Gamma)$ is a manifold, then the realization of its partial dual $|\Gamma^S|$ is also a manifold? How partial duality affects the (co)homology groups $H_*(\Delta(\Gamma))$?

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