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Symmetric Functions and Macdonald's Result for Top Connexion Coefficients in the Symmetric Group

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In unpublished work, Macdonald gave an indirect proof that the connexion coefficients for certain symmetric functions coincide with the connexion coefficients of the class algebra of the symmetric group. We give a direct proof of this result and demonstrate the use of these functions in a number of combinatorial questions associated with ordered factorisations of permutations into factors of specified cycle-type, including factorisations considered up to commutation in the symmetric group. Several related properties of the symmetric functions are given.

1. INTRODUCTION

Let $\mathscr{P}$ be the set of all partitions and if α is a partition of n we write $\alpha \vdash n$. Let $l(\alpha)$ be the number of positive parts of α and $m_i(\alpha)$ be the number of parts of α equal to i, for $i \geqslant 1$. Where convenient we write $\alpha = [1^{m_1(\alpha)} 2^{m_2(\alpha)} \cdots]$.

Let $\mathsf{K}_\alpha \in \mathbb{C}\mathfrak{S}_n$ be the formal sum of the elements in the conjugacy class indexed by $\alpha \vdash n$, consisting of all permutations in $\mathfrak{S}_n$ with *cycle-type* α. Let the coefficient of K_γ in the product $\mathsf{K}_\alpha \mathsf{K}_\beta$ be denoted by $c^\gamma_{\alpha,\beta}$, a *connexion coefficient* in the class algebra of $\mathfrak{S}_n$. These coefficients are rich in combinatorial information but their determination remains difficult in all but the most restricted cases. Clearly, from connectivity considerations, $c^\gamma_{\alpha,\beta}$ is zero unless $l(\alpha) + l(\beta) \leqslant l(\gamma) + n$, and we call $c^\gamma_{\alpha,\beta}$ a *top* coefficient in the case of equality, for $\alpha, \beta, \gamma \vdash n$. For example, explicit calculation yields, for $n \geqslant 7$,

$$\mathsf{K}_{[1^{n-4}2^2]}\mathsf{K}_{[1^{n-3}3]} = 3(n-3)\,\mathsf{K}_{[1^{n-3}3]} + 4(n-4)\,\mathsf{K}_{[1^{n-4}2^2]} + (4\mathsf{K}_{[1^{n-6}2\,4]} + \mathsf{K}_{[1^{n-7}2^2\,3]} + 5\mathsf{K}_{[1^{n-5}5]}), \qquad (1)$$

where the top terms are enclosed in large round brackets.

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A compact and explicit expression for the top connexion coefficient $c_{\alpha,\beta}^{[n]}$ was given by Faharat and Higman [5]. Goupil and Bédard [9] rederived this result, and extended it to give an explicit, but cumbersome, expression for the arbitrary top connexion coefficient (see also [6]).

For partition α, let $\alpha - \mathbf{1}$ be the partition obtained by subtracting one from each of the positive parts of α (and suppressing resulting zeros). Macdonald [17] used Lagrange inversion to construct a basis $\{u_\lambda\}$ for the ring, $\Lambda_x = \mathbb{C}[[x_1, x_2, ...]]^{\mathfrak{S}}$, of symmetric functions in the indeterminates $x_1, x_2, ...$. Using a result in [5], he observed that the top connexion coefficients (indexed by λ) of the class algebra of $\mathfrak{S}_n$ are precisely the connexion coefficients (indexed by $\lambda - \mathbf{1}$) for these symmetric functions. For example (use Table I and $p_\lambda p_\mu = p_{\lambda+\mu}$ to verify this),

$$u_{[1^2]} u_{[2]} = 4u_{[1\,3]} + u_{[1^2\,2]} + 5u_{[4]},$$

where these coefficients correspond to the top terms, enclosed in brackets in (1). In this paper we give a direct proof of Macdonald's result which makes the essential role of Lagrange inversion clear and natural. We also give an inversion theorem that permits us to use the u_λ in a number of combinatorial applications. The u_λ have also occurred in recent work by Haiman [10] on the module of symmetric invariants of $\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$ under the diagonal action of $\mathfrak{S}_n$.

In Section 2, a combinatorial construction of Goulden and Jackson [6] is used to identify an arbitrary top connexion coefficient with the number of 2-coloured trees of a particular type. The appearance of trees is explained by the fact that, by the embedding theorem (see, for example, [15]), the product of a pair of permutations encodes the embedding of a 2-face-coloured map in an oriented surface. A tree is a map with one face, and is, of course, 2-vertex-colourable. The generating series for the set of such trees can be expressed as WB where W and B are solutions of the simultaneous functional equations

$$B = \xi(w_1 + w_2 W + w_3 W^2 + \cdots), \tag{2}$$

$$W = \xi(b_1 + b_2 B + b_3 B^2 + \cdots) \tag{3}$$

in two infinite sets of indeterminates $b_1, b_2, ...,$ and $w_1, w_2, ...$.

In Section 3, we transform these equations by replacing b_i and w_i with certain algebraically independent symmetric functions, thereby expressing the number of trees as a coefficient in a symmetric function. The algebraically independent symmetric functions form a basis for Λ_x whose dual is Macdonald's $\{u_\lambda\}$. A property of coefficients in dual bases then allows us to establish Macdonald's result. To this extent, the u_λ's may be said to be naturally related to the class algbra. For information on symmetric functions additional to that given in Section 3, the reader is directed to [18].

In Section 4 we give a general result for determining the coefficient of u_λ in certain functions of the u_α's as the solution of a functional equation. The solution can be obtained by Lagrange's inversion theorem [7].

Many combinatorial questions can be translated to the class algebra and the main theorem therefore gives a way in which some of these can be treated in the ring of symmetric functions. We have not studied this exhaustively, but instead illustrate the principle on some particular problems concerning ordered factorisations of permutations. Some typical questions of this type are considered in Section 5, where attention is restricted to factorisations of a generic cycle of length n in $\mathfrak{S}_n$. By using the class algebra, it is also possible to count *inequivalent* ordered factorisations, namely those distinguished only up to commutation of the factors. A bijective approach to a special instance of this question has been given by Eidswick [4] and by Longyear [16].

Although such questions can also be addressed using characters of irreducible representations of $\mathfrak{S}_n$, it does not seem possible to reduce the resulting expressions to the elementary forms which have been obtained in Section 5. In this sense, the role of u_λ appears to be significant. This point is amplified in Section 6.

It is shown in Section 7 that there is a particular involution, considered by Macdonald [17], defined by its action on the complete symmetric functions. It may be thought that other involutions on Λ_x can therefore serve a similar purpose. The fundamental involution is considered in this context. A brief table of resolutions of power sums in terms of the u_λ's and vice versa, is given.

Finally, no insight has been gained into constructing a comparable set of symmetric functions to account for the *nontop* terms in the product of K_α and K_β. For example, such terms appeared with coefficients $3(n-3)$ and $4(n-4)$ in (1). The effectiveness of the combinatorial construction used in Section 2 appears to be confined to the *top* case.

2. A Combinatorial Construction

Goulden and Jackson [6] have given a bijection between 2-coloured plane edge-rooted trees on $n+1$ vertices and pairs of permutations in $\mathfrak{S}_n$ whose product is an arbitrary permutation of cycle-type $[n]$, and whose cycle-types α, β, respectively, are such that $l(\alpha)+l(\beta)=n+1=n+l([n])$. This yielded (Theorem 2.2 [6]) an expression for the top connexion coefficient $[\mathsf{K}_{[n]}]\,\mathsf{K}_\alpha \mathsf{K}_\beta$, where $[A]\,B$ denotes the coefficient of A in B. The bijection easily can be extended to yield a bijection between pairs of permutations whose product is an arbitrary permutation of cycle-type $\gamma=(\gamma_1, \gamma_2, ...)$ and an ordered list of 2-coloured plane edge-rooted trees on

$\gamma_i + 1$ vertices, $i \geqslant 1$. Furthermore, as shown in [6], the generating function for these 2-coloured trees is WB, where W, B satisfy the Eqs. (2) and (3). Thus we deduce the following expression for an arbitrary top connexion coefficient.

LEMMA 2.1. *Let $\alpha, \beta, \gamma \vdash n$ with $l(\alpha) + l(\beta) = l(\gamma) + n$. Then*

$$[\mathsf{K}_\gamma]\, \mathsf{K}_\alpha \mathsf{K}_\beta = [w_\alpha b_\beta] \prod_{i \geqslant 1} [\xi^{\gamma_i + 1}]\, WB,$$

where $B = \xi(w_1 + w_2 W + w_3 W^2 + \cdots)$, $W = \xi(b_1 + b_2 B + b_3 B^2 + \cdots)$.

The bijection has been used by Hanlon *et al.* [11] to provide a combinatorial proof of a symmetric function result arising from an integral representation.

3. Symmetric Functions and the Main Result

The *complete* symmetric function of degree i in the set $x = (x_1, x_2, ...)$ of indeterminates is denoted by $h_i = h_i(x)$ so $\Lambda_x = \mathbb{Z}[h_1, h_2, ...]$. If $\alpha = (\alpha_1, \alpha_2, ...)$, let $h_\alpha = h_{\alpha_1} h_{\alpha_2} \cdots$. The h_i are algebraically independent and $\{h_\alpha : \alpha \in \mathscr{P}\}$ is a multiplicative basis of Λ_x. The generating series for the complete symmetric functions is $H(t; x) = \prod_{j \geqslant 1} (1 - t x_j)^{-1} = \sum_{i \geqslant 0} h_i(x)\, t^i$. Let $y = (y_1, y_2, ...)$ and $z = (z_1, z_2, ...)$ be sets of indeterminates. Clearly,

$$H(t; x, y) = H(t; x)\, H(t; y). \tag{4}$$

The *monomial* symmetric functions are denoted by m_λ, and form a basis of Λ_x. For symmetric functions in x, the usual inner product $\langle\ ,\ \rangle_x$ on Λ_x is defined by $\langle m_\lambda, h_\mu \rangle_x = \delta_{\lambda, \mu}$, making $\{h_\alpha : \alpha \in \mathscr{P}\}$ and $\{m_\alpha : \alpha \in \mathscr{P}\}$ *dual bases* of Λ_x. When the context permits, the suffix x of $\langle\ ,\ \rangle_x$ and Λ_x, and the argument of the symmetric functions are suppressed.

The *power sum* symmetric functions, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$, are orthogonal with respect to this inner product, and $\langle p_\alpha, p_\beta \rangle = g^{-1}(\alpha)\, \delta_{\alpha, \beta}$, where $g(\alpha) = \prod_{i \geqslant 1} (i^{m_i(\alpha)} m_i(\alpha)!)^{-1}$. The *Schur* symmetric functions, s_α, are orthonormal with respect to this inner product.

Let u and t be related by

$$u = tH(t; x). \tag{5}$$

Then t is implicitly a power series in u so we may write

$$t = uH^*(u; x), \tag{6}$$

where $H^*(u; x) = \sum_{i \geqslant 0} h_i^*(x) u^i$ is uniquely defined. Note that H^* does not satisfy (4), and that

$$H^*(u; x) = H(t, x)^{-1}. \tag{7}$$

Now

$$h_n^* = h_n^*(x) = [u^{n+1}]\, t = \frac{1}{n+1} [t^n]\, H(t)^{-n-1}, \tag{8}$$

by Lagrange's implicit function theorem applied to (5), so is a symmetric function of degree n.

It follows from (8) that $h_n^* = (1/(n+1))[s^n] \exp(-(n+1) \sum_{k \geqslant 1} (1/k) p_k s^k)$, so

$$h_n^* = \prod_{\lambda \vdash n} (-1)^{l(\lambda)} (n+1)^{l(\lambda)-1} g(\lambda)\, p_\lambda. \tag{9}$$

The h_i^* are algebraically independent. Let $h_\lambda^* = h_{\lambda_1}^* h_{\lambda_2}^* \cdots$. Then $\{h_\lambda^* : \lambda \in \mathscr{P}\}$ is a basis of Λ.

Let $\{u_\lambda : \lambda \in \mathscr{P}\}$ be the basis dual to $\{h_\lambda^* : \lambda \in \mathscr{P}\}$, so $\langle u_\lambda, h_\mu^* \rangle = \delta_{\lambda, \mu}$. Note that $a_{\lambda + \mu} = a_\lambda a_\mu$ for $a_\lambda = h_\lambda, p_\lambda, h_\lambda^*$, but not for $m_\lambda, s_\lambda, \mu_\lambda$.

Our main interest in this paper is the connexion coefficients for the u_λ. The following result expresses connexion coefficients for an arbitrary set of symmetric functions in terms of operations on the dual basis.

PROPOSITION 3.1. *Let $\{a_\alpha : \alpha \in \mathscr{P}\}$, $\{b_\alpha : \alpha \in \mathscr{P}\}$ be dual bases of Λ. Then, for λ, μ, $\nu \in \mathscr{P}$,*

$$[a_\lambda(x)]\, a_\mu(x)\, a_\nu(x) = [b_\mu(y)\, b_\nu(z)]\, b_\lambda(y, z).$$

Proof. First note that $\sum_{\rho \in \mathscr{P}} a_\rho(x)\, b_\rho(y) = \prod_{i, j \geqslant 1} (1 - x_i y_j)^{-1}$, by Cauchy's theorem [18]. Then

$$\begin{aligned}
&[a_\lambda(x)]\, a_\mu(x)\, a_\nu(x) \\
&\quad = \langle b_\lambda(x), a_\mu(x)\, a_\nu(x) \rangle_x \\
&\quad = [b_\mu(y)\, b_\nu(z)] \left\langle b_\lambda(x), \sum_{\rho \in \mathscr{P}} a_\rho(x)\, b_\rho(y) \sum_{\omega \in \mathscr{P}} a_\omega(x)\, b_\omega(z) \right\rangle_x \\
&\quad = [b_\mu(y)\, b_\nu(z)] \left\langle b_\lambda(x), \prod_{i, j \geqslant 1} (1 - x_i y_j)^{-1} (1 - x_i z_j)^{-1} \right\rangle_x \\
&\quad = [b_\mu(y)\, b_\nu(z)]\, b_\lambda(y, z). \quad \blacksquare
\end{aligned}$$

The following result of Macdonald [17] states that the top connexion coefficients in the class algebra of $\mathfrak{S}_n$ are the connexion coefficients for the symmetric functions u_λ.

THEOREM 3.2. *Let $\lambda, \mu, \nu \vdash n$, with $l(\mu)+l(\nu)=n+l(\lambda)$. Then*

$$[\mathsf{K}_\lambda]\, \mathsf{K}_\mu \mathsf{K}_\nu = [u_{\lambda-1}]\, u_{\mu-1} u_{\nu-1}.$$

Proof. Let B, W be as defined in Lemma 2.1. Since the $h_i^*(z)$ are algebraically independent, let $w_i = h_{i-1}^*(z)$ so $B = \zeta H^*(W; z)$ since $B = \zeta(w_1 + w_2 W + \cdots)$. Let $t = WH^*(W; z)$. Then $WB = \zeta t$, and $W = tH(t; z)$.

Since the $h_i^*(y)$ are algebraically independent, let $b_i = h_{i-1}^*(y)$ so $W = \zeta H^*(B; y)$ since $W = \zeta(b_1 + b_2 B + \cdots)$. Let $t' = BH^*(B; y)$. Then $WB = \zeta t'$, so $t' = t$, and $B = tH(t; y)$.

Now $\zeta t = WB = t^2 H(t; y)\, H(t; z)$ so $\zeta = tH(t; y, z)$, from (4), whence $t = \zeta H^*(\zeta; y, z)$. It follows that $WB = \zeta^2 H^*(\zeta; y, z)$ so $[\zeta^{\gamma_i+1}]\, WB = h_{\gamma_i-1}^*(y, z)$. This allows us to deduce from Lemma 2.1 that $[\mathsf{K}_\gamma]\, \mathsf{K}_\alpha \mathsf{K}_\beta = [h_{\alpha-1}^*(z)\, h_{\beta-1}^*(y)]\, h_{\gamma-1}^*(y, z)$, and the result follows from Proposition 3.1. ∎

In combinatorial applications, it will be convenient to use the following equivalent form of Theorem 3.2.

COROLLARY 3.3. *For a polynomial Φ and $\alpha \vdash n$,*

$$[t^{n-l(\alpha)} \mathsf{K}_\alpha]\, \Phi(t^{n-l(\lambda)} \mathsf{K}_\lambda \colon \lambda \vdash n) = [u_{\alpha-1}]\, \Phi(u_{\lambda-1} \colon \lambda \vdash n).$$

Proof. Note that the restriction on $\lambda, \mu, \nu \vdash n$ in the statement of Theorem 3.2 can be written as $(n-l(\mu)) + (n-l(\nu)) = (n-l(\lambda))$, so applying Theorem 3.2 iteratively gives

$$[\mathsf{K}_\alpha]\, \mathsf{K}_{\lambda_1} \cdots \mathsf{K}_{\lambda_m} = [u_{\alpha-1}]\, u_{\lambda_1-1} \cdots u_{\lambda_m-1}$$

for $n - l(\alpha) = \sum_{i=1}^m (n - l(\lambda_i))$. The result follows by linearity, since Φ is a linear combination of terms of the form $\mathsf{K}_{\lambda_1} \cdots \mathsf{K}_{\lambda_m}$. ∎

4. RESOLUTION WITH RESPECT TO THE NEW BASIS

In applications of this theory to combinatorial questions, through Corollary 3.3, we shall need a few technical results about the symmetric functions u_λ. The first is an explicit expression in terms of power sums for u_λ when λ has a single part.

PROPOSITION 4.1.

$$u_{[n]} = -p_n.$$

Proof. $[u_\lambda]\, p_n = \langle h_\lambda^*, p_n \rangle = \langle \delta_{\lambda,[n]} h_n^*, p_n \rangle$ from (9), since h_λ^* is multiplicative. The result follows from (9). ∎

The next results reduce the extraction of the coefficient of u_λ from symmetric functions of a particular form to solving a functional equation. The functional equation is precisely of the type to which Lagrange's implicit function theorem can be applied. The mappings used here between power series rings are defined elementwise, and extend to homomorphisms. The coefficient rings under this action are recoverable trivially from the appropriate ring isomorphism and, for brevity, these, the domain and codomain are suppressed. Such isomorphisms are generalisations of the natural isomorphism $\mathsf{R}[[x_1, x_2]] \cong (\mathsf{R}[[x_1]])[[x_2]]$, where R is a ring.

LEMMA 4.2. *Let $F(t)$ be a power series with constant term equal to* 1. *Then*

$$[u_\lambda(x)] \prod_{i \geq 1} F(x_i) = c_\lambda,$$

where $\sum_{i \geq 0} c_i s^{i+1} = v$ satisfies the functional equation $v = sF^{-1}(v)$, and $c_\lambda = c_{\lambda_1} c_{\lambda_2} \cdots$, $c_0 = 1$.

Proof. Let $F(s) = 1 + f_1 s + f_2 s^2 + \cdots$ and $f_\mu = f_{\mu_1} f_{\mu_2} \cdots$, where we may regard the f_i as indeterminates. Then

$$\begin{aligned}[u_\lambda(x)] \prod_{i \geq 1} F(x_i) &= [u_\lambda(x)] \sum_{\mu \in \mathscr{P}} f_\mu m_\mu(x) \\ &= \sum_{\mu \in \mathscr{P}} f_\mu \langle h_\lambda^*(x), m_\mu(x) \rangle \\ &= \sum_{\mu \in \mathscr{P}} f_\mu [h_\mu(x)]\, h_\lambda^*(x) = \theta h_\lambda^*(x),\end{aligned}$$

where $\theta: h_i(x) \mapsto f_i$, extended as a homomorphism to Λ_x. Now define $w = sH^*(s; x)$ so $v = \theta s H^*(s; x) = \sum_{i \geq 0} s^{i+1} \theta h_i^*(x)$ where $v = \theta w$. But $s = wH(w; x)$ and, under the action of θ, this gives $s = vF(v)$, a functional equation for v with a unique solution, and the result follows with $c_i = \theta h_i^*(x)$, $i \geq 1$. ∎

THEOREM 4.3. *Let $a_\lambda = a_{\lambda_1} a_{\lambda_2} \cdots$, $a_0 = 1$. Let $A(t) = 1 + a_1 t + a_2 t^2 + \cdots$ and $1 + c_1 s + c_2 s^2 + \cdots = A^w(r)$, where r satisfies $r = sA^{w-1}(r)$. Then, for an indeterminate w,*

$$\left(\sum_{\lambda \in \mathscr{P}} a_\lambda u_\lambda(x) \right)^w = \sum_{\lambda \in \mathscr{P}} c_\lambda u_\lambda(x).$$

Proof. In Lemma 4.2, let $F(x_i) = H(x_i; y)$. Then $v = sH(v; y)^{-1}$, so $v = sH^*(s; y) = \sum_{i \geqslant 0} c_i(y) s^{i+1}$ whence $c_i(y) = h_i^*$. Let $\xi : h_i^*(y) \mapsto a_i$, extended as a homomorphism. Then $\xi c_\lambda(y) = a_\lambda$. From Lemma 4.2 we have, $\sum_{\lambda \in \mathscr{P}} c_\lambda(y) u_\lambda(x) = \prod_i H_i(x_i; y)$. Thus, applying ξ to this, $\sum_{\lambda \in \mathscr{P}} a_\lambda u_\lambda(x) = \xi \prod_{i \geqslant 1} H(x_i; y)$. Then $[u_\mu](\sum_{\lambda \in \mathscr{P}} a_\lambda u_\lambda(x))^w = [u_\mu] \prod_{i \geqslant 1} \xi H(x_i; y)^w = c_\mu$, from Lemma 4.2, where $\sum_{i \geqslant 0} c_i s^{i+1} = v$ and

$$v = s\xi H(v, y)^{-w} \tag{10}$$

Note that v does not depend on y, and is therefore invariant under ξ. Let

$$t = vH(v; y) \tag{11}$$

so $v = tH^*(t; y)$. Applying ξ to this gives

$$v = rA(r), \tag{12}$$

where $r = \xi t$. But, from (11), $r = v\xi H(v; y)$, so eliminating $\xi H(v; y)$ between this and (10) gives $r^w = sv^{w-1}$. Finally, eliminate v between this and (12) to give $r = sA^{w-1}(r)$. ∎

5. Ordered Factorisations of Permutations

If $\rho, \sigma_1, ..., \sigma_l$ are elements of $\mathfrak{S}_n$ such that $\rho = \sigma_1 \cdots \sigma_l$, we say that $(\sigma_1, ..., \sigma_l)$ is an *ordered factorisation* of ρ, and the σ_i are the *factors*. Let the cycle-type of ρ be γ and the cycle-type of σ_i be α_i for $i = 1, ..., k$. Then, from connectivity considerations,

$$\sum_{i=1}^{l} (n - l(\alpha_i)) \geqslant n - l(\gamma)$$

and, in the case of equality, we say that the factorisation in *minimal.* Thus we can use Corollary 3.3 to calculate the number of minimal ordered factorisations by extracting a coefficient from symmetric functions $\{u_\lambda\}$.

Factors with cycle-type $[1^{n-k-1} k+1]$ are called $(k+1)$-cycles, and n-cycles are referred to as *full cycles.* We first consider minimal factorisations into $(k+1)$-cycles. The case of transpositions $(k = 1)$ was first solved by Dénes [3], who showed that the number of such factorisations is n^{n-2}, which is also the number of labelled trees on n vertices. Bijective proofs of this coincidence have been given by Moszkowski [19] and Goulden and Pepper [8].

COROLLARY 5.1. *The number of minimal ordered factorisations of a full cycle in $\mathfrak{S}_n$ into $(k+1)$-cycles is*

$$n^{m-1}$$

if $n-1=km$ for a positive integer m and zero otherwise.

Proof. The required number is

$$[u_{[n-1]}](1-u_{[k]})^{-1}=\langle h^*_{n-1}, u^m_{[k]}\rangle=\langle h^*_{n-1}, (-1)^m p^m_k\rangle$$

from Corollary 3.3 and Proposition 4.1, where $km=n-1$. The result follows from (9). ▮

COROLLARY 5.2. *The number of minimal ordered factorisations of a full cycle in $\mathfrak{S}_n$, into l factors which together contain i_j cycles of length $j+1$, $j\geqslant 1$, is*

$$\frac{l(n(l-1))!}{\{n(l-1)+1-\sum_{j\geqslant 1} i_j\}!\prod_{j\geqslant 1} i_j!},$$

where $\sum_{j\geqslant 1} ji_j=n-1$.

Proof. Let $a_1, a_2, \ldots$ be indeterminates. From Corollary 3.3 and Theorem 4.3 with $w=l$, the desired number is

$$[a_1^{i_1}a_2^{i_2}\cdots u_{[n-1]}(x)]\left\{\sum_{\lambda\in\mathscr{P}} a_\lambda u_\lambda(x)\right\}^l=[a_1^{i_1}a_2^{i_2}\cdots s^{n-1}]\,A^l(r),$$

where $r=sA^{l-1}(r)$. The result follows directly from Lagrange's implicit function theorem. ▮

COROLLARY 5.3. *The number of minimal ordered factorisations of a full cycle in $\mathfrak{S}_n$ into l factors containing cycles of length $k+1$ and 1 alone is*

$$\frac{l}{m}\binom{n(l-1)}{m-1},$$

if $n-1=km$ for a positive integer m, and zero otherwise.

Proof. Let $a_j=\delta_{k,j}$ for $j\geqslant 1$. Then, from Corollary 3.3 and Theorem 4.3 with $w=l$, the desired number is $[u_{[n-1]}(x)]\{\sum_{\lambda\in\mathscr{P}} a_\lambda u_\lambda(x)\}^l=[s^{n-1}](1+r^k)^l$ where $r=s(1+r^k)^{l-1}$. The result follows directly from Lagrange's implicit function theorem. ▮

We again consider the minimal ordered factorisations of a full cycle in $\mathfrak{S}_n$ into $(k+1)$-cycles, whose total number was given in Corollary 5.1. Two such factorisations are said to be *equivalent* if one can be transformed into the other by permissible commutation of the adjacent factors. Of course, two $(k+1)$-cycles commute only when there are no common elements in the cycles of length $k+1$. Let $d_{n,k}$ be the number of inequivalent minimal ordered factorisations of full cycles in $\mathfrak{S}_n$ into $(k+1)$-cycles. The case of transpositions, where $k=1$, has been considered by Eidswick [4] and Longyear [16].

First we show how Cartier and Foata's commutation monoid can be used to give an expression for $d_{n,k}$ in terms of the class algebra.

LEMMA 5.4.

$$d_{n,k} = [t^{n-1}\mathsf{K}_{[n]}]\left(\sum_{j=0}^{\lfloor n/(k+1)\rfloor} (-1)^j t^{jk} \mathsf{K}_{[1^{n-(k+1)j}(k+1)^j]}\right)^{-1}.$$

Proof. Let π_i be the ith $(k+1)$-cycle in a canonical ordering of the set of all $(k+1)$-cycles in $\mathfrak{S}_n$. Then $d_{n,k} = [\pi]\sum_{r\geqslant 0}\sum^{*}_{i_1,\ldots,i_r\geqslant 1}\pi_{i_1}\cdots\pi_{i_r}$ where π is the full cycle and $\sum^*$ is over all inequivalent products of π_i's. Let

$$\mathscr{C} = \{\{i_1, \ldots, i_s\} : s\geqslant 0, i_1 < \cdots < i_s, \pi_{i_j}\pi_{i_l} = \pi_{i_l}\pi_{i_j} \text{ for } 1\leqslant j\neq l\leqslant s\}.$$

For $\alpha = \{i_1, \ldots, i_s\}\in\mathscr{C}$, let $\pi_\alpha = \pi_{i_1}\cdots\pi_{i_s}$ and $c_s = \sum_{\alpha\in\mathscr{C},\, |\alpha| = s}\pi_\alpha$. Then, by a theorem of Cartier and Foata [2],

$$d_{n,k} = [\pi]\left(\sum_{j\geqslant 0}(-1)^j c_j\right)^{-1}.$$

But clearly, $c_s = \mathsf{K}_{[1^{n-(k+1)s}(k+1)^s]}$ so $\{\sum_{j\geqslant 0}(-1)^j c_j\}^{-1}\in\mathbb{C}\mathfrak{S}_n$ and $[\pi]$ can be replaced with $[\mathsf{K}_{[n]}]$. The result follows. ∎

We can now apply Corollary 3.3 to convert the determination of $d_{n,k}$ into a calculation with symmetric functions, and hence evaluate $d_{n,k}$.

THEOREM 5.5. *The number of inequivalent minimal ordered factorisations of a full cycle in $\mathfrak{S}_n$ into $(k+1)$-cycles is*

$$\frac{1}{m}\binom{(2k+1)m}{m-1}$$

if $n-1 = km$ for some positive integer m, and zero otherwise.

Proof. Let $a_j = -\delta_{k,j}$ for $j \geqslant 1$. Then from Lemma 5.4, Corollary 3.3 and Theorem 4.3 with $w = -1$,

$$d_{n,k} = [u_{[n-1]}(x)] \left(\sum_{j \geqslant 0} (-1)^j u_{[k^j]}(x) \right)^{-1}$$

$$= [u_{[n-1]}(x)] \left\{ \sum_{\lambda \in \mathscr{P}} a_\lambda u_\lambda(x) \right\}^{-1} = [s^{n-1}](1 - r^k)^{-1},$$

where r satisfies $r = s(1-r^k)^{-2}$. To solve this, let $1 + R = (1 - r^k)^{-1}$. Then $d_{n,k} = [s^{n-1}](1+R)$ where R satifies the functional equation

$$R = s^k (1+R)^{2k+1} \tag{13}$$

The result now follows from Lagrange's implicit function theorem. ∎

Eidswick [4] and Longyear [16] have determined $d_{n,1}$ by different means. When $k = 1$, (13) is the functional equation derived by Longyear [16] with a direct combinatorial construction. It is resonable to ask whether there is a generalisation of Longyear's construction to give a direct proof of (13). In looking for such a construction, it may be worthwhile to note that one combinatorial interpretation of (13) is that R is the generating series for planted plane trees, with vertices of degree 1 and $2k+1$ only, with respect to half the number of edges.

It should be noted that we have considered only minimal factorisations of a full cycle in the above example. However, the product form of the coefficients of u_λ in Theorem 4.3 make it clear that the number of minimal factorisations of an arbitrary permutation is simply a product of terms of the type given above. Thus there is no loss of generality, but rather only simplification in notation, in considering only full cycles in the above examples.

6. A Character Theoretic Expression

An alternative means of using the class algebra of $\mathfrak{S}_n$ to solve combinatorial problems such as those of Section 5 (see, for example, Jackson [12, 13, 14], Stanley [20]) is through orthogonal idempotents and the characters of irreducible representations of $\mathfrak{S}_n$.

Let χ_α^θ be the character of the ordinary irreducible representation, of degree f^θ, of $\mathfrak{S}_n$, indexed by $\theta \vdash n$, evaluated at an element of the conjugacy class indexed by $\alpha \vdash n$. The size of the conjugacy class indexed by θ is denoted by h^θ.

LEMMA 6.1. *The number of ordered factorisations of an arbitrary permutation with cycle structure γ into factors in the conjugacy classes indexed by $\alpha_1, ..., \alpha_p$ is*

$$[\mathsf{K}_\gamma]\,\mathsf{K}_{\alpha_1}\cdots\mathsf{K}_{\alpha_p}=\frac{1}{n!}h^{\alpha_1}\cdots h^{\alpha_p}\sum_{\theta\vdash n}(f^\theta)^{-(p-1)}\chi^\theta_{\alpha_1}\cdots\chi^\theta_{\alpha_p}\chi^\theta_\gamma.$$

Proof. Let $\mathsf{F}_\alpha=(f^\alpha/n!)\sum_{\theta\vdash n}\chi^\alpha_\theta\,\mathsf{K}_\theta$. $\{\mathsf{F}_\alpha:\alpha\vdash n\}$ is a set of orthogonal idempotents [1] spanning the centre of $\mathbb{C}\mathfrak{S}_n$. The inverse relation is $\mathsf{K}_\alpha=h^\alpha\sum_{\theta\vdash n}(1/f^\theta)\,\chi^\theta_\alpha\mathsf{F}_\theta$, and the result now follows. ∎

In attempting to apply this methodology through Lemma 6.1 to the examples given in Section 5, however, we have found that the resulting expressions are intractible. For example, consider the problem treated in Theorem 5.5. First, note [1] that

$$\sum_{r=0}^{n-1}\chi_\alpha^{[1^r n-r]}y^r=(1+y)^{-1}\prod_{i=1}^{n}(1-(-y)^i)^{m_i(\alpha)}.$$

In particular, $f^{[1^r n-r]}=\binom{n-1}{r}$. Also $\chi^\alpha_{[n]}=(-1)^k$ if $\alpha=[1^k\,n-k]$ and is 0 otherwise. Thus, from Lemmas 5.4 and 6.1, after manipulation

$$d_{n,1}=\frac{1}{n!}[t^{n-1}]$$
$$\times\sum_{k=0}^{n-1}\frac{(-1)^k\binom{n-1}{k}}{1+\binom{n-1}{k}^{-1}[y^ku^n]\,n!\,(1+y)^{n-1}e^u\left(\exp\left(\frac{1}{2}u^2t\,\frac{1+y}{1-y}\right)-1\right)}.$$

It is unclear how this expression can be be simplified to the one given by Theorem 5.5, so it appears that the simplification for minimal factorisations afforded by the use of the symmetric functions u_λ has been significant. This is particularly striking in view of the close connexion between characters and symmetric functions given by $\langle s_\theta, p_\alpha\rangle=\chi^\theta_\alpha$.

7. A PAIR OF INVOLUTIONS ON Λ

The following mapping on Λ was considered by Macdonald [17]. Let $\psi: h_i\mapsto h_i^*$, extended as a homomorphism to Λ. Then, from (5) and (6), $h_n=(1/(n+1))[t^n]\,H^*(t)^{-n-1}=\psi h_n^*$, from (8), so ψ is involutory and degree-preserving. The adjoint ψ^* of ψ is therefore also an involution, and is linear but not necessarily multiplicative. Now $\langle u_\lambda, \psi h_\mu\rangle=\delta_{\lambda,\mu}$ since $\{u_\lambda\}$ is dual to $\{h_\lambda^*\}$, so $\langle\psi^*u_\lambda, h_\mu\rangle=\delta_{\lambda,\mu}=\langle m_\lambda, h_\mu\rangle$. Thus $m_\lambda=\psi^*u_\lambda$, and since ψ^* is involutory we have

$$u_\lambda=\psi^*m_\lambda. \tag{14}$$

TABLE I

Representation of u_λ in terms of p_μ, and p_λ in terms of u_μ

λ	Rep. of u_λ in $\{p_\mu : \mu \in \mathscr{P}\}$	Rep. of p_λ in $\{u_\mu : \mu \in \mathscr{P}\}$
$[1]$	$-p_{[1]}$	$-u_{[1]}$
$[2]$	$-p_{[2]}$	$-u_{[2]}$
$[1^2]$	$\frac{3}{2}p_{[2]}+\frac{1}{2}p_{[1^2]}$	$3u_{[2]}+2u_{[1^2]}$
$[3]$	$-p_{[3]}$	$-u_{[3]}$
$[2\,1]$	$4p_{[3]}+p_{[2\,1]}$	$4u_{[3]}+u_{[2\,1]}$
$[1^3]$	$-\frac{10}{3}p_{[3]}-\frac{3}{2}p_{[2\,1]}-\frac{1}{6}p_{[1^3]}$	$-16u_{[3]}-9u_{[2\,1]}-6u_{[1^3]}$
$[4]$	$-p_{[4]}$	$-u_{[4]}$
$[3\,1]$	$5p_{[4]}+p_{[3\,1]}$	$5u_{[4]}+u_{[3\,1]}$
$[2^2]$	$\frac{5}{2}p_{[4]}+\frac{1}{2}p_{[2^2]}$	$5u_{[4]}+2u_{[2^2]}$
$[2\,1^2]$	$-15p_{[4]}-4p_{[3\,1]}-\frac{3}{2}p_{[2^2]}-\frac{1}{2}p_{[2\,1^2]}$	$-25u_{[4]}-8u_{[3\,1]}-6u_{[2^2]}-2u_{[2\,1^2]}$
$[1^4]$	$\frac{35}{4}p_{[4]}+\frac{10}{3}p_{[3\,1]}+\frac{9}{8}p_{[2^2]}$ $+\frac{3}{4}p_{[2\,1^2]}+\frac{1}{24}p_{[1^4]}$	$125u_{[4]}+64u_{[3\,1]}+54u_{[2^2]}$ $+36u_{[2\,1^2]}+24p_{[1^4]}$

This provides an alternative compact characterisation of the u_λ. From Proposition 4.1 and (14), $\psi^* p_n = -p_n$. It is readily shown that

$$\psi p_n = \sum_{\lambda \vdash n} (-n)^{l(\lambda)}\, g(\lambda)\, p_\lambda,$$

so $\psi p_n \neq \psi^* p_n$ if $n \geqslant 1$. Thus $\psi \neq \psi^*$ so ψ is not self-adjoint.

The following result gives the resolution of u_λ with respect to the power sum symmetric function basis. Table I gives u_λ and p_μ in terms of each other for $|\lambda| \leqslant 4$. Since the p_μ form an orthogonal multiplicative basis for symmetric functions, this may provided a convenient means of making calculations involving the u_λ.

LEMMA 7.1. *If* $\psi p_\alpha = \sum_\beta a^\alpha_\beta h_\beta$ *then* $u_\alpha = \sum_\beta g(\beta)\, a^\beta_\alpha p_\beta$.

Proof. Let $u_\lambda = \sum_\alpha c^\lambda_\alpha p_\alpha$, so $c^\lambda_\alpha = g(\alpha)\langle u_\lambda, p_\alpha\rangle$. But from (14), $\langle u_\lambda, p_\alpha\rangle = \langle \psi^* m_\lambda, p_\alpha\rangle = \langle m_\lambda, \psi p_\alpha\rangle$. Let $\psi p_\alpha = \sum_\beta a^\alpha_\beta h_\beta$. Then $\langle u_\lambda, p_\alpha\rangle = \langle m_\lambda, \psi p_\alpha\rangle = a^\alpha_\lambda$ whence $c^\lambda_\alpha = g(\alpha)\, a^\alpha_\lambda$. The result follows. ∎

Note that ψp_α can be computed in terms of the h_j by means of the following result.

LEMMA 7.2.

$$\psi p_\alpha = \prod_{j \geqslant 1} \left([s^j] \frac{1}{H^j(s)} \right)^{a_j}, \qquad \textit{where} \quad \alpha = [1^{a_1} 2^{a_2} \cdots].$$

Proof. First note that $(\sum_{j\geqslant 1} \frac{1}{j} p_j u^j) = \log H(u; x)$, so

$$\sum_{j\geqslant 1} \frac{1}{j} \psi p_j u^j = \log \psi H(u; x) = \log H^*(u; x) = -\log H(t, x)$$

from (7). Thus $\psi p_n = -n[u^n] \log H(t) = [s^n] H^{-n}(s)$, by Lagrange's implicit function theorem. ∎

We may regard the defining Eqs. (5) and (6) as the means by which the involution ψ is realised, and the main theorem as a combinatorial result about ψ. It is therefore appropriate to determine whether other involutions can be given a combinatorial interpretation in this way. The obvious candidate is the fundamental homomorphism [18], defined by $\omega: h_i \mapsto e_i$. It is degree-preserving and involutory, and although it shares these properties with ψ, we have been unable to obtain further combinatorial information by reversing the argument given here, with ω replacing ψ. It seems that the delicate functional equational structure in the symmetric function ring in the case of ψ disappears in the case of ω, since the latter is self-adjoint. We obtain simply that $[m_\lambda] m_\mu m_\nu = [f_\lambda] f_\mu f_\nu$, where f_λ is a *forgotten* symmetric function, and seem to have no prospect of providing a compact combinatorial interpretation of f_λ in this way.

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