$X = X^* = \frac{1}{\sqrt{N}} (x_{ij})$ is the $N \times N$ GUE, $\text{Tr}(A) = \sum_i a_{ii}$ is the un-normalized trace of $A = (a_{ij})$. $\mathcal{P}(r)$ is the partially order set of all partitions on the set $[r] = \{1, \ldots, r\}$. For $\mathcal{U}, \mathcal{V} \in \mathcal{P}(r)$, $\mu(\mathcal{U}, \mathcal{V})$ is the M"obius function of $\mathcal{P}(r)$ evaluated at the pair $(\mathcal{U}, \mathcal{V})$. $k_r(Y_1, \ldots, Y_r)$ is the $r$th cumulant of the random variables $Y_1, \ldots, Y_r$. For $\mathcal{U} \in \mathcal{P}(r)$ let the blocks of $\mathcal{U}$ be $\{U_1, \ldots, U_j\}$. In the equation below defining $E_\mathcal{U}$ we take the product over the blocks of $\mathcal{U}$.

$$E_\mathcal{U}(Y_1, \ldots, Y_r) = \prod_{U \in \mathcal{U}} E(Y_{i_1} \cdots Y_{i_k}).$$

Thus we have the first moment cumulant formula

$$k_r(Y_1, \ldots, Y_r) = \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) E_\mathcal{U}(Y_1, \ldots, Y_r).$$

If we define

$$k_\mathcal{U}(Y_1, \ldots, Y_r) = \prod_{U \in \mathcal{U}} k_j(Y_{i_1}, \ldots, Y_{i_j})$$

then we get the other moment cumulant formula which says that

$$E(Y_1 \cdots Y_r) = \sum_{\mathcal{U} \in \mathcal{P}(r)} k_\mathcal{U}(Y_1, \ldots, Y_r).$$

If $Y_1, \ldots, Y_r$ are independent centered Gaussians then only $k_2 \neq 0$, thus we have the (Wick) formula.

$$E(Y_{i_1} \cdots Y_{i_r}) = \sum_{\mathcal{U} \in \mathcal{P}_2(r)} k_\mathcal{U}(Y_{i_1}, \ldots, Y_{i_r}) = \sum_{\mathcal{U} \in \mathcal{P}_2(r)} \prod_{(s,t) \in \mathcal{U}} E(Y_{i_s} Y_{i_t}),$$

where $\mathcal{P}_2(r)$ is the pairings of $[r]$, i.e. all blocks of size 2.

Suppose we are given positive integers $n_1, \ldots, n_r$. Let $n = n_1 + \cdots + n_r$. Let $\gamma \in S_n$ be the permutation with cycles $(1, 2, \ldots, n) \cdots (n_1 + \cdots + n_{r-1} + 1, \ldots, n_1 + \cdots + n_r)$. Given a partition $\mathcal{U} \in \mathcal{P}(r)$ we construct $\tilde{\mathcal{U}}$ a partition in $\mathcal{P}(n)$ as follows. If $U = \{u_1, \ldots, u_k\}$ is a block of $\mathcal{U}$ we let $\tilde{U}$ be the block of $\tilde{\mathcal{U}}$ which is the union of the cycles of $\gamma$ labelled by $i_1, \ldots, i_k$. Namely

$$\tilde{U} = \bigcup_{j=1}^{k} \{n_1 + \cdots + n_{u_{j-1}} + 1, \ldots, n_1 + \cdots + n_{u_j}\}$$
\[ k_r(\text{Tr}(X^{n_1}), \text{Tr}(X^{n_2}), \ldots, \text{Tr}(X^{n_r})) \]
\[ = \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) E_{\mathcal{U}}(\text{Tr}(X^{n_1}), \ldots, \text{Tr}(X^{n_r})) \]
\[ = N^{-n/2} \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) \sum_{i_1, \ldots, i_n = 1}^{N} \sum_{\mathcal{V} \leq \mathcal{U}} k_{\mathcal{V}}(x_{i_1 i_{\gamma(1)}}, \ldots, x_{i_n i_{\gamma(n)}}) \]
\[ = N^{-n/2} \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) \sum_{i_1, \ldots, i_n = 1}^{N} \sum_{\mathcal{V} \leq \mathcal{U}} k_{\mathcal{V}}(x_{i_1 i_{\gamma(1)}}, \ldots, x_{i_n i_{\gamma(n)}}) \]

Now if \( \mathcal{V} \in \mathcal{P}_2(n) \) is a pairing \( k_{\mathcal{V}}(x_{i_1 i_{\gamma(1)}}, \ldots, x_{i_n i_{\gamma(n)}}) \neq 0 \) only if for each pair \((s, t)\) of \( \mathcal{V} \) we have \( E(x_{i_s i_{\gamma(s)}}, x_{i_t i_{\gamma(t)}}) \neq 0 \), i.e. \( i_s = i_{\gamma(t)} \) and \( i_t = i_{\gamma(s)} \). So let us think of \( \mathcal{V} \) as a permutation of \([n]\) of order 2 but without any fixed points. If \( \pi \) is a such a permutation then \( i_s = i_{\gamma(t)} \) and \( i_t = i_{\gamma(s)} \) means that \( i = i \circ \gamma \pi \), i.e. \( i \) is constant on the orbits of \( \gamma \pi \). Hence we may resume our calculation.

\[ k_r(\text{Tr}(X^{n_1}), \text{Tr}(X^{n_2}), \ldots, \text{Tr}(X^{n_r})) \]
\[ = N^{-n/2} \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) \sum_{i_1, \ldots, i_n = 1}^{N} \sum_{\mathcal{V} \leq \mathcal{U}} k_{\mathcal{V}}(x_{i_1 i_{\gamma(1)}}, \ldots, x_{i_n i_{\gamma(n)}}) \]
\[ = N^{-n/2} \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) \sum_{\pi \in \mathcal{P}_2(n)} N^\#(\gamma \pi) \]
\[ = \sum_{\pi \in \mathcal{P}_2(n)} N^\#(\gamma \pi) - n/2 \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) \cdot \]

The final step is to find for which \( \pi \in \mathcal{P}_2(n) \) the sum \( \sum_{\mathcal{U} \in \mathcal{P}(r)} \mu(\mathcal{U}, 1_r) \neq 0 \).

Recall that (by definition) the sum of the Möbius function over an interval is always 0 unless the interval is a single point. Thus the only \( \pi \)'s which count are those for which \( \mathcal{U} \geq \pi \) implies that \( \mathcal{U} = 1_r \), i.e. \( \pi \) must satisfy
\( \pi \lor \gamma = 1_n \). This means that \( \pi \) connects the cycles of \( \gamma \). Thus

\[
k_r(\text{Tr}(X^{n_1}), \text{Tr}(X^{n_2}), \ldots, \text{Tr}(X^{n_r})) = \sum_{\begin{subarray}{c} \pi \in \mathcal{P}_2(n) \\ \pi \lor \gamma = 1_n \end{subarray}} N^{\#(\gamma \pi) - n/2}.
\]

Finally we observe that by Euler’s formula we have that for \( \pi \in \mathcal{P}_2(n) \) with \( \pi \lor \gamma = 1_n \)

\[
\#(\pi \gamma) - n/2 \leq 2 - r
\]

with equality only of \( \pi \) is planar with respect to \( \gamma \).