

**SECOND ORDER FREENESS
AND FLUCTUATIONS OF RANDOM MATRICES:
I. GAUSSIAN AND WISHART MATRICES AND
CYCLIC FOCK SPACES**

JAMES A. MINGO ^(*) AND ROLAND SPEICHER ^{(*)(†)}

ABSTRACT. We extend the relation between random matrices and free probability theory from the level of expectations to the level of fluctuations. We introduce the concept of “second order freeness” and interpret the global fluctuations of Gaussian and Wishart random matrices by a general limit theorem for second order freeness. By introducing cyclic Fock space, we also give an operator algebraic model for the fluctuations of our random matrices in terms of the usual creation, annihilation, and preservation operators. We show that orthogonal families of Gaussian and Wishart random matrices are asymptotically free of second order.

1. INTRODUCTION

Free probability has at least three basic facets: operator algebras, random matrices, and the combinatorics of non-crossing diagrams. This can be seen very clearly in Voiculescu’s generalization of Wigner’s semi-circle law to the case of several independent matrices [Voi1]. The distribution arising in this limit of random matrices can be modelled by a sum of creation and annihilation operators on full Fock spaces and described nicely in terms of non-crossing partitions.

On the random matrix side, Voiculescu’s theorem describes the leading contribution to the large N -limit of expectations of traces of Gaussian random matrices. However, in the random matrix literature there are many investigations on more refined questions in this context. On one side, subleading contributions to the large N -limit are of interest and have to be understood up to some point for dealing with questions concerning the largest eigenvalue of such random matrices. On

* Research supported by Discovery Grants and a Leadership Support Initiative Award from the Natural Sciences and Engineering Research Council of Canada.

† Research supported by a Premier’s Research Excellence Award from the Province of Ontario.

the other side, there has also been a lot of interest in leading contributions to other important quantities, like, e.g., global fluctuations (i.e., variance of two traces) of the considered random matrices. One should note that there are relations between these two questions. We are not going to explore these relations here, but we want to direct the reader's attention to the so-called "loop equations" in the physical literature (see, e.g. [Eyn]) and to the "master equation" in [HT].

We will concentrate in this paper on the second kind of question. As is well-known from the physical literature, in many cases these leading contributions are given by planar (or genus zero) diagrams and thus have quite a bit the flavour of the combinatorics of free probability. In the recent paper [MN] this description was made precise for the global fluctuations in the case of Wishart matrices, and in particular the relevant set of planar diagrams ("annular non-crossing permutations") was introduced and examined. However, this description of the fluctuations in the large N -limit was on a purely combinatorial level. Since it looks quite similar to the description of free Poisson distributions in terms of non-crossing partitions, one expects to find some genuine free probability behind these results. In particular, one would expect to have a description on the level of operator algebras and to have also a precise statement of the kind of "freeness" that arises here.

In this paper we will show that this is indeed the case. On one hand, using the notion of a cyclic Fock space, we can formulate the fluctuations in terms of the usual creation, annihilation, and preservation operators. On the other hand, we will also introduce an abstract "freeness" property for bilinear tracial functionals, which not only give us a conceptual understanding, but, on the other hand, is also crucial for proving our main theorems on the fluctuations.

Second order freeness, while stronger than the freeness of Voiculescu, nevertheless appears to be a central feature of ensembles of random matrices. Indeed, in this paper we prove that two standard examples of random matrix ensembles exhibit second order freeness: orthogonal families of Gaussian random matrices and orthogonal families of Wishart random matrices are asymptotically free of second order. Moreover in [MSS] we show that independent Haar distributed random unitary matrices are asymptotically free of second order.

The main results of the paper are thus. In section 5 (with proofs in section 7) we show that semi-circular and compound Poisson families on the full Fock space are free of second order. In section 6 we establish the basic properties of second order freeness and prove a general limit theorem. In section 8 we diagonalize the fluctuations in the Gaussian and Wishart case, thus recovering and extending results of

Cabanal-Duvillard [C-D]. In section 9 we prove asymptotic freeness of second order for orthogonal families of Gaussian and Wishart random matrices.

2. PRELIMINARIES

Here we collect some general notation and concepts which we will use in the following.

Our presentation should be, by and large, self-contained, however, it will rely of course on the basic ideas and concepts of free probability. For more details on this, one should consult [VDN, Voi2, NSp, HP]. Furthermore, the concepts of annular non-crossing permutations and partitions will play a crucial role. We will provide all relevant information on them in the text. However, our presentation will be quite condensed, and for further details one should consult the original paper [MN].

2.1. Some general notation. For natural numbers $n, m \in \mathbb{N}$ with $n < m$, we denote by $[n, m]$ the interval of natural numbers between n and m , i.e.,

$$[n, m] := \{n, n+1, n+2, \dots, m-1, m\}.$$

For a matrix $A = (a_{ij})_{i,j=1}^N$, we denote by Tr the un-normalized trace and by tr the normalized trace,

$$\text{Tr}(A) := \sum_{i=1}^N a_{ii}, \quad \text{tr}(A) := \frac{1}{N} \text{Tr}(A).$$

For an $n \in \mathbb{N}$, we will denote by $P(n)$ the set of *partitions* of $[1, n]$, i.e., $\sigma = \{B_1, \dots, B_r\} \in P(n)$ is a decomposition of $[1, n]$ into disjoint subsets B_i : $B_i \neq \emptyset$ for $i = 1, \dots, r$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and

$$[1, n] = \bigcup_{i=1}^r B_i.$$

The elements B_i of σ will be addressed as *blocks* of σ .

Given a mapping $i : [1, n] \rightarrow [1, N]$, the *kernel*, $\ker(i)$, is defined as the partition of $[1, n]$, such that two numbers $k, l \in [1, n]$ belong to the same block if and only if $i(k) = i(l)$.

If we are considering classical random variables on some probability space, then we denote by \mathbb{E} the expectation with respect to the corresponding probability measure and by k_r the corresponding classical cumulants (as multi-linear functionals in r arguments); in particular,

$$k_1\{a\} = \mathbb{E}\{a\} \quad \text{and} \quad k_2\{a_1, a_2\} = \mathbb{E}\{a_1 a_2\} - \mathbb{E}\{a_1\}\mathbb{E}\{a_2\}.$$

2.2. Annular non-crossing permutations and partitions. The leading asymptotics of various random matrix quantities can be described in terms of special “planar” objects (see, e.g., [Eyn, Zvo]). There are two equivalent ways of formulating these results: a geometric “genus”-expansion, expressed by a sum over surfaces where the planar part corresponds then to sums over surfaces of genus zero; an algebraic description, where instead of using surfaces one can sum over permutations and planarity is then a geodesic-like condition on a length-function of these permutations. If one prefers to associate partition like pictures with permutations, then planarity is a condition that these partitions have non-crossing diagrams (where, however, one has to be careful about which drawings are allowed).

We prefer to think in terms of permutations and partitions. Let us recall the relevant definitions and results.

Let, for $r \geq 1$, natural numbers $n(1), \dots, n(r)$ be fixed. Consider a partition $\sigma \in P(n(1) + \dots + n(r))$. In [MN], the class of “multi-annular non-crossing partitions” $NC(n(1), \dots, n(r))$ was defined and, for $r = 2$ (“annular” case), an extensive study of various characterizations of such non-crossing partitions was made. We will not go into details here, but we only want to state the characterization which we will use. It will be the case $r = 2$ which is relevant for us; so let us use the notation $NC(n, m)$ for the (n, m) -annular non-crossing partitions. It is a good picture to think of two concentric circles, with n points on the outer and with m points on the inner. We put $[1, n]$ in clockwise order on the outer circle and $[n + 1, n + m]$ in counter-clockwise order on the inner one. Adopting this convention will require that in some of our formulas the indices corresponding to the outer circle run in the opposite direction from the indices on the inner circle.

Figure 1. On the left is the annular non-crossing permutation $(1, 2, 12, 9, 8)(3, 4)(5, 10, 11)(6)(7)$. On the right is the permutation after the blocks that are contained in one circle have been removed.

Consider a $\sigma \in P(n+m)$. We shall give a recursive procedure for deciding if σ is annular non-crossing. Suppose σ has a block which is contained either in $[1, n]$ or in $[n+1, n+m]$ and which consists of cyclically consecutive numbers; then we remove this block and repeat the process until we get a partition $\sigma' \in P(n'+m')$ with no blocks which are contained in either $[1, n]$ or $[n+1, n+m]$ and which consist of cyclically consecutive elements. Then, by definition, σ will be in $NC(n, m)$ if and only if $n', m' \geq 1$ and $\sigma' \in NC(n', m')$. Thus it suffices to say when σ is in $NC(n, m)$ for σ with no blocks which are contained in either $[1, n]$ or $[n+1, n+m]$ and which consist of cyclically consecutive elements.

The characterizing property of such σ is the following: If we write the blocks $B \in \sigma$ in the form $B = B' \cup B''$, where $B' \subset [1, n]$ and $B'' \subset [n+1, n+m]$, then, for all blocks B of σ , both parts, B' and B'' must be non-empty and each of them must consist of cyclically consecutive numbers. Furthermore, the cyclic order of the restrictions, B'_1, B'_2, \dots, B'_k , of the blocks of σ to the interval $[1, n]$ must be the reverse of the cyclic order of the restrictions, $B''_1, B''_2, \dots, B''_k$, of the blocks to the interval $[n+1, n+m]$. Note that this characterization contains the statement that a $\sigma \in NC(n, m)$ is connected in the sense that at least one block of σ contains elements both from $[1, n]$ and from $[n+1, n+m]$ (i.e., σ connects the two circles).

In the context of random matrices, it is permutations, not partitions, which appear in calculations. In order to go over from partitions σ to permutations π one has to choose a cyclic order on each of the blocks of σ . Choosing such an order for each block will produce an “annular non-crossing permutation” out of an annular non-crossing partition. The set of annular non-crossing permutations is denoted by $S_{NC}(n, m)$ – and by $S_{NC}(n(1), \dots, n(r))$ in the multi-annular case – and it is this set which was the main object of interest in [MN]. In [MN, §6] it was shown that a permutation τ is in $S_{NC}(n, m)$ if two conditions are satisfied. The first condition is *connectedness*: at least one cycle of τ connects the two circles and the second is *planarity*: the *geodesic condition* must be satisfied: $\#(\pi) + \#(\pi^{-1}\gamma) = m + n$, where $\#(\pi)$ denotes the number of cycles of π and γ is the permutation with two cycles: $\gamma = \gamma_{n,m} = (1, 2, 3, \dots, n)(n+1, \dots, n+m)$

We wish to describe what it means to be a non-crossing permutation on an r multi-annulus. By an r multi-annulus we mean a collection of r circles with $n(1)$ points on the first circle, $n(2)$ points on the second circle, \dots , $n(r)$ points on the r^{th} circle. Connectivity of τ means that every pair of circles is connected by at least one cycle of τ . The planarity of τ is defined using a geodesic condition. Let $\gamma_{n(1),n(2),\dots,n(r)}$ be the permutation of $[n(1) + \dots + n(r)]$ with r cycles — the r^{th} cycle being $(n(1) + \dots + n(r-1) + 1, \dots, n(1) + \dots + n(r))$. τ will be planar if τ satisfies the geodesic condition $\#(\tau) + \#(\gamma_{n(1),n(2),\dots,n(r)}\tau^{-1}) = n(1) + \dots + n(r) + 2 - r$.

As observed in [MN], there is not necessarily a unique choice of a cyclic order on a block of σ ; to put it another way, the mapping from π to σ (which consists in forgetting the order on the cycles) is not injective. However, this deviation from injectivity is not too bad. Let us consider a block $B \in \sigma$, and denote by $B' := B \cap [1, n]$ and $B'' := B \cap [n+1, n+m]$ the parts of B lying on the first and on the second circle, respectively. On each of the two circles we respect the given cyclic order on $(1, \dots, n)$ and on $(n+1, \dots, n+m)$. The allowed orders on B thus consist of choosing a ‘first’ element of B' and a ‘first’ element of B'' ; then the order on B is obtained by running through B' from the first to the last element, then going over to the first element in B'' and continuing in B'' to the last element. Hence the only freedom we have is the choice of first elements in B' and in B'' .

Let us call a block $B \in \sigma$ a *through-block*, if both $B' := B \cap [1, n]$ and $B'' := B \cap [n+1, n+m]$ are non-empty. Then only σ with exactly one through-block have two or more π 's in $S_{NC}(n, m)$ as preimages. Namely, if $B = B' \cup B''$ is the unique through-block of such a σ , then every element from B' can be chosen as first element, and the same for B'' , thus there are exactly $|B'| \cdot |B''|$ possible choices of cyclic orders for B . If, however, there is more than one through-block, then the first element on each component of them is uniquely determined and there is exactly one possible order for each block.

For Gaussian random matrices only non-crossing pairings will play a role. These are those annular non-crossing partitions for which each block consists of exactly two elements. One should note that in this case the distinction between permutations and partitions vanishes, because for pairings there is always exactly one possibility for putting an order on blocks. We will denote the set of annular non-crossing pairings by $NC_2(n, m)$; and, for the multi-annular situation, by $NC_2(n(1), \dots, n(r))$. In the multi-annular case the geodesic condition can be written $\#(\gamma_{n_1, \dots, n_r} \pi) = 2 - r + (n_1 + \dots + n_r)/2$.

3. COMBINATORIAL DESCRIPTION OF GLOBAL FLUCTUATIONS

We are interested in the fluctuations of various types of $N \times N$ random matrices around their large N -limit. Here, we are going to consider two classes of random matrices, namely Gaussian random matrices and (a generalization of) Wishart matrices. Let us fix the notation for our investigations.

3.1. Semi-circular case. Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of $N \times N$ Hermitian Gaussian random matrices. Then, in the limit $N \rightarrow \infty$, X_N converges to a semi-circular variable s . Let us consider directly the case of several such Gaussian random matrices. The entries of different random matrices need not be independent from each other, but they have to form a Gaussian family. A convenient way to describe such a situation is to index the matrices by elements from some real Hilbert space $\mathcal{H}_{\mathbb{R}}$, such that the covariance between entries from $X_N(f)$ and $X_N(g)$ is given by the inner product $\langle f, g \rangle$. More precisely we say that $\{X_n(f)\}_{f \in \mathcal{H}_{\mathbb{R}}}$ is a *family* of Hermitian Gaussian random matrices if $X_N(f) = (x_{i,j}(f))_{i,j=1}^N$ and the entries $\{x_{i,j}(f) \mid 1 \leq i, j \leq N, f \in \mathcal{H}_{\mathbb{R}}\}$ form a Gaussian family with covariance given by

$$E\{x_{ij}(f)x_{kl}(g)\} = 0 \text{ for } i < j, k < l, \text{ and } f, g \in \mathcal{H}_{\mathbb{R}} \text{ and}$$

$$E\{x_{ij}(f)\bar{x}_{kl}(g)\} = \delta_{ik}\delta_{jl} \cdot \frac{1}{N} \langle f, g \rangle \text{ for } i \leq j, k \leq l, \text{ and } f, g \in \mathcal{H}_{\mathbb{R}}$$

By Wick's formula (see e.g. [J, Sections 1.3 and 1.4]) we have

$$E\{x_{i_1, j_1}(f_1)x_{i_2, j_2}(f_2) \cdots x_{i_{2k}, j_{2k}}(f_{2k})\} = \sum_{\pi \in \mathcal{P}_2(2k)} \prod_{(r, s) \in \pi} \langle x_{i_r, j_r}(f_r), x_{i_s, j_s}(f_s) \rangle$$

where the sum is over all pairings π of $[2k]$ and the contribution of each pairing is the product of $\langle x_{i_r, j_r}(f_r), x_{i_s, j_s}(f_s) \rangle$ over all pairs (r, s) of π .

Then, Voiculescu's extension of Wigner's theorem to this multi-dimensional case states that, for $N \rightarrow \infty$, such a family of random matrices converges to a semi-circular system with the same covariance. We want to look more closely on that convergence and investigate the "global fluctuations" around this semi-circular limit; this means, we want to understand the asymptotic behaviour of traces of products of our random matrices. It turns out that, with the right scaling with N , these random variables converge to a Gaussian family and thus the main information about them is contained in their covariance.

If one invokes the usual genus expansions for expectations of Gaussian random matrices then one gets quite easily the following theorem. It turns out that the leading orders are given by planar pairings. Since

we are looking on cumulants and not just moments, the relevant pairings also have to connect the r circles.

Theorem 3.1. *Let $X_N(f)$ ($f \in \mathcal{H}_{\mathbb{R}}$) be a family of Hermitian Gaussian random matrices. Let k_r denote the r^{th} classical cumulant (considered as multi-linear mapping of r arguments), then for $f_1, \dots, f_{n_1+\dots+n_r} \in \mathcal{H}_{\mathbb{R}}$, the leading order of the cumulants of the random variables*

$$(1) \quad \left\{ \text{Tr}(X_N(f_1) \cdots X_N(f_{n(1)})), \dots, \right. \\ \left. \text{Tr}(X_N(f_{n(1)+\dots+n(r-1)+1}) \cdots X_N(f_{n(1)+\dots+n(r)})) \right\}$$

are given by

$$(2) \quad k_r \left\{ \text{Tr}[X_N(f_1) \cdots X_N(f_{n(1)})], \dots, \right. \\ \left. \text{Tr}[X_N(f_{n(1)+\dots+n(r-1)+1}) \cdots X_N(f_{n(1)+\dots+n(r)})] \right\} \\ = N^{2-r} \cdot \sum_{\pi \in NC_2(n(1), \dots, n(r))} \prod_{(i,j) \in \pi} \langle f_i, f_j \rangle + O(N^{-r}).$$

Proof: Let $n = n(1) + \dots + n(r)$, γ be the permutation of $[n_1 + \dots + n_r]$ with the r cycles $(1, \dots, n_1)(n_1 + 1, \dots, n_1 + n_2) \cdots (n_1 + \dots + n_{r-1} + 1, \dots + n_1 + \dots + n_r)$, and $Y_i = \text{tr}(X_N(f_{n(1)+\dots+n(i-1)+1}) \cdots X_N(f_{n(1)+\dots+n(i)}))$. Then by Wick's formula

$$\begin{aligned} \mathbb{E}(Y_1 \cdots Y_r) &= \sum_{i_1, \dots, i_n=1}^N \mathbb{E}(x_{i_1, i_{\gamma(1)}}(f_1) \cdots x_{i_n, i_1}(f_n)) \\ &= \sum_{i_1, \dots, i_n=1}^N \sum_{\pi \in \mathcal{P}_2(n)} \prod_{(k,l) \in \pi} \langle f_k, f_l \rangle \delta_{i_k, i_{\gamma(l)}} \delta_{i_l, i_{\gamma(k)}} \\ &= \sum_{\pi \in \mathcal{P}_2(n)} \prod_{(k,l) \in \pi} \langle f_k, f_l \rangle \sum_{i_1, \dots, i_n=1}^N \delta_{i_k, i_{\gamma(l)}} \delta_{i_l, i_{\gamma(k)}} \\ &= \sum_{\pi \in \mathcal{P}_2(n)} \prod_{(k,l) \in \pi} \langle f_k, f_l \rangle N^{\#(\gamma\pi)} \end{aligned}$$

Following the argument in [MN, proof of Proposition 9.3] we have

$$k_r(Y_1, \dots, Y_r) = \sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \pi \text{ is connected}}} \prod_{(k,l) \in \pi} \langle f_k, f_l \rangle N^{\#(\gamma\pi)}$$

The terms of highest order are the planar ones thus we obtain equation (2). \square

This theorem contains all relevant combinatorial information about the asymptotic behaviour of our traces. Since an increase of the number of arguments of the cumulants corresponds to a decrease in the exponent of N , a cumulant k_r will always dominate a cumulant k_p if $r < p$. So in leading order only the first cumulant survives in the limit, which gives us the following statement analogous to the law of large numbers.

Corollary 3.2. *For each $f_1, \dots, f_n \in \mathcal{H}_{\mathbb{R}}$, the random variables*

$$\left\{ \text{tr} \left(X_N(f_1) \cdots X_N(f_n) \right) \right\}_N$$

converge in distribution to the constant random variables $\alpha(f_1, \dots, f_n) \cdot 1$, where

$$\alpha(f_1, \dots, f_n) = \sum_{\pi \in NC_2(n)} \prod_{(i,j) \in \pi} \langle f_i, f_j \rangle.$$

This corollary is of course just a reformulation of Voiculescu's result that $E\{\text{tr}[X_N(f_1) \cdots X_N(f_n)]\}$ has, in the limit $N \rightarrow \infty$, to agree with the corresponding moments of a semi-circular family.

But we can now go a step further. If we subtract the mean of the random variables, then the first cumulants are shifted to zero and it will be the second cumulants which survive – after the right rescaling. Since higher cumulants vanish compared to the second ones, we get Gaussian variables in the limit.

Corollary 3.3. *Consider the (magnified) fluctuations around the limit expectation,*

$$\begin{aligned} F_N(f_1, \dots, f_n) &:= N \cdot \left(\text{tr}[X_N(f_1) \cdots X_N(f_n)] - \alpha(f_1, \dots, f_n) \right) \\ &= \text{Tr}[X_N(f_1) \cdots X_N(f_n)] - N\alpha(f_1, \dots, f_n). \end{aligned}$$

The family of all fluctuations $(F_N(f_1, \dots, f_n))_{n \in \mathbb{N}, f_i \in \mathcal{H}_{\mathbb{R}}}$ converges in distribution towards $(F(f_1, \dots, f_n))_{n \in \mathbb{N}, f_i \in \mathcal{H}_{\mathbb{R}}}$, a centered Gaussian family with covariance given by

$$E\{F(f_1, \dots, f_n) \cdot F(f_{n+1}, \dots, f_{n+m})\} = \sum_{\pi \in NC_2(n,m)} \prod_{(i,j) \in \pi} \langle f_i, f_j \rangle.$$

Our goal now is to present a conceptual understanding of this form of the covariance; in particular one that would easily diagonalize it. In principle, this is a purely combinatorial problem. However, our point of view is that limits of random matrices which have the flavour of free combinatorics should also have a description in terms of the operator side of free probability, i.e., operators on full Fock spaces.

We will provide such a description and show that it diagonalizes our covariance.

3.2. Compound Poisson case. Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of $N \times N$ complex Gaussian random matrices (i.e. the entries of X_N are independent centered complex Gaussians with variance $1/N$) and let $(D_N)_{n \in \mathbb{N}}$ be a sequence of $N \times N$ non-random matrices for which a limit distribution exists as $N \rightarrow \infty$. Then, in the limit $N \rightarrow \infty$, $\{X_N, X_N^*, D_N\}$ converges in distribution to $\{c, c^*, d\}$, where c is a circular element, d has the limit distribution of the D_N , and $\{c, c^*\}$ and d are free. In particular, $X_N^* D_N X_N$ converges to $c^* d c$, which is a free compound Poisson element, see [Sp2, 4.4]. We shall discuss the fluctuations of the random matrices

$$P_N := X_N^* D_N X_N$$

around the limit $c^* d c$. Since P_N is a generalization of a Wishart matrix, we will call it in the following a *compound Wishart matrix*.

As we shall see it is appropriate to consider a more general situation. Namely, consider not just a single non-random matrix D_N , but also all its powers D_N^k at the same time, or more generally, let us consider a family $\{D_1^{(N)}, D_2^{(N)}, D_3^{(N)}, \dots, D_p^{(N)}\}_N$ of $N \times N$ complex matrices. We shall say the family *converges in distribution* if there are operators $d_1, d_2, d_3, \dots, d_p$ and a tracial state ψ on \mathcal{D} , the complex $*$ -algebra generated by $\{1, d_1, d_2, \dots, d_p\}$, such that

$$\lim_{N \rightarrow \infty} \text{tr}[D_{i_1}^{(N)} \cdots D_{i_k}^{(N)}] = \psi(d_{i_1} \cdots d_{i_k})$$

for all i_1, i_2, \dots, i_k .

We are again interested in global fluctuations of these matrices in the limit $N \rightarrow \infty$; i.e., we want to consider the asymptotic behaviour of mixed moments of our random matrices. Again the key point is the understanding of the leading order of the cumulants in these traces. This leading order is described by summing over non-crossing permutations, but in contrast to the semi-circular case, all permutations contribute, not just pairings. In order to describe the contribution of such a general non-crossing permutation, we need the following notation.

Notation 3.4. Let (\mathcal{A}, ψ) be a unital algebra with a tracial state ψ ; for each $\pi \in S_p$ we shall define a p -linear functional, ψ_π , on $\mathcal{A} \times \cdots \times \mathcal{A}$. Write $\pi = c_1 \cdot c_2 \cdots c_k$ as a product of disjoint cycles, and for each i , $c_i = (r_{i,1}, \dots, r_{i,l_i})$. Then define the p -linear functional ψ_π by

$$\psi_\pi(a_1, a_2, a_3, \dots, a_p) = \prod_{i=1}^k \psi(a_{r_{i,1}} \cdots a_{r_{i,l_i}})$$

Note that we need ψ to be a trace, because a cycle c comes only with a cyclic order.

An example of this notation is the following, take

$$\pi = \{(1, 2, 6), (3, 4, 5)\} \in S_{NC}(3, 3).$$

Then

$$\psi_\pi(a_1, a_2, a_3, a_4, a_5, a_6) = \psi(a_1 a_2 a_6) \cdot \psi(a_3 a_4 a_5).$$

Note also that the cyclic order is important. In $S_{NC}(2, 1)$ consider

$$\pi_1 = \{(1, 2, 3)\} \quad \text{and} \quad \pi_2 = \{(1, 3, 2)\}.$$

Although their block structure is the same, as permutations they are different elements from $S_{NC}(2, 1)$ and we have

$$\psi_{\pi_1}(a_1, a_2, a_3) = \psi(a_1 a_2 a_3) \quad \text{and} \quad \psi_{\pi_2}(a_1, a_2, a_3) = \psi(a_1 a_3 a_2).$$

We shall denote the number of cycles in the permutation π by $\#(\pi)$.

Let us now state the basic combinatorial description of the leading order of cumulants in traces of products of our compound Wishart matrices. For the usual Wishart matrices this was derived in [MN]. Our more general version follows by the same kind of calculations (*c.f.* Capitaine and Casalis [CC, §5]).

Theorem 3.5. *Let $\{X_N\}_N$ be a sequence of complex Gaussian random matrices. Put*

$$P_N(D_i) := X_N^* D_i^{(N)} X_N$$

Let $(k_r)_{r \in \mathbb{N}}$ denote the classical cumulants, then we have for all $r \in \mathbb{N}$

$$(3) \quad \lim_N N^{r-2} k_r \left\{ \text{Tr}[P_N(D_1) \cdots P_N(D_{n_1})], \dots, \right. \\ \left. \text{Tr}[P_N(D_{n_1+\dots+n_{r-1}+1}) \cdots P_N(D_{n_1+\dots+n_r})] \right\} \\ = \sum_{\pi \in S_{NC}(n(1), \dots, n(r))} \psi_\pi(d_1, \dots, d_{n_1+\dots+n_r})$$

Proof: Let $n = n_1 + \dots + n_r$ and let γ be the permutation with r cycles: $(1, \dots, n_1)(n_1 + 1, \dots, n_1 + n_2) \cdots (n_1 + \dots + n_{r-1} + 1, \dots, n_1 + \dots + n_r)$. Let

$$Y_l = \text{Tr}(P_N(D_{n_1+\dots+n_{l-1}+1}) \cdots P_N(D_{n_1+\dots+n_l}))$$

By [GLM, Theorem 2]

$$(4) \quad \mathbb{E}(Y_1 \cdots Y_r) = \sum_{\sigma \in S_n} N^{\#(\sigma^{-1}\gamma) - n} \text{Tr}_\sigma(D_1, D_2, \dots, D_n)$$

For $\sigma \in S_n$ let $\sigma \vee \gamma$ be the partition of $[n]$ whose blocks are the orbits of the group generated by σ and γ . $\sigma \vee \gamma$ also defines a partition of the

r cycles of γ . Let us denote this partition of $[r]$ by $A_\gamma(\sigma)$. Conversely let s be the number of cycles of σ , $\sigma \vee \gamma$ determines a partition of the cycles of σ ; we shall denote this by $A_\sigma(\gamma)$. Note that if $A_\gamma(\sigma) = 1_r$ then $\sigma \vee \gamma = 1_n$ and thus $A_\sigma(\gamma) = 1_s$.

For a partition $A = \{A_1, \dots, A_k\}$ of $[r]$ let

$$E_A(Y_1, \dots, Y_r) = \prod_{l=1}^k E\left(\prod_{i \in A_l} Y_i\right)$$

If $A = 1_r$ then $E_A(Y_1, \dots, Y_r) = E(Y_1 \cdots Y_r)$. Equation (4) can now be extended easily to obtain

$$E_A(Y_1, \dots, Y_r) = \sum_{\substack{\sigma \in S_n \\ A_\gamma(\sigma) \leq A}} N^{\#(\sigma^{-1}\gamma) - n} \text{Tr}_\sigma(D_1, \dots, D_n)$$

Let $\mu(A, B)$ be the Möbius function of the lattice of partitions; in particular $\mu(A, 1_r) = (-1)^{\#(A)-1} (\#(A) - 1)!$. Note that

$$\sum_{B \leq A \leq 1_r} \mu(A, 1_r) = \begin{cases} 1 & B = 1_r \\ 0 & B < 1_r \end{cases}$$

$$\begin{aligned} & k_r(Y_1, \dots, Y_r) \\ &= \sum_{A \in \mathcal{P}(r)} \mu(A, 1_r) E_A(Y_1, \dots, Y_r) \\ &= \sum_{A \in \mathcal{P}(r)} \mu(A, 1_r) \sum_{\substack{\sigma \in S_n \\ A_\gamma(\sigma) \leq A}} N^{\#(\sigma^{-1}\gamma) - n} \text{Tr}_\sigma(D_1, \dots, D_n) \\ &= \sum_{\sigma \in S_n} N^{\#(\sigma^{-1}\gamma) - n} \text{Tr}_\sigma(D_1, \dots, D_n) \sum_{\substack{A \in \mathcal{P}(r) \\ A_\gamma(\sigma) \leq A}} \mu(A, 1_r) \\ &= \sum_{\substack{\sigma \in S_n \\ \sigma \vee \gamma = 1_n}} N^{\#(\sigma^{-1}\gamma) - n} \text{Tr}_\sigma(D_1, \dots, D_n) \\ &= \sum_{\substack{\sigma \in S_n \\ \sigma \vee \gamma = 1_n}} N^{\#(\sigma^{-1}\gamma) - n} \text{Tr}_\sigma(D_1, \dots, D_n) \end{aligned}$$

Recall that for $\sigma \in S_n$ with $\sigma \vee \gamma = 1_n$ there is an integer $g = g(\sigma)$ such that $\#(\sigma) + \#(\sigma^{-1}\gamma) + \#(\gamma) = n + 2(1 - g)$ and that $\sigma \in S_{NC}(n_1, \dots, n_r)$ means that $\sigma \vee \gamma = 1_n$ and $g(\sigma) = 0$.

$$\begin{aligned}
 k_r(Y_1, \dots, Y_r) &= N^{-n} \sum_{\substack{\sigma \in S_n \\ \sigma \vee \gamma = 1_n}} N^{\#(\sigma^{-1}\gamma)} N^{\#(\sigma)} \text{tr}_\sigma(D_1, \dots, D_n) \\
 &= \sum_{\substack{\sigma \in S_n \\ \sigma \vee \gamma = 1_n}} N^{2-r-2g(\sigma)} \text{tr}_\sigma(D_1, \dots, D_n) \\
 &= N^{2-r} \sum_{\sigma \in S_{NC}(n_1, \dots, n_r)} \text{tr}_\sigma(D_1, \dots, D_n) + O(N^{-r})
 \end{aligned}$$

Since $\lim_N \text{tr}_\sigma(D_1, \dots, D_n) = \psi_\sigma(d_1, \dots, d_n)$ we have the required result. \square

This theorem contains again all relevant information about the limit behaviour of the random variables $\text{Tr}(P_N(D_{i_1}) \cdots P_N(D_{i_n}))$. First, we have the following statement analogous to the law of large numbers.

Corollary 3.6. *The random variables $\{\text{tr}[P_N(D_{i_1}) \cdots P_N(D_{i_n})]\}_{i_1, \dots, i_n}$ converge in distribution to constant random variables $\beta(d_{i_1}, \dots, d_{i_n}) \cdot 1$, where*

$$(5) \quad \beta(d_{i_1}, \dots, d_{i_n}) := \sum_{\pi \in NC(n)} \psi_\pi(d_{i_1}, \dots, d_{i_n}).$$

The form of $\beta(d_{i_1}, \dots, d_{i_n})$ is, of course, in agreement with the fact that $E\{\text{tr}[P_N(D_{i_1}) \cdots P_N(D_{i_n})]\}$ has, in the limit $N \rightarrow \infty$, to agree with the corresponding moment of the compound free Poisson variables,

$$\psi(c^* d_{i_1} c \cdot c^* d_{i_2} c \cdots c^* d_{i_n} c)$$

where c is a circular random variable $*$ -free from $\{d_1, \dots, d_p\}$. Again, we magnify the fluctuations around that limit, thus shifting the first cumulants to zero and getting only a non-vanishing limit for the second cumulants – hence getting normal limit fluctuations.

Corollary 3.7. *Consider the (magnified) fluctuations around the limit value,*

$$\begin{aligned}
 (6) \quad F_N(D_{i_1}, \dots, D_{i_n}) &:= N \cdot \left(\text{tr}[P_N(D_{i_1}) \cdots P_N(D_{i_n})] - \beta(d_{i_1}, \dots, d_{i_n}) \right) \\
 &= \text{Tr}[P_N(D_{i_1}) \cdots P_N(D_{i_n})] - N\beta(d_{i_1}, \dots, d_{i_n}).
 \end{aligned}$$

The family of all fluctuations $(F_N(D_{i_1}, \dots, D_{i_n}))_{n \in \mathbb{N}}$ converges in distribution towards a centered Gaussian family $\left((F(d_{i_1}, \dots, d_{i_n}))_{i_1, \dots, i_n} \right)$,

with covariance given by

$$(7) \quad E\{F(d_{i_1}, \dots, d_{i_m}) \cdot F(d_{i_{m+1}}, \dots, d_{i_{m+n}})\} \\ = \sum_{\pi \in S_{NC}(m, n)} \psi_{\pi}(d_{i_1}, \dots, d_{i_{m+n}}).$$

Again, it remains to understand this covariance and we will be aiming at a more operator-algebraic description of these fluctuations in order to attack this combinatorial problem.

4. REALIZATION OF SEMI-CIRCULAR AND FREE COMPOUND POISSON ELEMENTS ON FOCK SPACES

The main theme of our investigations is the conviction that wherever planar or non-crossing objects arise, there is some free probability lurking behind the picture. Since the fluctuations of our Gaussian and Wishart random matrices can be described combinatorially in terms of non-crossing permutations, we expect also some operator-algebraic or some more abstract “free” description of this situation. Our main results in the coming sections will provide these descriptions. Let us begin by recalling the realization of a semi-circle and a compound Poisson distribution on a full Fock space by using creation, annihilation, and preservation operators.

4.1. Semi-circular case. For a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with complexification \mathcal{H} , we consider the full Fock space

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} = \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \dots$$

and define, for $f \in \mathcal{H}$, the *creation operator* $l(f)$ by

$$l(f)\Omega = f$$

and

$$l(f)f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n$$

and the *annihilation operator* $l^*(f)$, by

$$l^*(f)\Omega = 0$$

and

$$l^*(f)f_1 \otimes \dots \otimes f_n = \langle f_1, f \rangle f_2 \otimes \dots \otimes f_n$$

($n \in \mathbb{N}$, $f_1, \dots, f_n \in \mathcal{H}$).

For $f \in \mathcal{H}_{\mathbb{R}}$, we put

$$\omega(f) := l(f) + l^*(f)$$

and we will denote by $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$ the complex unital $*$ -algebra generated by all $\omega(f)$ for $f \in \mathcal{H}_{\mathbb{R}}$. Note that all $\omega(f)$ are self-adjoint and that the vector Ω is cyclic and separating for the algebra $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$.

If we define on \mathcal{H} an involution $f \mapsto \bar{f}$ by

$$\overline{f_1 + if_2} := f_1 - if_2 \quad \text{for } f_1, f_2 \in \mathcal{H}_{\mathbb{R}},$$

then $f \mapsto \omega(f)$ extends from a real linear mapping on $\mathcal{H}_{\mathbb{R}}$ to a complex linear mapping on \mathcal{H} with

$$\omega(f) = l(f) + l^*(\bar{f}) \quad (f \in \mathcal{H}).$$

Note that the unital $*$ -algebra generated by all $\omega(f)$ with $f \in \mathcal{H}$ is just $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$.

It is well known (see, e.g., [VDN]) that these operators $\omega(f)$ have a semi-circular distribution and thus the asymptotics of the expectation of traces of Gaussian random matrices can also be stated as follows.

Proposition 4.1. *Let $X_N(f)$ ($f \in \mathcal{H}_{\mathbb{R}}$) be a family of Hermitian Gaussian random matrices. Then for all $f_1, \dots, f_n \in \mathcal{H}_{\mathbb{R}}$*

$$(8) \quad \lim_{N \rightarrow \infty} E\{\text{tr}[X_N(f_1) \cdots X_N(f_n)]\} = \langle \omega(f_1) \cdots \omega(f_n) \Omega, \Omega \rangle.$$

Let us in this context also recall the definition of the Wick products.

Definition 4.2. For $f_1, \dots, f_n \in \mathcal{H}$ the *Wick product* $W(f_1 \otimes \cdots \otimes f_n)$ is the unique element of $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$ such

$$(9) \quad W(f_1 \otimes \cdots \otimes f_n) \Omega = f_1 \otimes \cdots \otimes f_n.$$

For $n = 0$, this has to be understood as $W(\Omega) = 1$.

Since Ω is cyclic and separating for $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$, these Wick products exist and are uniquely determined.

From the definition of the creation and annihilation operators it is clear that these Wick products satisfy for all $f, f_1, \dots, f_n \in \mathcal{H}$ the relation

$$\omega(f)W(f_1 \otimes \cdots \otimes f_n) = W(f \otimes f_1 \otimes \cdots \otimes f_n) + \langle f_1, \bar{f} \rangle W(f_2 \otimes \cdots \otimes f_n).$$

This can also be used as a recursive definition for the Wick products and shows that $W(f_1 \otimes \cdots \otimes f_n)$ is a polynomial in $\omega(f_1), \dots, \omega(f_n)$.

In the case $f = f_1 = \cdots = f_n$ this reduces to the three-term recurrence relation for the Chebyshev polynomials and shows that

$$W(f^{\otimes n}) = U_n(\omega(f)/2),$$

where U_n is the n^{th} Chebyshev polynomial of the second kind.

4.2. Compound Poisson case. In this case we start with a unital $*$ -algebra \mathcal{D} equipped with a tracial state ψ and represent \mathcal{D} , via the GNS-representation, on $\mathcal{H} := \overline{\mathcal{D}}^{\langle \cdot, \cdot \rangle}$, where the inner product on \mathcal{H} is given by

$$\langle d_1, d_2 \rangle := \psi(d_2^* d_1).$$

Then we take the full Fock space $\mathcal{F}(\mathcal{H})$ and consider there as before the creation and annihilation operators $l(d)$ and $l^*(d)$, respectively. But now we have, in addition, also to consider the *preservation* (or *gauge*) operator $\Lambda(d)$ ($d \in \mathcal{D}$) which is defined by

$$\Lambda(d)\Omega = 0$$

and

$$\Lambda(d)f_1 \otimes \cdots \otimes f_n = (df_1) \otimes f_2 \otimes \cdots \otimes f_n$$

for $f_1, \dots, f_n \in \mathcal{H}$. Note that the multiplication $\mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ extends to a module action

$$\begin{aligned} \mathcal{D} \times \mathcal{H} &\rightarrow \mathcal{H} \\ (d, f) &\mapsto df. \end{aligned}$$

For $d \in \mathcal{D}$ we define now

$$(10) \quad p(d) := l(d) + l^*(d^*) + \Lambda(d) + \psi(d)1,$$

and we will denote by $\mathcal{A}(\mathcal{D})$ the unital $*$ -algebra generated by all $p(d)$ for $d \in \mathcal{D}$. Note that we have

$$p(d)^* = p(d^*) \quad \text{for } d \in \mathcal{D}.$$

One knows (see, e.g., [GSS, Sp1, NSp]) that these operators $p(d)$ give a realization of compound Poisson elements, i.e., their moments are given by Eq. (5). Thus we can state the asymptotics of the expected value of traces of our compound Wishart matrices also in the following form.

Proposition 4.3. *Suppose the family $\{D_1^{(N)}, \dots, D_p^{(N)}\}$ converges in distribution to $\{d_1, \dots, d_p\}$ in (\mathcal{D}, ψ) . Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of $N \times N$ Hermitian Gaussian random matrices. Then*

$$(11) \quad \lim_{N \rightarrow \infty} E\{tr[P_N(D_{i_1}) \cdots P_N(D_{i_n})]\} = \langle p(d_{i_1}) \cdots p(d_{i_n})\Omega, \Omega \rangle.$$

Again, Wick products will play a role in this context. As before, these should be polynomials in the $\{p(d) \mid d \in \mathcal{D}\}$ with the defining property that

$$W(d_1 \otimes \cdots \otimes d_n)\Omega = d_1 \otimes \cdots \otimes d_n.$$

However, in contrast to the semi-circular case, the multiplication of d 's in the arguments (under the action of Λ) has the effect that in order

to produce counter-terms for $p(d_1) \cdots p(d_n)\Omega$ to get $d_1 \otimes \cdots \otimes d_n$ one also has to involve operators like $p(d_1 d_2)$ etc. This means that $W(d^{\otimes n})$ is in general not just a polynomial in $p(d)$, but some polynomial in all $\{p(d^k) \mid k \leq n\}$. In particular, in general there is no relation between Wick polynomials $W(d^{\otimes n})$ and the orthogonal polynomials with respect to the distribution of $p(d)$. From the point of view of Levy processes the occurrence of $p(d^k)$ in $W(d^{\otimes n})$ is not very surprising, because this corresponds to the higher diagonal measures (Ito-formulas) and a Levy process should come along with its higher variations.

It appears that Anshelevich [Ans] was the first to introduce and investigate these polynomials in this generality (and also some q -deformations thereof). Since these polynomials appear implicitly in the classical case in a paper of Kailath and Segall, he called them free Kailath-Segall polynomials.

By taking into account the action of our operators on the full Fock space one sees quite easily that these Wick products should be defined as follows.

Definition 4.4. For a given algebra \mathcal{D} with state ψ , the *Wick products* or *free Kailath-Segall polynomials* of the corresponding compound Poisson distribution are recursively defined by $(d, d_1, \dots, d_n \in \mathcal{D})$

$$W(d) = p(d) - \psi(d)1$$

and

$$\begin{aligned} W(d \otimes d_1 \otimes \cdots \otimes d_n) &= p(d)W(d_1 \otimes d_2 \otimes \cdots \otimes d_n) \\ &\quad - \psi(dd_1)W(d_2 \otimes d_3 \otimes \cdots \otimes d_n) \\ &\quad - W(dd_1 \otimes d_2 \otimes \cdots \otimes d_n) \\ &\quad - \psi(d)W(d_1 \otimes d_2 \otimes \cdots \otimes d_n) \end{aligned}$$

5. CYCLIC FOCK SPACE

Our main aim is to express the formulas for the limit fluctuations of Gaussian random matrices and of compound Wishart matrices also with the help of the operators $\omega(f)$ and $p(d)$, respectively. In order to do so we have, however, to introduce another variant of a Fock space. Whereas the elements in the full Fock space, $f_1 \otimes \cdots \otimes f_n$, are linear kind of objects – with a beginning and an end – we are looking on traces and thus should identify the beginning and the end in a cyclic way.

Here are two versions of such a cyclic Fock space, the first one over arbitrary Hilbert spaces \mathcal{H} and suited for semi-circular systems, and the second one over an algebra \mathcal{D} and suited for compound Poisson systems.

Since for the calculation of our moments we only have to deal with elements in the algebraic Fock space (without taking a Hilbert space completion), we will restrict ourselves to this case in the following in order to avoid technicalities about unbounded operators.

5.1. Semi-circular case. For a Hilbert space \mathcal{H} , the algebraic full Fock space

$$\mathcal{F}_{\text{alg}}(\mathcal{H}) := \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \dots$$

is generated by tensors $f_1 \otimes \dots \otimes f_n$, where we can think of the f_1, \dots, f_n as being arranged on a linear string. To stress this linear nature of the usual full Fock space, we will address it in the following as *linear Fock space* $\mathcal{F}_{\text{lin}}(\mathcal{H})$. In our tracial context, however, we should consider circular tensors, where we think of the f_1, \dots, f_n as being arranged around a circle. We will denote these circular tensors by $[f_1 \otimes \dots \otimes f_n]$ and the corresponding n -th particle space by $\mathcal{H}_{\text{cyc}}^{\otimes n}$. If we pair two circles, then we have the freedom of rotating them against each other, so the canonical inner product for this situation is given as follows.

Definition 5.1. The *cyclic Fock space* is the algebraic direct sum

$$(12) \quad \mathcal{F}_{\text{cyc}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\text{cyc}}^{\otimes n}$$

equipped with an inner product given by linear extension of

$$(13) \quad \langle [f_1 \otimes \dots \otimes f_n], [g_1 \otimes \dots \otimes g_m] \rangle_{\text{cyc}} := \delta_{nm} \cdot \sum_{k=0}^{n-1} \langle f_1, g_{1+k} \rangle \cdot \langle f_2, g_{2+k} \rangle \cdots \langle f_n, g_{n+k} \rangle,$$

where we count modulo n in the indices of g .

Note that one can also embed the full Fock space into the cyclic one via

$$[f_1 \otimes \dots \otimes f_n] = \frac{1}{\sqrt{n}} \sum_{k=1}^n f_k \otimes f_{k+1} \otimes \dots \otimes f_{k-1}.$$

In order to write our formula for the fluctuations in terms of moments of operators we still use the operators on the linear Fock space, but we will in the end make things cyclic by mapping the linear Fock space onto the cyclic one.

Definition 5.2. We consider the mapping \mathbf{c} between linear and cyclic Fock space,

$$\mathbf{c} : \mathcal{F}_{\text{lin}}(\mathcal{H}) \rightarrow \mathcal{F}_{\text{cyc}}(\mathcal{H}),$$

which is given recursively by

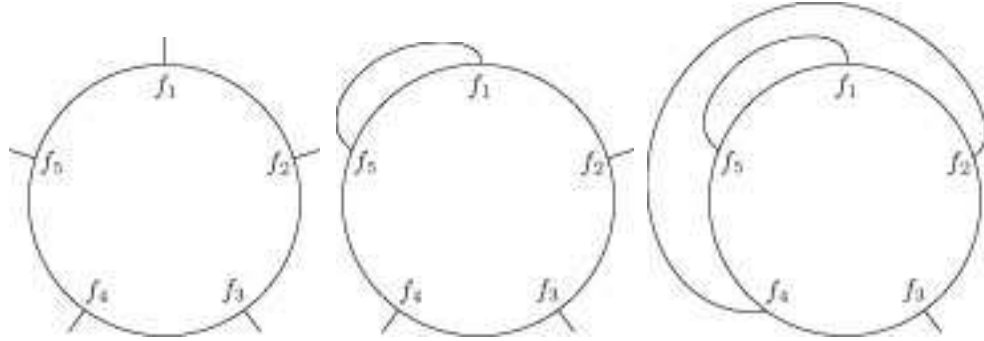
$$(14) \quad \mathbf{c} \Omega = 0, \quad \mathbf{c}(f) = [f],$$

and

$$(15) \quad \mathbf{c}(f_1 \otimes \cdots \otimes f_n) = [f_1 \otimes \cdots \otimes f_n] + \langle f_1, \bar{f}_n \rangle \cdot \mathbf{c}(f_2 \otimes \cdots \otimes f_{n-1})$$



Figure 2.



$$[f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5] \quad \langle f_1, \bar{f}_5 \rangle [f_2 \otimes f_3 \otimes f_4] \quad \langle f_1, \bar{f}_5 \rangle \langle f_2, \bar{f}_4 \rangle [f_3]$$

Figure 3

Figure 4

Figure 5

Illustration of equation (15). Elements of the full Fock space, $f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5$ are represented by linear “half pairings” (figure 2). The operator \mathbf{c} takes a linear half pairing and wraps it around into a circle (figure 3). Then \mathbf{c} pairs off the f 's until either one or none remains (figures 4 and 5). The idea of a half pairing is a special case of a half permutation explained fully in [KMS].

Of course, one can also write down this explicitly, here are just two examples:

$$(16) \quad \mathbf{c}(f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5) = [f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5] \\ + \langle f_1, \bar{f}_5 \rangle \cdot [f_2 \otimes f_3 \otimes f_4] + \langle f_1, \bar{f}_5 \rangle \cdot \langle f_2, \bar{f}_4 \rangle \cdot [f_3]$$

and

$$(17) \quad \mathbf{c}(f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 \otimes f_6) = [f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 \otimes f_6] \\ + \langle f_1, \bar{f}_6 \rangle \cdot [f_2 \otimes f_3 \otimes f_4 \otimes f_5] + \langle f_1, \bar{f}_6 \rangle \cdot \langle f_2, \bar{f}_5 \rangle \cdot [f_3 \otimes f_4]$$

Let us now consider the relation between this cyclic Fock space and fluctuations of Gaussian random matrices. So for the following, let

$X_N(f)$ be our Gaussian random matrices which converge, for $N \rightarrow \infty$, in distribution to a semi-circular family, given by $\omega(f) := l(f) + l^*(f)$ realized on the full Fock space.

Our main point is now that we can express the fluctuations of the Gaussian matrices via the operators $\omega(f)$.

Theorem 5.3. *Let $X_N(f)$ ($f \in \mathcal{H}_{\mathbb{R}}$) be a family of Hermitian Gaussian random matrices. Then for all $n, m \in \mathbb{N}$ and all $f_1, \dots, f_n, g_1, \dots, g_m \in \mathcal{H}_{\mathbb{R}}$*

$$(18) \quad \lim_{N \rightarrow \infty} k_2 \{ \text{Tr}[X_N(f_1) \cdots X_N(f_n)], \text{Tr}[X_N(g_1) \cdots X_N(g_m)] \} \\ = \langle \mathbf{c} \omega(f_1) \cdots \omega(f_n) \Omega, \mathbf{c} \omega(g_m) \cdots \omega(g_1) \Omega \rangle_{\text{cyc}}$$

Note that the inversion of indices in the g 's is forced upon us by the fact that our expression in random matrices is linear in both its traces, whereas our cyclic Fock space inner product is anti-linear in the second argument

Remark 5.4. One might wonder whether the right-hand side of our Eq. (18) should not also have the structure of a variance. This is indeed the case, but is somehow hidden in our definition that $\mathbf{c} \Omega = 0$. If, instead of \mathbf{c} , we use the mapping $\tilde{\mathbf{c}}$, given as follows

$$\tilde{\mathbf{c}} \eta := \mathbf{c} \eta + \langle \eta, \Omega \rangle \Omega,$$

then the right-hand sided of (18) has the form

$$\langle \mathbf{c} \eta_1, \mathbf{c} \eta_2 \rangle_{\text{cyc}} = \langle \tilde{\mathbf{c}} \eta_1, \tilde{\mathbf{c}} \eta_2 \rangle_{\text{cyc}} - \langle \tilde{\mathbf{c}} \eta_1, \Omega \rangle_{\text{cyc}} \cdot \langle \Omega, \tilde{\mathbf{c}} \eta_2 \rangle_{\text{cyc}}.$$

We will prove Theorem 5.3 later as a corollary of a general limit theorem. For the moment, we will be content with checking the consistency of our statement with respect to traciality. Since the left hand side is tracial in the arguments of the traces, the right hand side should be tracial, too. Recall that $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$ is the unital $*$ -algebra generated by all $\omega(f) = l(f) + l^*(f)$ with $f \in \mathcal{H}$.

Lemma 5.5. *The mapping*

$$(19) \quad \mathcal{A}(\mathcal{H}_{\mathbb{R}}) \rightarrow \mathcal{F}_{\text{cyc}}(\mathcal{H})$$

$$(20) \quad a \mapsto \mathbf{c} a \Omega$$

is tracial, i.e., for all $a, b \in \mathcal{A}(\mathcal{H}_{\mathbb{R}})$ we have

$$\mathbf{c} a b \Omega = \mathbf{c} b a \Omega.$$

Proof. Since Ω is cyclic and separating for the algebra $\mathcal{A}(\mathcal{H}_{\mathbb{R}})$, it suffices to show that

$$\mathbf{c} \omega(f) W(f_1 \otimes \cdots \otimes f_n) \Omega = \mathbf{c} W(f_1 \otimes \cdots \otimes f_n) \omega(f) \Omega$$

for all $f, f_1, \dots, f_n \in \mathcal{H}$. On the left side, we have

$$\mathbf{c} \omega(f) W(f_1 \otimes \dots \otimes f_n) \Omega = \mathbf{c} (f \otimes f_1 \otimes \dots \otimes f_n) + \langle f_1, \bar{f} \rangle \cdot \mathbf{c} (f_2 \otimes \dots \otimes f_n).$$

For the right side, it follows from the relation¹ $W(f_1 \otimes \dots \otimes f_n)^* = W(\bar{f}_n \otimes \dots \otimes \bar{f}_1)$ that

$$W(f_1 \otimes \dots \otimes f_n) \omega(f) = W(f_1 \otimes \dots \otimes f_n \otimes f) + \langle f_n, \bar{f} \rangle W(f_1 \otimes \dots \otimes f_{n-1}),$$

which yields

$$\mathbf{c} W(f_1 \otimes \dots \otimes f_n) \omega(f) \Omega = \mathbf{c} (f_1 \otimes \dots \otimes f_n \otimes f) + \langle f_n, \bar{f} \rangle \cdot \mathbf{c} (f_1 \otimes \dots \otimes f_{n-1}).$$

From the definition of \mathbf{c} we see that both sides are the same. \square

5.2. Compound Poisson case. Let us now consider the case where we have a $*$ -algebra \mathcal{D} with trace ψ . We denote by

$$\mathcal{F}_{\text{lin}}(\mathcal{D}) = \bigoplus_{n=0}^{\infty} \mathcal{D}^{\otimes n}$$

the algebraic linear Fock space and by

$$\mathcal{F}_{\text{cyc}}(\mathcal{D}) = \bigoplus_{n=0}^{\infty} \mathcal{D}_{\text{cyc}}^{\otimes n}$$

the algebraic cyclic Fock space.

Since in this case we also have actions of our operators which multiply inside the argument, we have to take this into account when we glue the beginning and end of the tensors together. Thus we have to change the definition of the map \mathbf{c} as follows.

Definition 5.6. We consider the linear mapping

$$\mathbf{c} : \mathcal{F}_{\text{lin}}(\mathcal{D}) \rightarrow \mathcal{F}_{\text{cyc}}(\mathcal{D}),$$

given recursively by

$$\mathbf{c} \Omega := 0 \text{ and } \mathbf{c}(d) = [d]$$

¹As we have been unable to locate a proof of this in the literature one is given in [KMS, §10]. The main idea is to write $\omega(f_1)\omega(f_2)\dots\omega(f_n)$ as a linear combination of Wick polynomials. In fact we may write $\omega(f_1)\omega(f_2)\dots\omega(f_n) = \sum_{\pi} W_{\pi}(f_1 \otimes f_2 \otimes \dots \otimes f_n)$ (*) where the sum is over all non-crossing “half-pairings” of $[n]$, that is non-crossing pairings of $[n]$ with only singletons and pairs such that the singletons are not enclosed by any pair, and $W_{\pi}(f_1 \otimes \dots \otimes f_n) = \langle f_{i_1}, \bar{f}_{j_1} \rangle \dots \langle f_{i_k}, \bar{f}_{j_k} \rangle W(f_{l_1} \otimes \dots \otimes f_{l_m})$ where the pairs of π are $(i_1, j_1), \dots, (i_k, j_k)$ and the singletons are $(l_1), \dots, (l_m)$. Note that in (*) there is only one term involving a Wick polynomial on n f 's. So that if we know that $W(f_1 \otimes \dots \otimes f_m)^* = W(\bar{f}_m \otimes \dots \otimes \bar{f}_1)$ for $m < n$ they we apply induction to the adjoint of (*).

and

$$(21) \quad \mathbf{c}(d_1 \otimes \cdots \otimes d_n) := [d_1 \otimes \cdots \otimes d_n] + [d_n d_1 \otimes d_2 \otimes \cdots \otimes d_{n-1}] \\ + \psi(d_1 d_n) \cdot \mathbf{c}(d_2 \otimes \cdots \otimes d_{n-1})$$

For example

$$(22) \quad \mathbf{c}(d_1 \otimes d_2 \otimes d_3) = [d_1 \otimes d_2 \otimes d_3] + [d_3 d_2 \otimes d_2] + \psi(d_3 d_1)[d_2]$$

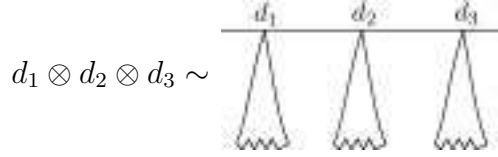


Figure 6.

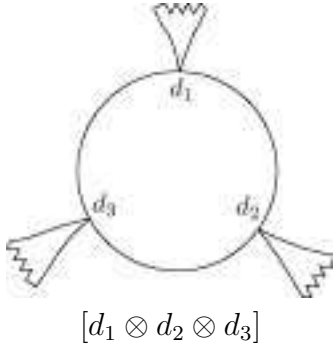


Figure 7.

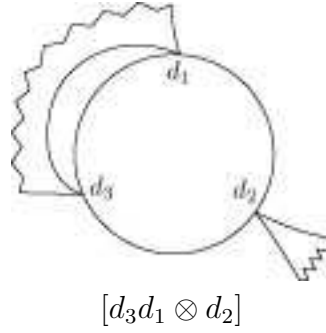


Figure 8.

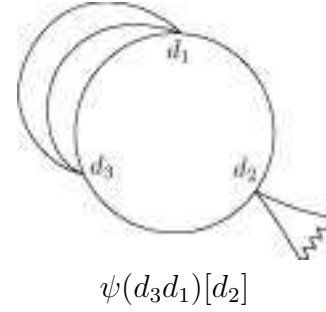


Figure 9.

Illustration of Equation (22) The vector $d_1 \otimes d_2 \otimes d_3$ is represented by a linear half permutation (figure 6) with one open block for each factor in the tensor product (*c.f.* [KMS, §10]). $[d_1 \otimes d_2 \otimes d_3]$ is represented by a circular half permutation with one open block for each factor in the tensor product. $\psi(d_3 d_1)[d_2]$ is represented by a circular half permutation with one closed block and one open block (figure 9). The operator \mathbf{c} first turns the linear half permutation into a circular half permutation (figure 7). Then \mathbf{c} fuses a pair of open blocks (figure 8) and then closes the just formed open block (figure 9). This process continues until either one or zero open blocks remain.

Then we claim that one can express the fluctuations of our compound Wishart matrices also by calculations in terms of the corresponding operators $p(d)$.

Theorem 5.7. *Suppose $\{D_1^{(N)}, \dots, D_p^{(N)}\}$ converges in distribution to $\{d_1, \dots, d_p\}$ in (\mathcal{D}, ψ) and $(X_N)_{N \in \mathbb{N}}$ is a sequence of $N \times N$ Hermitian*

Gaussian random matrices. We put $P_N(D_i) := X_N D_i^{(N)} X_N$ and let $p(d_i)$ be our operators on the full Fock space, then we have for all $m, n \in \mathbb{N}$ that

$$(23) \quad \lim_{N \rightarrow \infty} k_2 \{ \text{Tr}[P_N(D_{i_1}) \cdots P_N(D_{i_n})], \overline{\text{Tr}[P_N(D_{i_{n+1}}) \cdots P_N(D_{i_{n+m}})]} \} \\ = \langle \mathbf{c} p(d_{i_1}) \cdots p(d_{i_n}) \Omega, \mathbf{c} p(d_{i_{n+1}}) \cdots p(d_{i_{n+m}}) \Omega \rangle_{\text{cyc}}.$$

Again, we check only the traciality of the right hand side and postpone the proof of the statement until we have proved our general limit theorem. Recall that we denote by $\mathcal{A}(\mathcal{D})$ the unital $*$ -algebra generated by all $p(d)$ for $d \in \mathcal{D}$.

Lemma 5.8. *The mapping*

$$\begin{aligned} \mathcal{A}(\mathcal{D}) &\rightarrow \mathcal{F}_{\text{cyc}}(\mathcal{D}) \\ a &\mapsto \mathbf{c} a \Omega \end{aligned}$$

is tracial.

Proof. Since Ω is cyclic and separating for $\mathcal{A}(\mathcal{D})$ (see [Ans]), it suffices to check for $d, d_1, \dots, d_n \in \mathcal{D}$ that

$$\mathbf{c} p(d) W(d_1 \otimes \cdots \otimes d_n) \Omega = \mathbf{c} W(d_1 \otimes \cdots \otimes d_n) p(d) \Omega.$$

For $n = 0$, i.e., $W(\Omega) = 1$, this is surely true. In general we have for the left hand side

$$\begin{aligned} \mathbf{c} p(d) W(d_1 \otimes \cdots \otimes d_n) \Omega &= \mathbf{c} p(d) d_1 \otimes \cdots \otimes d_n \\ &= \mathbf{c} (d \otimes d_1 \otimes \cdots \otimes d_n + \psi(dd_1) d_2 \otimes \cdots \otimes d_n \\ &\quad + dd_1 \otimes d_2 \otimes \cdots \otimes d_n + \psi(d) d_1 \otimes \cdots \otimes d_n) \end{aligned}$$

By using the identity² $W(d_1 \otimes \cdots \otimes d_n)^* = W(d_n^* \otimes \cdots \otimes d_1^*)$ we have

$$\begin{aligned} W(d_1 \otimes \cdots \otimes d_n) p(d) &= W(d_1 \otimes \cdots \otimes d_n \otimes d) \\ &\quad + \psi(d_n d) W(d_1 \otimes \cdots \otimes d_{n-1}) \\ &\quad + W(d_1 \otimes \cdots \otimes d_{n-1} \otimes d_n d) \\ &\quad + \psi(d) W(d_1 \otimes \cdots \otimes d_n), \end{aligned}$$

Thus the right hand side becomes

$$\begin{aligned} \mathbf{c} W(d_1 \otimes \cdots \otimes d_n) p(d) \Omega &= \mathbf{c} (d_1 \otimes \cdots \otimes d_n \otimes d \\ &\quad + \psi(d_n d) d_1 \otimes \cdots \otimes d_{n-1} \\ &\quad + d_1 \otimes \cdots \otimes d_{n-1} \otimes d_n d \\ &\quad + \psi(d) d_1 \otimes \cdots \otimes d_n) \end{aligned}$$

²The proof is very similar to that sketched in the footnote on page 21. A detailed proof is provided in [KMS, §10].

So it remains to show that

$$\begin{aligned} & \mathbf{c} \left(d \otimes d_1 \otimes \cdots \otimes d_n + \psi(dd_1)d_2 \otimes \cdots \otimes d_n + dd_1 \otimes d_2 \otimes \cdots \otimes d_n \right) \\ &= \mathbf{c} \left(d_1 \otimes \cdots \otimes d_n \otimes d + \psi(d_n d)d_1 \otimes \cdots \otimes d_{n-1} + d_1 \otimes \cdots \otimes d_{n-1} \otimes d_n d \right) \end{aligned}$$

This can be checked directly by applying the definition of the mapping \mathbf{c} . \square

6. SECOND ORDER FREENESS AND ABSTRACT LIMIT THEOREMS

We shall derive our main theorems, 5.3 and 5.7, from a general limit theorem, very much in the same spirit as one can get the distribution of the semi-circle and the compound free Poisson distributions from free limit theorems, see [Sp1]. The crucial idea is the notion of second order freeness which we introduce below.

Definition 6.1. A *second order non-commutative probability space* $(\mathcal{A}, \varphi, \rho)$ consists of a unital algebra \mathcal{A} , a tracial linear functional

$$\varphi : \mathcal{A} \rightarrow \mathbb{C} \quad \text{with} \quad \varphi(1) = 1$$

and a bilinear functional

$$\rho : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C},$$

which is tracial in both arguments and which satisfies

$$\rho(a, 1) = 0 = \rho(1, b) \quad \text{for all } a, b \in \mathcal{A}.$$

Notation 6.2. Let unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$ be given.

- 1) We say that a tuple (a_1, \dots, a_n) ($n \geq 1$) of elements from \mathcal{A} is *cyclically alternating* if, for each k , we have an $i(k) \in \{1, \dots, r\}$ such that $a_k \in \mathcal{A}_{i(k)}$ and, if $n \geq 2$, we have $i(k) \neq i(k+1)$ for all $k = 1, \dots, n$. We count indices in a cyclic way modulo n , i.e., for $k = n$ the above means $i(n) \neq i(1)$. Note that for $n = 1$ we mean that a_1 is in some \mathcal{A}_i .
- 2) We say that a tuple (a_1, \dots, a_n) of elements from \mathcal{A} is *centered* if we have

$$\varphi(a_k) = 0 \quad \text{for all } k = 1, \dots, n.$$

Definition 6.3. Let $(\mathcal{A}, \varphi, \rho)$ be a second order non-commutative probability space. We say that unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$ are *free with respect to* (φ, ρ) or *free of second order*, if they are free with respect to φ and whenever we have centered and cyclically alternating tuples (a_1, \dots, a_n) and (b_1, \dots, b_m) from \mathcal{A} then we have:

- i) $\rho(a_1 \cdots a_n, b_1 \cdots b_m) = 0$ for $n \neq m$;
- ii) $\rho(a, b) = 0$ for $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, and $i \neq j$;

iii) if $n = m \geq 2$, then

$$\rho(a_1 \cdots a_n, b_1 \cdots b_n) = \sum_{k=0}^{n-1} \varphi(a_1 b_{n+k}) \cdot \varphi(a_2 b_{(n-1)+k}) \cdots \varphi(a_n b_{1+k}).$$

Note that in the sum the indices of the a 's increase, whereas those of the b 's decrease; one should think of two concentric circles with the a 's on one of them and the b 's on the other. However, whereas on one circle we have a clockwise orientation of the points, on the other circle the orientation is counter-clockwise. Thus, in order to match up these points modulo a rotation of the circles, we have to pair the indices as in the sum above.

Condition (*iii*) is the annular version of the disc picture of first order freeness: suppose $a_k \in \mathcal{A}_{i_k}$ with $\phi(a_k) = 0$ for $1 \leq k \leq n$ and we arrange the elements a_1, \dots, a_n around the boundary of a disc (figure 10). The only non-crossing partition of $[n]$ that only connects elements from the same algebra consists of all singletons and since ϕ of a singleton is 0, we have that $\phi(a_1 \cdots a_n) = 0$.

In the annular case we put the centered and cyclically alternating elements (a_1, \dots, a_n) and (b_1, \dots, b_n) around the boundary of an annulus. We only connect elements from the same algebra and as the elements are centered we have no singletons; so we must connect in pairs elements from opposite circles in all possible ways (figure 11). This is the meaning of condition (*iii*).

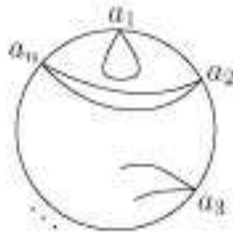


Figure 10.

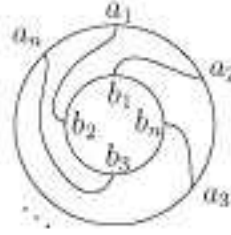


Figure 11.

Note that, as in the case of freeness, the trick of writing elements a as

$$a = a^\circ + \varphi(a) \cdot 1, \quad \text{where } \varphi(a^\circ) = 0,$$

allows us to calculate ρ in terms of φ and ρ restricted to the subalgebras. However, whereas the formulas for φ of mixed moments contain only φ applied to the subalgebras, ρ of mixed moments has in general to be expressed in both φ and ρ restricted to the subalgebras.

For example, assume we have two subalgebras \mathcal{A}_1 and \mathcal{A}_2 , and elements $a_1, a_2 \in \mathcal{A}_1$ and $b_1, b_2 \in \mathcal{A}_2$. Then we have

$$(24) \quad \rho(a_1, b_1) = 0,$$

$$(25) \quad \rho(a_1 b_2, a_2) = \varphi(b_2) \cdot \rho(a_1, a_2)$$

or

$$(26) \quad \begin{aligned} \rho(a_1 b_2, a_2 b_2) &= \varphi(a_1 a_2) \varphi(b_1 b_2) - \varphi(a_1 a_2) \varphi(b_1) \varphi(b_2) \\ &\quad - \varphi(a_1) \varphi(a_2) \varphi(b_1 b_2) + \varphi(a_1) \varphi(a_2) \varphi(b_1) \varphi(b_2) \\ &\quad + \rho(a_1, a_2) \varphi(b_1) \varphi(b_2) + \varphi(a_1) \varphi(a_2) \rho(b_1, b_2). \end{aligned}$$

One should note that each variable appearing in the arguments of ρ on the left-hand side of these examples has to appear exactly once in each product on the right-hand side. Let us formalize this in the following definition.

Notation 6.4. Let $(\mathcal{A}, \varphi, \rho)$ be a second order non-commutative probability space with subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$, and consider elements $a_1, \dots, a_n \in \cup_{i=1}^r \mathcal{A}_i$. A *balanced expression* (with respect to the subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r$) in a_1, \dots, a_n is a product of factors $\varphi(a_{i_1} \cdots a_{i_t})$ and $\rho(a_{i_1} \cdots a_{i_s}, a_{j_1} \cdots a_{j_t})$ where each a_i has to appear exactly once among all arguments and the argument of each φ or the arguments of each ρ contains only a_i 's from a single \mathcal{A}_j .

For example, balanced expressions in a_1, a_2, a_3, a_4 are

$$\varphi(a_1 a_3) \varphi(a_2 a_4) \text{ if } a_1, a_3 \in \mathcal{A}_1 \text{ and } a_2, a_4 \in \mathcal{A}_2$$

$$\text{or } \varphi(a_1) \varphi(a_4) \rho(a_2, a_3) \text{ if } a_1, a_4 \in \mathcal{A}_1 \text{ and } a_2, a_3 \in \mathcal{A}_2$$

Every summand on the right-hand side of Eq. (26) is a balanced expression in a_1, a_2, b_1, b_2 if $a_1, a_2 \in \mathcal{A}_1$ and $b_1, b_2 \in \mathcal{A}_2$.

Lemma 6.5. *Let $\mathcal{A}_1, \dots, \mathcal{A}_r$ in $(\mathcal{A}, \varphi, \rho)$ be free with respect to (φ, ρ) . Suppose we have cyclically alternating (a_1, \dots, a_n) and (b_1, \dots, b_m) and denote by s the number of different subalgebras appearing in $\{a_1, \dots, a_n, b_1, \dots, b_m\}$. Then $\rho(a_1 \cdots a_n, b_1 \cdots b_m)$ is either 0 or can be written as a sum of balanced expressions in $a_1, \dots, a_n, b_1, \dots, b_m$, such that each of these balanced expressions has at least s factors and contains at most one ρ -factor.*

Thus any expression of the form $\rho(a_1 \cdots a_n, b_1 \cdots b_m)$ for cyclically alternating (a_1, \dots, a_n) and (b_1, \dots, b_m) is determined by the value of φ restricted to $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_r$ and by the value of ρ restricted to $(\mathcal{A}_1 \times \mathcal{A}_1) \cup \cdots \cup (\mathcal{A}_r \times \mathcal{A}_r)$.

Proof. We will prove this by induction on $n + m$. The case $n + m = 2$, i.e., $n = m = 1$, is clear.

So consider $n + m \geq 3$. Put

$$a_k^\circ := a_k - \varphi(a_k) \cdot 1, \quad b_l^\circ := b_l - \varphi(b_l) \cdot 1$$

for $k = 1, \dots, n$ and $l = 1, \dots, m$. Then we have

$$\begin{aligned} & \rho(a_1 \cdots a_n, b_1 \cdots b_m) \\ &= \rho((a_1^\circ + \varphi(a_1)) \cdots (a_n^\circ + \varphi(a_n)), (b_1^\circ + \varphi(b_1)) \cdots (b_m^\circ + \varphi(b_m))) \\ &= \sum_{p,q} \varphi(a_{p(1)}) \cdots \varphi(a_{p(k)}) \cdot \varphi(b_{q(1)}) \\ (27) \quad & \times \cdots \times \varphi(b_{q(l)}) \cdot \rho(a_{\bar{p}(1)}^\circ \cdots a_{\bar{p}(n-k)}^\circ, b_{\bar{q}(1)}^\circ \cdots b_{\bar{q}(m-l)}^\circ), \end{aligned}$$

where the sum runs over all partitions

$$((p(1), \dots, p(k)), (\bar{p}(1), \dots, \bar{p}(n-k))) \quad \text{of the set } [1, n]$$

and

$$((q(1), \dots, q(l)), (\bar{q}(1), \dots, \bar{q}(m-l))) \quad \text{of the set } [1, m]$$

into two ordered subsets (with $k = 0, \dots, n$ and $l = 0, \dots, m$). The term corresponding to $k = l = 0$ is, by Definition 6.3, either 0 (when $m \neq n$) or is a balanced expression in the centered elements with at least one factor for each occurring subalgebra. Now note that a balanced expression in centered elements can be rewritten as a sum of balanced expressions in the original elements and that the number of factors can only increase by doing so.

For the other terms with $k + l \geq 1$, $(a_{\bar{p}(1)}^\circ, \dots, a_{\bar{p}(n-k)}^\circ)$ and $(b_{\bar{q}(1)}^\circ, \dots, b_{\bar{q}(m-l)}^\circ)$ may no longer be cyclically alternating. So we group together adjacent elements from the same algebra to produce a cyclically alternating tuple with at least $\max\{1, s - (k + l)\}$ subalgebras appearing, and so we can apply our induction hypothesis. Indeed, the term

$$(28) \quad \rho(a_{\bar{p}(1)}^\circ \cdots a_{\bar{p}(n-k)}^\circ, b_{\bar{q}(1)}^\circ \cdots b_{\bar{q}(m-l)}^\circ)$$

contains elements from at least $s - (k + l)$ different subalgebras; by our induction hypothesis, we may write it as the sum of balanced expressions in the a° 's and b° 's, each product containing at least $s - (k + l)$ factors. Again we write a balanced expression in centered elements as a sum of balanced expressions in the original elements. This means we can write the term (28) as a sum of balanced expressions in

$$a_{\bar{p}(1)}, \dots, a_{\bar{p}(n-k)}, b_{\bar{q}(1)}, \dots, b_{\bar{q}(m-l)}$$

with at least $s - (k + l)$ factors for each product. Together with the $k + l$ factors

$$\varphi(a_{p(1)}) \cdots \varphi(a_{p(k)}) \cdot \varphi(b_{q(1)}) \cdots \varphi(b_{q(l)})$$

this gives the assertion. Note that in all our steps balancedness is preserved and that at most one ρ -term can occur in all the reductions. \square

A very special case of such a factorization is given in the next lemma.

Lemma 6.6. *Let $(\mathcal{A}, \varphi, \rho)$ be a second order non-commutative probability space and let $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$ be free with respect to (φ, ρ) . Consider cyclically alternating (a_1, \dots, a_n) and (b_1, \dots, b_m) from \mathcal{A} .*

1) *Assume that the subalgebra of a_1 appears only once. Then we have*

$$\rho(a_1 \cdots a_n, b_1 \cdots b_m) = \varphi(a_1) \rho(a_2 \cdots a_n, b_1 \cdots b_m).$$

2) *Assume that the subalgebra of b_1 appears only once. Then we have*

$$\rho(a_1 \cdots a_n, b_1 \cdots b_m) = \varphi(b_1) \rho(a_1 \cdots a_n, b_2 \cdots b_m).$$

Proof. We only prove the first part. Put

$$a_k^\circ := a_k - \varphi(a_k)1, \quad b_l^\circ := b_l - \varphi(b_l)1.$$

We have

$$\begin{aligned} \rho(a_1 \cdots a_n, b_1 \cdots b_m) &= \rho((a_1^\circ + \varphi(a_1)1) a_2 \cdots a_n, b_1 \cdots b_m) \\ &= \rho(a_1^\circ a_2 \cdots a_n, b_1 \cdots b_m) + \varphi(a_1) \rho(a_2 \cdots a_n, b_1 \cdots b_m). \end{aligned}$$

We shall show that the first term is 0.

Indeed, we shall show that if (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_m) are cyclically alternating and the algebra of a_1 appears only once then $\rho(a_1^\circ a_2 \cdots a_n, b_1 \cdots b_m) = 0$. We shall do this by induction on $m + n$. By equations (24) and (25) we have $\rho(a_1^\circ a_2 \cdots a_n, b_1 \cdots b_m) = 0$ when $m + n = 2$ or 3. Suppose we have proved the result for $m + n < j$; we shall prove it for $m + n = j$.

We shall use the expansion in equation (27) and show that for $1 \leq k \leq n - 1$ and $1 \leq l \leq m$, $\rho(a_1^\circ a_{p(1)}^\circ \cdots a_{p(k)}^\circ, b_{q(1)}^\circ \cdots b_{q(l)}^\circ) = 0$ for all subsets $\{p(1), \dots, p(k)\} \subset \{1, 2, 3, \dots, n - 1\}$ and $\{q(1), \dots, q(l)\} \subset \{1, 2, 3, \dots, m\}$.

When $k = n - 1$ and $l = m$ we have that $(a_1^\circ, \dots, a_n^\circ)$ and $(b_1^\circ, \dots, b_m^\circ)$ are centered and cyclically alternating. If $m \neq n$ we have $\rho(a_1^\circ a_2^\circ \cdots a_n^\circ, b_1^\circ \cdots b_m^\circ) = 0$ by (i) of Definition 6.3. If $m = n$ then by (iii), we have $\rho(a_1^\circ a_2^\circ \cdots a_n^\circ, b_1^\circ \cdots b_m^\circ) = 0$ because $\varphi(a_1^\circ b_i^\circ) = 0$ for all i .

Suppose next that $k + l \leq m + n - 2$. We can no longer expect $(a_1^\circ, a_{p(1)}^\circ, \dots, a_{p(k)}^\circ)$ and $(b_{q(1)}^\circ, \dots, b_{q(l)}^\circ)$ to be cyclically alternating; so

we group adjacent terms from the same algebra and write $a_1^{\circ} a_{p(1)}^{\circ} \cdots a_{p(k)}^{\circ} = a_1^{\circ} c_1 \cdots c_s$ and $b_{q(1)}^{\circ} \cdots b_{q(l)}^{\circ} = d_1 \cdots d_t$ with (c_1, \dots, c_s) and (d_1, \dots, d_t) cyclically alternating and neither involving the algebra of a_1 . Now $s + t + 1 \leq k + l + 1 \leq m + n - 1 \leq j - 1$, so by our induction hypothesis $\rho(a_1^{\circ} a_{p(1)}^{\circ} \cdots a_{p(k)}^{\circ}, b_{q(1)}^{\circ} \cdots b_{q(l)}^{\circ}) = \rho(a_1^{\circ} c_1 \cdots c_s, d_1 \cdots d_t) = 0$ \square

If by the lemma above we successively remove all subalgebras which occur only once and multiply together cyclic neighbours from the same subalgebra, then we arrive finally at $\rho(a, b)$ for a, b from one of the subalgebras (both from the same, in order to get a non-vanishing contribution) or at $\rho(a_1 \cdots a_n, b_1 \cdots b_m)$ where both arguments are cyclically alternating and in addition each involved subalgebra appears at least twice. In the latter case we have either a very special matching of the involved subalgebras or we can strengthen Lemma 6.5 to obtain at least one more φ -factor.

Lemma 6.7. *Let $(\mathcal{A}, \varphi, \rho)$ be a second order non-commutative probability space and let $\mathcal{A}_1, \dots, \mathcal{A}_r \subset \mathcal{A}$ be free with respect to (φ, ρ) . Suppose (a_1, \dots, a_n) and (b_1, \dots, b_m) are cyclically alternating and denote by s the number of different subalgebras appearing in $\{a_1, \dots, a_n, b_1, \dots, b_m\}$. Suppose also that each involved subalgebra appears at least twice.*

Then $\rho(a_1 \cdots a_n, b_1 \cdots b_m)$ can be written as a sum of balanced expressions with at least $s + 1$ factors unless the following conditions are satisfied:

$$(*) \begin{cases} - m = n = s; \\ - \text{for each } k \text{ there is } k' \text{ such that } a_k \text{ and } b_{k'} \text{ are from the same} \\ \text{subalgebra; and} \\ - \text{there is } q \text{ such that for all } k, k' = -k + q \pmod{n}. \end{cases}$$

In this case $\rho(a_1 \cdots a_n, b_1 \cdots b_m) = \varphi(a_1 b_{1'}) \cdots \varphi(a_n b_{n'}) + S$, where S is a sum of balanced expressions with at least $s + 1$ factors.

Proof. Let us look again at the expansion

$$\begin{aligned} & \rho(a_1 \cdots a_n, b_1 \cdots b_m) \\ &= \sum_{p,q} \varphi(a_{p(1)}) \cdots \varphi(a_{p(k)}) \cdot \varphi(b_{q(1)}) \cdots \varphi(b_{q(l)}) \\ (29) \quad & \cdot \rho(a_{\bar{p}(1)}^{\circ} \cdots a_{\bar{p}(n-k)}^{\circ}, b_{\bar{q}(1)}^{\circ} \cdots b_{\bar{q}(m-l)}^{\circ}). \end{aligned}$$

First, consider a term with $k+l \geq 1$. Then there are two possibilities. If all $\{a_{p(1)}, \dots, a_{p(k)}, b_{q(1)}, \dots, b_{q(l)}\}$ belong to different subalgebras, then there must be exactly s subalgebras in $\{a_{\bar{p}(1)}, \dots, a_{\bar{p}(n-k)}, b_{\bar{q}(1)}, \dots, b_{\bar{q}(m-l)}\}$ because each involved subalgebra appears at least twice.

If we group together any adjacent terms that may come from the same subalgebra we obtain cyclically alternating arguments and so can apply Lemma 6.5. According to Lemma 6.5 we can write $\rho(a_{\bar{p}(1)}^\circ \cdots a_{\bar{p}(n-k)}^\circ, b_{\bar{q}(1)}^\circ \cdots b_{\bar{q}(m-l)}^\circ)$ as a sum of balanced expressions with at least s factors. Combining these with the $k+l$ factors $\varphi(a_{p(1)}) \cdots \varphi(a_{p(k)}) \varphi(b_{q(1)}) \cdots \varphi(b_{q(l)})$ we have that every term with $k+l \geq 1$ can be written as a sum of balanced expressions with at least $s+1$ factors.

Second, consider the term $\rho(a_1^\circ \cdots a_n^\circ, b_1^\circ \cdots b_m^\circ)$ corresponding to $k=l=0$. If $m \neq n$ we have this is zero by Definition 6.3. So suppose $m=n$. Again by Definition 6.3 $\rho(a_1^\circ \cdots a_n^\circ, b_1^\circ \cdots b_n^\circ) = \sum_{k=0}^{n-1} \varphi(a_1^\circ b_{n+k}^\circ) \cdots \varphi(a_n^\circ b_{k+1}^\circ)$.

If $s < n$ then each term $\varphi(a_1^\circ b_{n+k}^\circ) \cdots \varphi(a_n^\circ b_{k+1}^\circ)$ has $n \geq s+1$ factors or is zero if for some factor $\varphi(a_r^\circ b_{n+k-r}^\circ)$, a_r and b_{n+k-r} come from different algebras. Thus we get either 0 or a balanced expression with at least $s+1$ factors.

Finally assume that $s=m=n$. Each subalgebra must appear exactly twice, so for each k there is k' such that a_k and $b_{k'}$ are from the same subalgebra, or else for all q , $\varphi(a_1^\circ b_{n+q-1}^\circ) \cdots \varphi(a_n^\circ b_q^\circ) = 0$. Again we will have $\varphi(a_1^\circ b_{n+q-1}^\circ) \cdots \varphi(a_n^\circ b_q^\circ) = 0$ unless $k' = -k+q \pmod n$, for some q . For this q we have $\rho(a_1^\circ \cdots a_n^\circ, b_1^\circ \cdots b_n^\circ) = \varphi(a_1^\circ b_{1'}^\circ) \cdots \varphi(a_n^\circ b_{n'}^\circ)$.

By substituting $\varphi(a_k^\circ b_{k'}^\circ) = \varphi(a_k b_{k'}) - \varphi(a_k) \varphi(b_{k'})$ into $\varphi(a_1^\circ b_{1'}^\circ) \cdots \varphi(a_n^\circ b_{n'}^\circ)$ we may write $\rho(a_1^\circ \cdots a_n^\circ, b_1^\circ \cdots b_n^\circ)$ as $\varphi(a_1 b_{1'}) \cdots \varphi(a_n b_{n'})$ plus a sum of balanced expressions with at least $n+1 = s+1$ factors. \square

We are now almost ready for the main limit theorem of second order freeness. It will turn out that moments of the limit can be calculated in terms of annular non-crossing objects. However, in this setting we will not arrive directly at permutations (as in the fluctuation formulas for random matrices), but – as is much more natural in the context of limit theorems – at partitions. In the random matrix setting of section 3 we got contributions of the form ψ_π for non-crossing permutations π . So we have to define the analogous object $\check{\psi}_\sigma$ for non-crossing partitions σ . However, for non-crossing partitions, the contribution to $\check{\psi}_\sigma$ of a block which is the only through-block will require special treatment. We will need two different types of functions, $\check{\psi}_1$ in the case of multiple through-blocks and $\check{\psi}_2$ in the case of a single through-block.

Notation 6.8. Let T be an index set and let two functions

$$\check{\psi}_1 : \bigcup_{n \in \mathbb{N}} T^n \rightarrow \mathbb{C}, \quad (t_1, \dots, t_n) \mapsto \check{\psi}_1(t_1, \dots, t_n)$$

and

$$\check{\psi}_2 : \bigcup_{n,m \in \mathbb{N}} T^n \times T^m \rightarrow \mathbb{C}$$

$$(t_1, \dots, t_n) \times (t_{n+1}, \dots, t_{n+m}) \mapsto \check{\psi}_2(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m})$$

be given. Assume that $\check{\psi}_1$ is tracial in its arguments, i.e., for all $n \in \mathbb{N}$ and all $t_1, \dots, t_n \in T$ we have

$$\check{\psi}_1(t_1, t_2, \dots, t_n) = \check{\psi}_1(t_2, \dots, t_n, t_1),$$

and that $\check{\psi}_2$ is tracial in each of its groups of arguments, i.e., for all $n, m \in \mathbb{N}$ and all t_1, \dots, t_{n+m} we have

$$\check{\psi}_2(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) = \check{\psi}_2(t_2, \dots, t_n, t_1; t_{n+1}, \dots, t_{n+m})$$

and

$$\check{\psi}_2(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) = \check{\psi}_2(t_1, \dots, t_n; t_{n+2}, \dots, t_{n+m}, t_{n+1}).$$

Fix $n, m \in \mathbb{N}$ and consider an annular non-crossing partition $\sigma \in NC(n, m)$. Then, for given $t_1, \dots, t_n, t_{n+1}, \dots, t_{n+m} \in T$ we define $\check{\psi}_\sigma(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m})$ as follows: If B is not the only through-block of σ then we choose the unique cyclic order on B (c.f. section 2.2) and, writing it as a cycle $B = (i(1), \dots, i(k))$, we put

$$(30) \quad \check{\psi}_B(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) := \check{\psi}_1(t_{i(1)}, t_{i(2)}, \dots, t_{i(k)}).$$

If B is the only through-block of σ , then we write it as $B = B_1 \cup B_2$ with $B_1 = (i(1), \dots, i(k)) \subset [1, n]$ and $B_2 = (j(1), \dots, j(l)) \subset [n+1, n+m]$, where we induce the cyclic order of $[1, n]$ on B_1 and the cyclic order of $[n+1, n+m]$ on B_2 . For such a block B we put

$$(31) \quad \check{\psi}_B(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) := \check{\psi}_2(t_{i(1)}, \dots, t_{i(k)}; t_{j(1)}, \dots, t_{j(l)}).$$

Finally, we define

$$(32) \quad \check{\psi}_\sigma(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) := \prod_{B \in \sigma} \check{\psi}_B(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}).$$

Here are some examples of this notation: Consider $n = m = 2$ and

$$\sigma_1 = \{(1, 3), (2, 4)\} \quad \sigma_2 = \{(1, 2, 3), (4)\}.$$

Then σ_1 has two through-blocks so

$$(33) \quad \check{\psi}_{\sigma_1}(t_1, t_2; t_3, t_4) = \check{\psi}_1(t_1, t_3)\check{\psi}_1(t_2, t_4)$$

whereas σ_2 has one through-block so

$$(34) \quad \check{\psi}_{\sigma_2}(t_1, t_2; t_3, t_4) = \check{\psi}_1(t_4)\check{\psi}_2(t_1, t_2; t_3).$$

Theorem 6.9. *Let $(\mathcal{A}_N, \varphi_N, \rho_N)$ ($N \in \mathbb{N}$) be second order non-commutative probability spaces and let, for each $N \in \mathbb{N}$, unital subalgebras $\mathcal{A}_N^1, \dots, \mathcal{A}_N^N \subset \mathcal{A}_N$ be given which are free with respect to (φ_N, ρ_N) . Let T be an index set and assume that we have, for each $t \in T$ and each $N \in \mathbb{N}$, elements*

$$q_N^i(t) \in \mathcal{A}_N^i \quad (i = 1, \dots, N),$$

such that the following properties are satisfied:

(a) *The distribution of the $q_N^i(t)$ under (φ_N, ρ_N) is invariant under permutations of the upper indices, i.e., for all $N \in \mathbb{N}$, and all permutations $\pi : [1, N] \rightarrow [1, N]$ we have for all $n, m \in \mathbb{N}$, $t_1, \dots, t_{n+m} \in T$ and all $i(1), \dots, i(n+m) \in [1, N]$ that*

$$\varphi_N(q_N^{i(1)}(t_1) \cdots q_N^{i(n)}(t_n)) = \varphi_N(q_N^{\pi \circ i(1)}(t_1) \cdots q_N^{\pi \circ i(n)}(t_n))$$

and

$$\begin{aligned} & \rho_N(q_N^{i(1)}(t_1) \cdots q_N^{i(n)}(t_n), q_N^{i(n+1)}(t_{n+1}) \cdots q_N^{i(n+m)}(t_{n+m})) \\ &= \rho_N(q_N^{\pi \circ i(1)}(t_1) \cdots q_N^{\pi \circ i(n)}(t_n), q_N^{\pi \circ i(n+1)}(t_{n+1}) \cdots q_N^{\pi \circ i(n+m)}(t_{n+m})) \end{aligned}$$

(b) *For all $n, m \in \mathbb{N}$ and all $t_1, \dots, t_n, t_{n+1}, \dots, t_{n+m} \in T$ there exist constants $\check{\psi}_1(t_1, \dots, t_n)$ and $\check{\psi}_2(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m})$ such that*

$$(35) \quad \lim_{N \rightarrow \infty} N \cdot \varphi_N(q_N^i(t_1) \cdots q_N^i(t_n)) = \check{\psi}_1(t_1, \dots, t_n)$$

and

$$(36) \quad \begin{aligned} \lim_{N \rightarrow \infty} N \cdot \rho_N(q_N^i(t_1) \cdots q_N^i(t_n), q_N^i(t_{n+1}) \cdots q_N^i(t_{n+m})) \\ = \check{\psi}_2(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}). \end{aligned}$$

For $t \in T$ and $N \in \mathbb{N}$ let

$$S_N(t) := q_N^1(t) + \cdots + q_N^N(t) \in \mathcal{A}_N.$$

Then we have

$$(37) \quad \begin{aligned} \lim_{N \rightarrow \infty} \rho_N(S_N(t_1) \cdots S_N(t_n), S_N(t_{n+1}) \cdots S_N(t_{n+m})) \\ = \sum_{\sigma \in NC(n, m)} \check{\psi}_\sigma(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) \end{aligned}$$

Note that the left-hand side of the expressions (35) and (36) are independent of the value of the index i , and that the functions $\check{\psi}_1$ and $\check{\psi}_2$ defined there have the traciality properties which are required in Notation 6.8.

Proof. For better legibility, we will suppress in the following the index N at φ_N and ρ_N and just write φ and ρ , respectively.

We have

$$\begin{aligned} & \rho(S_N(t_1) \cdots S_N(t_n), S_N(t_{n+1}) \cdots S_N(t_{n+m})) \\ &= \sum_{i:[1,n+m] \rightarrow [1,N]} \rho(q_N^{i(1)}(t_1) \cdots q_N^{i(n)}(t_n), q_N^{i(n+1)}(t_{n+1}) \cdots q_N^{i(n+m)}(t_{n+m})) \end{aligned}$$

Because of our invariance assumption (a), the value of the term

$$(38) \quad \rho(q_N^{i(1)}(t_1) \cdots q_N^{i(n)}(t_n), q_N^{i(n+1)}(t_{n+1}) \cdots q_N^{i(n+m)}(t_{n+m}))$$

depends on i only through the information where these indices are the same and where they are different. As usual, this information is encoded in a partition σ of the set $[1, n+m]$, and we denote the common value of (38) for all i with $\ker(i) = \sigma$ by

$$(39) \quad \rho_\sigma(q_N(t_1) \cdots q_N(t_n), q_N(t_{n+1}) \cdots q_N(t_{n+m})).$$

Then we can continue our calculation as follows:

$$\begin{aligned} & \rho(S_N(t_1) \cdots S_N(t_n), S_N(t_{n+1}) \cdots S_N(t_{n+m})) \\ &= \sum_{\sigma \in \mathcal{P}(n+m)} \sum_{\substack{i:[1,n+m] \rightarrow [1,N] \\ \ker(i) = \sigma}} \rho(q_N^{i(1)}(t_1) \cdots q_N^{i(n)}(t_n), \\ & \quad q_N^{i(n+1)}(t_{n+1}) \cdots q_N^{i(n+m)}(t_{n+m})) \\ &= \sum_{\sigma \in \mathcal{P}(n+m)} \rho_\sigma(q_N(t_1) \cdots q_N(t_n), q_N(t_{n+1}) \cdots q_N(t_{n+m})) \cdot (N)_{|\sigma|}, \end{aligned}$$

because the number of $i : [1, n+m] \rightarrow [1, N]$ with the property $\ker(i) = \sigma$ is given by

$$N(N-1) \cdots (N - |\sigma| + 1) =: (N)_{|\sigma|}.$$

We have now to examine the contributions for different σ . Let us first assume that σ contains a block B which is either contained in $[1, n]$ or contained in $[n+1, \dots, n+m]$ and all of whose elements are consecutive in the induced cyclic order. Because of traciality of ρ it suffices to consider the case $B = [1, s]$ for some s with $1 \leq s \leq n$. By Lemma 6.6, this implies

$$\begin{aligned} & \rho_\sigma(q_N(t_1) \cdots q_N(t_s) \cdots q_N(t_n), q_N(t_{n+1}) \cdots q_N(t_{n+m})) \\ &= \varphi(q_N(t_1) \cdots q_N(t_s)) \cdot \rho_{\sigma'}(q_N(t_{s+1}) \cdots q_N(t_n), q_N(t_{n+1}) \cdots q_N(t_{n+m})), \end{aligned}$$

where σ' is that partition which results from σ by removing the block $B = [1, s]$ and relabelling elements. Since

$$\lim_{N \rightarrow \infty} N \cdot \varphi(q_N(t_1) \cdots q_N(t_s)) = \check{\psi}_1(t_1, \dots, t_s),$$

the block B makes exactly the contribution to the final result as claimed in Eq. (37). Thus, by successively removing such blocks, it suffices to consider σ 's which have no blocks which are contained in either $[1, n]$ or $[n + 1, n + m]$ and which consist of cyclically consecutive elements.

So let us now assume that σ contains no blocks which are contained in either $[1, n]$ or $[n + 1, n + m]$ and which consist of cyclically consecutive elements, and consider (39). By multiplying together neighbouring elements corresponding to the same block of σ we can rewrite the two arguments of ρ in a cyclically alternating form. The fact that σ contains no blocks of the form treated above implies that after this rewriting of arguments each involved subalgebra occurs at least twice. But then Lemma 6.7 implies that, unless condition (*) is satisfied, we can write all these terms as sums of products of at least $|\sigma| + 1$ factors. By our assumption, each of these factors multiplied by N converges to a finite number; however, since we have more than $|\sigma|$ factors, this product multiplied by $N^{|\sigma|}$ will vanish in the limit $N \rightarrow \infty$. This means that we can only get a non-vanishing limit for a σ which satisfies condition (*) of Lemma 6.7. However, these are exactly the cases where each block B of σ is of the form $B = B_1 \cup B_2$, where $B_1 \subset [1, n]$ and $B_2 \subset [n + 1, n + m]$, are non-empty, and each consists of consecutive numbers with respect to the inherited order. Furthermore, the cyclic order of the restrictions of all blocks to the interval $[1, n]$ must be the inversion of the cyclic order of the restrictions of all blocks to the interval $[n + 1, n + m]$. In this case (39) calculates as follows. If we have only one block in σ , then our assumption, Equation (36), gives, in the limit, for such a σ the contribution

$$\check{\psi}_2(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}).$$

If, on the other side, σ has more than one block, then we get, according to the description of annular non-crossing partitions in section 2.2 and our assumption (36), the product of $\check{\psi}_B$ over all blocks B of σ .

Note that the reduction above leads to non-vanishing contributions exactly for non-crossing partitions σ from $NC(n, m)$ and each such partition σ contributes a term $\check{\psi}_\sigma(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m})$. \square

7. PROOFS OF THEOREMS 5.3 AND 5.7

Now we can prove our main theorems by reducing them to the situation covered in our limit theorem.

7.1. Proof of Theorem 5.3. We have to show that for all $n, m \in \mathbb{N}$ and $f_1, \dots, f_{n+m} \in \mathcal{H}_{\mathbb{R}}$

$$\begin{aligned} & \langle \mathbf{c}\omega(f_1) \cdots \omega(f_n)\Omega, \mathbf{c}\omega(f_{n+m}) \cdots \omega(f_{n+1})\Omega \rangle_{\text{cyc}} \\ &= \sum_{\pi \in NC_2(n,m)} \prod_{(i,j) \in \pi} \langle f_i, f_j \rangle. \end{aligned}$$

Note that we can, for any $N \in \mathbb{N}$, replace \mathcal{H} by

$$\bigoplus_{i=1}^N \mathcal{H} = \mathcal{H} \oplus \cdots \oplus \mathcal{H} \quad (N \text{ summands})$$

and $\omega(f)$ by

$$\frac{1}{\sqrt{N}}\omega(f \oplus \cdots \oplus f).$$

We can then put this into the framework of our general limit theorem by letting $T = \mathcal{H}_{\mathbb{R}}$,

$$\mathcal{A}_N = \mathcal{A}\left(\bigoplus_{i=1}^N \mathcal{H}\right), \quad \mathcal{A}_N^i = \mathcal{A}(0 \oplus \cdots \oplus \underset{i\text{-th}}{\mathcal{H}} \oplus \cdots \oplus 0)$$

$$\varphi_N(a) = \langle a\Omega, \Omega \rangle \quad (a \in \mathcal{A}_N)$$

and

$$\rho_N(a, b) = \langle \mathbf{c}a\Omega, \mathbf{c}b^*\Omega \rangle_{\text{cyc}} \quad (a, b \in \mathcal{A}_N)$$

and finally, for $f \in \mathcal{H}_{\mathbb{R}}$,

$$q_N^i(f) = \frac{1}{\sqrt{N}}\omega(0 \oplus \cdots \oplus \underset{i\text{-th}}{f} \oplus \cdots \oplus 0) \in \mathcal{A}_N^i.$$

Let us check that $\mathcal{A}_N^1, \dots, \mathcal{A}_N^N \subset \mathcal{A}_N$ are free with respect to (φ_N, ρ_N) : Freeness with respect to φ_N is well-known, so we only have to consider ρ_N . Take centered and cyclically alternating tuples (a_1, \dots, a_n) and (b_1, \dots, b_m) from \mathcal{A}_N . Let us only consider the case $n, m \geq 2$, the cases where at least one of them is 1 are similar. Note that the centeredness of the a_i implies that each $a_i\Omega$ has no component in the direction Ω and thus, by the fact that neighbours are from algebras with orthogonal Hilbert spaces, we have

$$a_1 a_2 \cdots a_n \Omega = (a_1 \Omega) \otimes (a_2 \Omega) \otimes \cdots \otimes (a_n \Omega).$$

Since also the first and the last element are orthogonal, the action of \mathbf{c} becomes in this case just

$$\mathbf{c} a_1 a_2 \cdots a_n \Omega = [a_1 \Omega \otimes a_2 \Omega \otimes \cdots \otimes a_n \Omega].$$

In the same way we have

$$\mathbf{c} b_m^* b_{m-1}^* \cdots b_1^* \Omega = [b_m^* \Omega \otimes b_{m-1}^* \Omega \otimes \cdots \otimes b_1^* \Omega].$$

If we take now the inner product in the cyclic Fock space between these two vectors, then we get

$$\begin{aligned}\rho_N(a_1 \cdots a_n, b_1 \cdots b_m) &= \delta_{nm} \sum_{k=0}^{n-1} \langle a_1 \Omega, b_{n+k}^* \Omega \rangle \cdots \langle a_n \Omega, b_{1+k}^* \Omega \rangle \\ &= \delta_{nm} \sum_{k=0}^{n-1} \varphi(a_1 b_{n+k}) \cdots \varphi(a_n b_{1+k}),\end{aligned}$$

as required by our Definition 6.3. Thus the subalgebras $\mathcal{A}_N^1, \dots, \mathcal{A}_N^N$ are free with respect to (φ_N, ρ_N) . The invariance assumption on the distribution with respect to (φ_N, ρ_N) is also easily verified and so we can apply our limit theorem.

Let $S_N(f) = q_N^1(f) + \cdots + q_N^N(f)$. Since

$$\begin{aligned}\langle \mathbf{c}\omega(f_1) \cdots \omega(f_n) \Omega, \mathbf{c}\omega(f_{n+m}) \cdots \omega(f_{n+1}) \rangle_{\text{cyc}} \\ = \rho_N(S_N(f_1) \cdots S_N(f_n), S_N(f_{n+1}) \cdots S_N(f_{n+m}))\end{aligned}$$

we can take the limit as $N \rightarrow \infty$ and apply Theorem 6.9. So it remains to identify the limits $\check{\psi}_1$ and $\check{\psi}_2$ in the assumption of that theorem. One sees easily that

$$\begin{aligned}\check{\psi}_1(f_1, \dots, f_n) &= \lim_{N \rightarrow \infty} N \cdot \varphi_N(q_N^i(f_1) \cdots q_N^i(f_n)) \\ &= \begin{cases} \langle f_1, f_2 \rangle & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and

$$\begin{aligned}\check{\psi}_2(f_1, \dots, f_n; g_1, \dots, g_m) \\ = \lim_{N \rightarrow \infty} N \cdot \rho_N(q_N^i(f_1) \cdots q_N^i(f_n), q_N^i(g_1) \cdots q_N^i(g_m)) \\ = \begin{cases} \langle f_1, g_1 \rangle & \text{if } n = 1 = m \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

This gives exactly our claim. \square

7.2. Proof of Theorem 5.7. From equation (3) we only have to prove that

$$\begin{aligned}(40) \quad \langle \mathbf{c}p(d_1) \cdots p(d_n) \Omega, \mathbf{c}p(d_{n+m}^*) \cdots p(d_{n+1}^*) \Omega \rangle_{\text{cyc}} \\ = \sum_{\pi \in S_{NC}(n, m)} \psi_\pi(d_1, \dots, d_n, d_{n+1}, \dots, d_{n+m}).\end{aligned}$$

Note that we can replace \mathcal{D} by $\mathcal{D} \otimes L^\infty[0, 1]$, ψ by $\psi \otimes \tau$, where τ is integration with respect to Lebesgue measure on $[0, 1]$, and for each $N \in \mathbb{N}$, $p(d)$ by

$$p(d \otimes \chi(0, 1)) = p_N^1(d) + p_N^2(d) + \cdots + p_N^N(d),$$

where we have put

$$p_N^i(d) := p(d \otimes \chi(I_N^i))$$

with $\chi(I_N^i)$ denoting the characteristic function of the interval

$$I_N^i = \left(\frac{i-1}{N}, \frac{i}{N} \right).$$

This fits into the framework of our general limit theorem by putting $T = \mathcal{D}$,

$$\mathcal{A}_N = \mathcal{A}(\mathcal{D} \otimes L^\infty(0, 1)), \quad \mathcal{A}_N^i = \mathcal{A}(\mathcal{D} \otimes L^\infty(I_N^i)),$$

$$\varphi(a) = \langle a\Omega, \Omega \rangle, \quad \rho(a, b) = \langle \mathbf{c} a \Omega, \mathbf{c} b^* \Omega \rangle_{\text{cyc}} \quad (a, b \in \mathcal{A}_N),$$

and finally

$$q_N^i(d) = p_N^i(d) \in \mathcal{A}_N^i.$$

One can check again by the same arguments as for the semi-circular case that $\mathcal{A}_N^1, \dots, \mathcal{A}_N^N \subset \mathcal{A}_N$ are free with respect to (φ_N, ρ_N) . Also the invariance assumption on the distribution with respect to (φ_N, ρ_N) is easily verified.

Since $p(d)$ has, for each N , the same moments with respect to φ_N and ρ_N as $S_N(d) = q_N^1(d) + \cdots + q_N^N(d)$, we can calculate the moments of $p(d)$ via $S_N(d)$ by sending $N \rightarrow \infty$ and invoking our limit theorem, Theorem 6.9. It only remains to identify the limits $\check{\psi}_1$ and $\check{\psi}_2$ from the hypothesis of the theorem, and show that

$$\begin{aligned} & \sum_{\sigma \in NC(n, m)} \check{\psi}_\sigma(d_1, \dots, d_n; d_{n+1}, \dots, d_{n+m}) \\ (41) \quad & = \sum_{\pi \in S_{NC}(n, m)} \psi_\pi(d_1, \dots, d_n, d_{n+1}, \dots, d_{n+m}) \end{aligned}$$

Note that each inner product appearing in the calculation of

$$\varphi_N(p_N^i(d_1) \cdots p_N^i(d_n))$$

gives a factor $1/N$; one inner product must be involved in any case to get a non-vanishing result, thus the sought limits single out exactly the contributions with one inner product. In the case of $\varphi_N(p_N^i(d_1) \cdots p_N^i(d_n))$ this means that $p_N^i(d_n)$ must act as a creation operator, $p_N^i(d_1)$

as an annihilation operator and all the other p 's as preservation operators, thus

$$\begin{aligned}\check{\psi}_1(d_1, \dots, d_n) &= \lim_{N \rightarrow \infty} N \cdot \varphi_N(p_N^i(d_1) \cdots p_N^i(d_n)) \\ &= \langle d_2 d_3 \cdots d_n, d_1^* \rangle = \psi(d_1 d_2 \cdots d_n)\end{aligned}$$

In the case of ρ_N one has to note that the only relevant contributions to $\mathbf{c} p_N^i(d_1) \cdots p_N^i(d_n) \Omega$ are of the form: $p_N^i(d_n)$ must act as creation operator; since $\mathbf{c} \Omega = 0$, no annihilation operator can appear, but since \mathbf{c} can also act by multiplication of arguments there might be a second action as creation operator (let's say of $p_N^i(d_k)$), all the other p have to act as preservation operators. Thus the relevant contributions of $\mathbf{c} p_N^i(d_1) \cdots p_N^i(d_n) \Omega$ are the terms with $k = 1, \dots, n$ of the form

$$\mathbf{c}(d_1 \cdots d_k \otimes d_{k+1} \cdots d_n).$$

Since we are looking for terms which give in the end exactly one inner product, the relevant action of \mathbf{c} is given by multiplying arguments and yields terms of the form

$$[d_{k+1} \cdots d_n d_1 \cdots d_k] \quad \text{for some } k = 1, \dots, n.$$

In the same way the relevant contributions of $\mathbf{c} p_N^i(d_{n+m}^*) \cdots p_N^i(d_{n+1}^*) \Omega$ are of the form

$$[d_{n+l-1}^* \cdots d_{n+1}^* d_{n+m}^* d_{n+m-1}^* \cdots d_{n+l}^*] \quad \text{for some } l = 1, \dots, m.$$

Thus we have

$$\begin{aligned}(42) \quad \check{\psi}_2(d_1, \dots, d_n; d_{n+1}, \dots, d_{n+m}) &= \lim_{N \rightarrow \infty} N \cdot \rho_N(p_N^i(d_1) \cdots p_N^i(d_n), p_N^i(d_{n+1}) \cdots p_N^i(d_{n+m})) \\ &= \lim_{N \rightarrow \infty} N \cdot \langle \mathbf{c} p_N^i(d_1) \cdots p_N^i(d_n) \Omega, \mathbf{c} p_N^i(d_{n+m}^*) \cdots p_N^i(d_{n+1}^*) \Omega \rangle_{\text{cyc}} \\ &= \sum_{k=1}^n \sum_{l=1}^m \langle [d_{k+1} \cdots d_n d_1 \cdots d_k], [d_{n+l-1}^* \cdots d_{n+1}^* d_{n+m}^* \cdots d_{n+l}^*] \rangle_{\text{cyc}} \\ &= \sum_{k=1}^n \sum_{l=1}^m \psi(d_{n+l} \cdots d_{n+m} d_{n+1} \cdots d_{n+l-1} d_{k+1} \cdots d_n d_1 \cdots d_k)\end{aligned}$$

Suppose $\sigma \in NC(n, m)$ has more than one through-block. Then for each block B , $\check{\psi}_B(d_1, \dots, d_n; d_{n+1}, \dots, d_{n+m}) = \psi_B(d_1, \dots, d_{n+m})$ by equation (30). Thus

$$\check{\psi}_\sigma(d_1, \dots, d_n; d_{n+1}, \dots, d_{n+m}) = \psi_\pi(d_1, \dots, d_n, d_{n+1}, \dots, d_{n+m})$$

where $\pi \in S_{NC}(n, m)$ is the unique permutation whose cycle decomposition is the partition π .

Figure 12. One of the possible cycles which contribute to the sum in $\check{\psi}_B$ in the case B is a unique through block.

Now suppose that σ has only one through-block, let $[\sigma]$ be the set of all $\pi \in S_{NC}(n, m)$ whose cycle decomposition gives the partition σ . If B is a block of σ which is not a through-block then again by equation (30) $\check{\psi}_B$ and ψ_B are equal. If B is the unique through-block then as in equation (31) write $B = \{j_1, \dots, j_r\} \cup \{j_{r+1}, \dots, j_{r+s}\}$. Then by (42)

$$\begin{aligned} \check{\psi}_B(d_{j_1}, \dots, d_{j_r}; d_{j_{r+1}}, \dots, d_{j_{r+s}}) &= \check{\psi}_2(d_{j_1}, \dots, d_{j_r}; d_{j_{r+1}}, \dots, d_{j_{r+s}}) \\ &= \sum_c \psi_c(d_{j_1}, \dots, d_{j_r}, d_{j_{r+1}}, \dots, d_{j_{r+s}}) \end{aligned}$$

where c runs over the cycles in $\pi \in [\sigma]$ which give the block B . Hence $\check{\psi}_\sigma = \sum_{\pi \in [\sigma]} \psi_\pi$ and thus equation (41) is proved. \square

8. DIAGONALIZATION OF FLUCTUATIONS

Let us now use our description of fluctuations of random matrices in terms of operators to diagonalize these fluctuations. The one-dimensional Gaussian case is well established in the physical and mathematical literature (see, e.g., [Pol, AJM, Joh]), whereas looking on the one-dimensional Wishart case and, in particular, on the multi-dimensional Gaussian case was initiated by Cabanal-Duvillard [C-D]. Indeed, trying to understand and reproduce the results of Cabanal-Duvillard was the original motivation for our investigations.

Since the fluctuations are given by taking inner products in cyclic Fock space, we can achieve such a diagonalization by taking functions of our operators which yield elementary tensors in cyclic Fock space. This means we are looking for a kind of cyclic Wick products.

8.1. Semi-circular case. We should look for cyclic analogues of the Wick products $W(f_1 \otimes \dots \otimes f_n)$. Let us denote them by $C(f_1 \otimes \dots \otimes f_n)$. They should be determined by the property that

$$\mathbf{c} C(f_1 \otimes \dots \otimes f_n) \Omega = [f_1 \otimes \dots \otimes f_n].$$

Notice that we have

$$\begin{aligned} \mathbf{c}W(f_1 \otimes \cdots \otimes f_n)\Omega &= \mathbf{c}(f_1 \otimes \cdots \otimes f_n) \\ &= [f_1 \otimes \cdots \otimes f_n] + \langle f_1, \bar{f}_n \rangle \cdot \mathbf{c}f_2 \otimes \cdots \otimes f_{n-1} \\ &= \mathbf{c}C(f_1 \otimes \cdots \otimes f_n)\Omega + \langle f_1, \bar{f}_n \rangle \cdot \mathbf{c}W(f_2 \otimes \cdots \otimes f_{n-1})\Omega, \end{aligned}$$

thus we could define these cyclic Wick products by the following recursion:

$$C(f_1 \otimes \cdots \otimes f_n) = W(f_1 \otimes \cdots \otimes f_n) - \langle f_1, \bar{f}_n \rangle \cdot W(f_2, \dots, f_{n-1}).$$

For $n = 1$, this means, of course,

$$C(f) = W(f) = \omega(f).$$

If we put

$$f := f_1 = f_2 = \cdots = f_n \quad (\text{with } \|f\| = 1),$$

then we know that

$$W(f^{\otimes n}) = U_n(\omega(f)/2),$$

where the $\{U_n\}_n$ are the Chebyshev polynomials of the second kind. Let $V_n(x) = U_n(x/2)$; then $V_n(\omega(f)) = W(f^{\otimes n})$. Now, if we write our cyclic Wick polynomials in this one-dimensional case as

$$C(f^{\otimes n}) = 2T_n(\omega(f)/2),$$

then these T_n must satisfy

$$2T_n = U_n - U_{n-2} \quad (n \geq 2)$$

and

$$T_1(x) = U_1(x)/2 = x.$$

This shows that the $\{T_n\}$ are Chebyshev polynomials of the first kind.

Let us now consider the multi-dimensional case. It is easy to see that if f_i is orthogonal to f_{i+1} for all $i = 1, \dots, k$, then we have for all $n(1), \dots, n(k) > 0$ that

$$W(f_1^{\otimes n(1)} \otimes f_2^{\otimes n(2)} \otimes \cdots \otimes f_k^{\otimes n(k)}) = W(f_1^{\otimes n(1)}) \cdot W(f_2^{\otimes n(2)}) \cdots W(f_k^{\otimes n(k)}).$$

If we assume in addition that also f_1 and f_k are orthogonal then we get for the corresponding C :

$$\begin{aligned} C(f_1^{\otimes n(1)} \otimes f_2^{\otimes n(2)} \otimes \cdots \otimes f_k^{\otimes n(k)}) &= W(f_1^{\otimes n(1)} \otimes f_2^{\otimes n(2)} \otimes \cdots \otimes f_k^{\otimes n(k)}) \\ &= W(f_1^{\otimes n(1)}) \cdot W(f_2^{\otimes n(2)}) \cdots W(f_k^{\otimes n(k)}). \end{aligned}$$

The covariance between such functions in our random matrices is given by the inner product in the cyclic Fock space. If we have $k, l \geq 2$ and

$f_1, \dots, f_k \in \mathcal{H}_{\mathbb{R}}$ and $g_1, \dots, g_l \in \mathcal{H}_{\mathbb{R}}$ such that $f_i \perp f_{i+1}$ for $i = 1, \dots, k$ and $g_i \perp g_{i+1}$ for $i = 1, \dots, l$ then we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} k_2 \left\{ \text{Tr}[V_{n(1)}(X_N(f_1)) \cdots V_{n(k)}(X_N(f_k)), \right. \\ & \qquad \qquad \qquad \left. \text{Tr}[V_{m(1)}(X_N(g_1)) \cdots V_{m(l)}(X_N(g_l))] \right\} \\ &= \langle [f_1^{\otimes n(1)} \otimes \cdots \otimes f_k^{\otimes n(k)}], [g_l^{\otimes m(l)} \otimes \cdots \otimes g_1^{\otimes m(1)}] \rangle_{\text{cyc}}. \end{aligned}$$

Thus we recover the results of Cabanal-Duvillard [C-D] for that case.

8.2. Compound Poisson case. Again, we are looking for polynomials $C(d_1 \otimes \cdots \otimes d_n)$ which have the property

$$\mathbf{c} C(d_1 \otimes \cdots \otimes d_n) \Omega = [d_1 \otimes \cdots \otimes d_n].$$

We have

$$\begin{aligned} \mathbf{c} W(d_1 \otimes \cdots \otimes d_n) \Omega &= \mathbf{c} (d_1 \otimes \cdots \otimes d_n) \\ &= [d_1, \dots, d_n] + [d_n d_1, d_2, \dots, d_{n-1}] + \psi(d_1 d_n) \mathbf{c} (d_2 \otimes \cdots \otimes d_{n-1}) \\ &= \mathbf{c} C(d_1 \otimes \cdots \otimes d_n) \Omega + \mathbf{c} C(d_n d_1 \otimes d_2 \otimes \cdots \otimes d_{n-1}) \Omega \\ &\quad + \psi(d_1 d_n) \mathbf{c} W(d_2 \otimes \cdots \otimes d_{n-1}) \Omega \end{aligned}$$

Thus we define the C 's in the following recursive way:

$$(43) \quad \begin{aligned} W(d_1 \otimes \cdots \otimes d_n) &= C(d_1 \otimes \cdots \otimes d_n) \\ &\quad + C(d_n d_1 \otimes d_2 \otimes \cdots \otimes d_{n-1}) + \psi(d_1 d_n) W(d_2 \otimes \cdots \otimes d_{n-1}) \end{aligned}$$

There does not seem to be a nice closed form for this in the one-dimensional case.

Let us also look at the multi-dimensional situation. We model this by assuming that we have elements $d_1, \dots, d_r \in \mathcal{D}$ such that $d_i d_j = 0$ for $i \neq j$. Then we have again for $i(j) \neq i(j+1)$ ($j = 1, \dots, n$) and $k(1), \dots, k(n) > 0$ that

$$W(d_{i(1)}^{\otimes k(1)} \otimes \cdots \otimes d_{i(n)}^{\otimes k(n)}) = W(d_{i(1)}^{\otimes k(1)}) \cdots W(d_{i(n)}^{\otimes k(n)}).$$

If also $i(1) \neq i(n)$, then we have again equality between W and C , i.e.

$$\begin{aligned} C(d_{i(1)}^{k(1)} \otimes \cdots \otimes d_{i(n)}^{k(n)}) &= W(d_{i(1)}^{\otimes k(1)} \otimes \cdots \otimes d_{i(n)}^{\otimes k(n)}) \\ &= W(d_{i(1)}^{\otimes k(1)}) \cdots W(d_{i(n)}^{\otimes k(n)}). \end{aligned}$$

8.3. Poisson case. Let us specialize the general compound Poisson case to the usual Poisson case.

The usual Poisson case is special within the class of compound ones by a very special state on \mathcal{D} . Restrict for the moment to one random matrix, i.e., the algebra \mathcal{D} is generated by one element d . Then the fact that we have a free Poisson variable $p(d)$ means that this d is a projection and thus

$$\psi(d^k) = \psi(d) =: \lambda.$$

So we can identify $p(d) = p(d^2) = \dots$ and everything reduces again to polynomials in just one variable $p(d)$. Again one knows (see [Ans, Theorem 4.11]) that the linear Wick polynomials $W_n(d) := W(d^{\otimes n})$ are given by the orthogonal polynomials with respect to the distribution of $p(d)$ (i.e. with respect to the Marchenko-Pastur = free Poisson distribution). Let us denote these polynomials by Π_n , then we have

$$W_n(d) = \Pi_n(p(d)).$$

If we put $C_n(d) := C(d^{\otimes n})$, then the general relation between W and C becomes in this case:

$$W_n(d) = C_n(d) + C_{n-1}(d) + \lambda W_{n-2}.$$

If we put $C_n(d) = \Gamma_n(p(d))$ for some polynomials Γ_n , then the above tells us that

$$\Pi_n - \lambda \Pi_{n-2} = \Gamma_n + \Gamma_{n-1}.$$

This gives us exactly the polynomials $\{\Gamma_n\}$ which appear in Cabanal-Duvillard's results [C-D].

As an extension of this, we also get the multi-dimensional Poisson case: There the "diagonalizing polynomials" in more than one variable are given by alternating products in the one-dimensional linear polynomials $\{\Pi_n\}$.

A more detailed investigation of this diagonalization of fluctuations will be presented in [KMS].

9. ASYMPTOTIC FREENESS OF GAUSSIAN AND CONSTANT MATRICES

Our results about compound Wishart matrices can be considered as describing the limiting relation between Gaussian random matrices and constant matrices for special moments – namely those with patterns of the form $X^*D_1XX^*D_2X \cdots X^*D_nX$. This raises, of course, the question whether we can say something substantial about the general relation between Gaussian and constant matrices. In view of the basic theorem of Voiculescu that Gaussian random matrices and constant matrices are asymptotically free, we would expect that we should

have the same kind of statement also on the level of fluctuations. We want to indicate here that this is indeed the case, thus providing strong evidence that our notion of “second order freeness” is indeed the correct concept. Note that in the following definition we make a quite strong requirement on the vanishing of the higher order cumulants. This is however in accordance with the observation that in many cases the unnormalized traces converge to Gaussian random variables. Of course, if we have a non-probabilistic ensemble of constant matrices, then the only requirement is the convergence of k_1 ; all other cumulants are automatically zero.

Definition 9.1. 1) Let $\{A_1, \dots, A_s\}_N$ be a sequence of $N \times N$ -random matrices. We say that they have a *second order limit distribution* if there exists a second order non-commutative probability space $(\mathcal{A}, \varphi, \rho)$ and $a_1, \dots, a_s \in \mathcal{A}$ such that for all polynomials p_1, p_2, \dots in s non-commuting indeterminates we have

$$(44) \quad \lim_{N \rightarrow \infty} k_1 \{ \text{tr}[p_1(A_1, \dots, A_s)] \} = \varphi(p_1(a_1, \dots, a_s)),$$

$$(45) \quad \lim_{N \rightarrow \infty} k_2 \{ \text{Tr}[p_1(A_1, \dots, A_s)], \text{Tr}[p_2(A_1, \dots, A_s)] \} = \rho(p_1(a_1, \dots, a_s); p_2(a_1, \dots, a_s)),$$

and, for $r \geq 3$,

$$(46) \quad \lim_{N \rightarrow \infty} k_r \{ \text{Tr}[p_1(A_1, \dots, A_s)], \dots, \text{Tr}[p_r(A_1, \dots, A_s)] \} = 0.$$

2) We say that two sequences of $N \times N$ -random matrices, $\{A_1, \dots, A_s\}_N$ and $\{B_1, \dots, B_t\}_N$, are *asymptotically free of second order* if the sequence $\{A_1, \dots, A_s, B_1, \dots, B_t\}_N$ has a second order limit distribution, given by $(\mathcal{A}, \varphi, \rho)$ and $a_1, \dots, a_s, b_1, \dots, b_t \in \mathcal{A}$, and if the unital algebras

$$\mathcal{A}_1 := \text{alg}(1, a_1, \dots, a_s) \quad \text{and} \quad \mathcal{A}_2 := \text{alg}(1, b_1, \dots, b_t)$$

are free with respect to (φ, ρ) .

Remark 9.2. Corollary 3.3 shows that a family $\{X_N(f)\}_{f \in \mathcal{H}_{\mathbb{R}}}$ of Hermitian Gaussian random matrices has a second order limit distribution. Theorem 5.3 identifies the limiting distribution in terms of cyclic Fock space, and in the proof of Theorem 5.3 we have in addition shown that the limiting distribution is free of second order in that if $\mathcal{K}_1, \dots, \mathcal{K}_n \subset \mathcal{H}$ are orthogonal subspaces and \mathcal{A}_i is the algebra generated by $\{\omega(f) \mid f \in \mathcal{K}_i\}$ then $\mathcal{A}_1, \dots, \mathcal{A}_n$ are free with respect to (φ, ρ) where $\varphi(a) = \langle a\Omega, \Omega \rangle$ and $\rho(a_1, a_2) = \langle \mathbf{c} a_1 \Omega, \mathbf{c} a_2^* \Omega \rangle_{\text{cyc}}$. Thus

we have shown orthogonal families of Gaussian random matrices are asymptotically free of second order.

Remark 9.3. Corollary 3.7 showed that if $\{X_N\}_N$ is a sequence of complex Gaussian random matrices and $P_N(D_i) = X_N^* D_i^{(N)} X_N$ where $\{D_1^{(N)}, D_2^{(N)}, D_3^{(N)}, \dots, D_p^{(N)}\}_N$ is a sequence of $N \times N$ complex matrices which converges in distribution to (d_1, d_2, \dots, d_p) in (\mathcal{D}, ψ) then the family $\{P_N(D_i)\}_i$ has a limiting distribution. Theorem 5.7 calculates the limiting distribution in terms of cyclic Fock space. In the proof of Theorem 5.7 we have shown that the limiting distribution is free of second order in that if $d_i d_j = 0$ for $i \neq j$ and \mathcal{A}_i is the algebra generated by $p(d_i)$ then $\mathcal{A}_1, \dots, \mathcal{A}_p$ are free with respect to (φ, ρ) where $\varphi(a) = \langle a\Omega, \Omega \rangle$ and $\rho(a_1, a_2) = \langle \mathbf{c} a_1 \Omega, \mathbf{c} a_2^* \Omega \rangle_{\text{cyc}}$. Thus we have shown orthogonal families of Wishart random matrices are asymptotically free of second order.

Now we can address the question of the relation between Gaussian random matrices and constant matrices. We can even be more general for the latter and consider random matrices which are independent from the Gaussian ones.

Let, as usual, $X_N(f)$ ($f \in \mathcal{H}_{\mathbb{R}}$) be a family of Hermitian Gaussian random matrices

$$X_N(f) = (x_{ij}(f))_{i,j=1}^N,$$

as in section 3.1

Theorem 9.4. *Let $\{X_N(f) \mid f \in \mathcal{H}_{\mathbb{R}}\}_N$ be a sequence of Hermitian Gaussian $N \times N$ -random matrices and $\{A_1, \dots, A_s\}_N$ a sequence of $N \times N$ -random matrices which has a second order limit distribution. If $\{X_N(f) \mid f \in \mathcal{H}_{\mathbb{R}}\}_N$ and $\{A_1, \dots, A_s\}_N$ are independent, then they are asymptotically free of second order.*

The proof of this theorem relies on the same kind of calculations as, for example, in [MN]. Since we do not want to go into random matrix calculations here, we defer more details about this to [KMS].

If the random matrices $\{A_1, \dots, A_s\}$ are non-random constant matrices with limiting distribution with respect to the trace, then all k_r vanish identically for $r \geq 2$, thus they have a second order limit distribution, and we get as a corollary of the above that the asymptotic freeness between Gaussian random matrices and constant matrices remains also true on the level of fluctuations, i.e., with respect to our concept of second order freeness.

A more systematic investigation of this concept will be pursued in forthcoming publications. In particular, fluctuations of Haar distributed unitary random matrices from this point of view will be treated in [MSS].

REFERENCES

- [Ans] M. Anshelevich: Appell polynomials and their relatives, *Int. Math. Res. Notices*, 2004, no. 65, 3469-3531
- [AJM] J. Ambjorn, J. Jurkiewicz and Y. Makeenko, Multiloop correlators for two-dimensional quantum gravity, *Phys. Lett. B* 251 (1990), 517 - 524.
- [C-D] T. Cabanal-Duvillard: Fluctuations de la loi empirique de grandes matrices aleatoires. *Ann. Inst. H. Poincare Probab. Statist.* 37 (2001), 373-402.
- [CC] M. Capitaine and M. Casalis, Asymptotic Freeness by Generalized Moments for Gaussian and Wishart Matrices. Application to Beta matrices. *Indiana Univ. Math. J.*, 53 (2004), 397-431.
- [Eyn] B. Eynard: Random Matrices, Cours de Physique Theorique de Saclay, 2000, (<http://www-spht.cea.fr/articles/t01/014>).
- [GSS] P. Glockner, M. Schürmann, and R. Speicher: Realization of free white noises. *Arch. Math.* 58 (1992), 407-416.
- [GLM] P. Graczyk, G. Letac, H. Massam, The Complex Wishart Distribution and the Symmetric Group, *Ann. of Statistics*, 31, (2003), 287-309.
- [HP] F. Hiai and D. Petz: *The semicircle law, free random variables and entropy.* AMS, Providence, RI, 2000.
- [HT] U. Haagerup and S. Thorbjørnsen: A new application of random matrices: $\text{Ext}(C_{\text{red}}^*(F_2))$ is not a group, *Annals of Math.*, 162 (2005), 711-775.
- [J] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Tracts in Mathematics, vol. 129, Cambridge University Press, Cambridge, 1997.
- [Joh] K Johansson: On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* 91 (1998), 151-204.
- [KMS] T. Kusalik, J. Mingo, and R. Speicher, Orthogonal Polynomials and Fluctuations of Random Matrices, preprint, math.OA/0503169.
- [MN] J. Mingo and A. Nica: Annular non-crossing permutations and partitions, and second-order asymptotics for random matrices, *Inter. Math. Res. Notices*, 28 (2004), 1413 - 1460.
- [MSS] J. Mingo, P. Śniady, and R. Speicher: Second Order Freeness and Fluctuations of Random Matrices: II. Unitary Random Matrices, preprint, math.OA/0405258.
- [NSp] A. Nica and R. Speicher: *Lectures on the Combinatorics of Free Probability Theory*, Paris, 1999
- [Pol] H. D. Politzer: Random matrix description of the distribution of mesoscopic conductance, *Phy. Rev. B*, 40 (1989), 11917 - 11919.
- [Sp1] R. Speicher: A new example of "Independence" and "White Noise". *Probab. Th. Rel. Fields* 84 (1990), 141-159.
- [Sp2] R. Speicher: Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. *Memoirs of the AMS* 132 (627), 1998

- [Voi1] D. Voiculescu: Limit laws for random matrices and free products, *Invent. Math.* 104 (1991), 201 - 220.
- [Voi2] D. Voiculescu: Lectures on free probability theory. *Lecture Notes in Math.* 1738 (Springer, 2000), 279-349.
- [VDN] D. Voiculescu, K. Dykema, and A. Nica: *Free Probability Theory*, Providence, RI, Amer. Math. Soc., 1991.
- [Zvo] A. Zvonkin: Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling* 26 (1997), 281-304.

QUEEN'S UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS, JEFFERY HALL, KINGSTON, ON K7L 3N6, CANADA

E-mail address: mingo@mast.queensu.ca

E-mail address: speicher@mast.queensu.ca

