Generating functions for weighted Hurwitz numbers*

Mathieu Guay-Paquet[1] and J. Harnad[2,3]

[1] *Université du Québec à Montréal*
201 Av du Président-Kennedy, Montréal QC, Canada H2X 3Y7
email: mathieu.guaypaquet@lacim.ca

[2] *Centre de recherches mathématiques, Université de Montréal*
C. P. 6128, succ. centre ville, Montréal, QC, Canada H3C 3J7
e-mail: harnad@crm.umontreal.ca

[3] *Department of Mathematics and Statistics, Concordia University*
7141 Sherbrooke W., Montréal, QC Canada H4B 1R6

Abstract

Weighted Hurwitz numbers for n-sheeted branched coverings of the Riemann sphere are introduced, together with associated weighted paths in the Cayley graph of S_n generated by transpositions. The generating functions for these, which include all formerly studied cases, are 2D Toda τ-functions of generalized hypergeometric type. Two new types of weightings are defined by coefficients in the Taylor expansion of the exponentiated quantum dilogarithm function. These are shown to provide q-deformations of strictly monotonic and weakly monotonic path enumeration generating functions. The standard double Hurwitz numbers are recovered from both types in the classical limit. By suitable interpretation of the parameter q, the corresponding statistical mechanics of random branched covers is related to that of Bose gases.

1 Introduction

In [7, 9], a method was developed for constructing generating functions for different variants of Hurwitz numbers, counting branched coverings of the Riemann sphere satisfying certain specified conditions. These were all shown to be 2D Toda τ-functions [19, 21, 20] of the special hypergeometric type [18]. A natural combinatorial implementation of the construction was shown to lead to equivalent interpretations of these in terms of path-counting in the Cayley graph of the symmetric group generated by transpositions. All previously known cases of generating functions for Hurwitz numbers were placed within this approach, and several new examples were studied, both from the enumerative geometrical and the combinatorial viewpoint.

In the present work, this approach is extended to include *any* 2D Toda τ-function of hypergeometric type, interpreted as a generating function for some suitably defined problem

*Work supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds de recherche du Québec – Nature et technologies (FRQNT).

of weighted enumeration of paths in the Cayley graph of S_n. In several examples, it is shown that this may equivalently be interpreted as weighted enumeration of branched covers of the Riemann sphere. The only additional ingredient needed is the specification of a suitably defined weighting for the branch points, depending on their ramification indices.

It is straightforward to recover all previous cases studied in [17, 5, 7, 9, 22, 2, 1, 11], and add to these a number of new ones that were never previously considered. One special case leads to the notion of q-deformed Hurwitz numbers or quantum Hurwitz numbers, of which we provide two possible variants. These may be seen as q-deformations of the previously studied generating functions for strictly monotonic and weakly monotonic path counting, which correspond in the Hurwitz number framework respectively to Belyi curves [22, 11, 2] and to signed counting, at fixed genus. The classical limit of both these weighted enumerations is easily seen to be the standard double Hurwitz number generating functions introduced by Okounkov [17]. In the quantum case the number of branch points is a random variable, as is the profile of any individual branch point. The τ-function may be viewed as a generating function for the expectation values of the Hurwitz numbers.

2 Generating functions for hypergeometric τ-functions

2.1 Weight generating functions and Jucys-Murphy elements

Let

$$G(z) = \sum_{k=0}^{\infty} G_k z^k \tag{2.1}$$

be the Taylor series of a complex analytic function in a neighbourhood of the origin, with $G(0) = 1$. In what follows, this will be referred to as the *weight generating function* since, it will determine the weights associated with path enumeration in the Cayley graph of S_n generated by the transpositions, and also, in the cases considered, those associated to the ramification structure of branched covers of the Riemann sphere. Developing further the methods introduced in [7, 9], we will show how to use such functions to construct generating functions for *weighted* Hurwitz numbers that are 2D Toda τ-functions [19, 21, 20]. As with ordinary Hurwitz numbers, these are also interpretable as weighted enumeration of certain paths in the Cayley graph of the symmetric group S_n generated by transpositions. We refer the reader to [7, 9], for details of the construction, notation and further examples.

Let $(a, b) \in S_n$ denote the transposition interchanging the elements a and b, and

$$\mathcal{J}_b := \sum_{a=1}^{b-1} (a, b), \quad b = 1, \dots, n \tag{2.2}$$

the Jucys-Murphy elements [12, 15, 3] of the group algebra $\mathbf{C}[S_n]$, which generate a maximal commutative subalgebra. We associate an element $G(z, \mathcal{J})$ of the completion of the center of the group algebra $\mathbf{Z}(\mathbf{C}[S_n])$) by forming the product

$$G(z, \mathcal{J}) := \prod_{a=1}^{n} G(z\mathcal{J}_a). \tag{2.3}$$

It follows that under multiplication such elements determine endomorphisms of $\mathbf{Z}(\mathbf{C}[S_n]))$ that are diagonal in the basis $\{F_\lambda\}$ of orthogonal idempotents

$$G(z, \mathcal{J})F_\lambda = r_\lambda^G(z)F_\lambda, \tag{2.4}$$

where the eigenvalue is given by the content product formula [18, 7, 9],

$$r_\lambda^{G(z)} := \prod_{(i,j)\in\lambda} G(z(j-i)) \tag{2.5}$$

taken over the elements of the Young diagram of the partition λ of weight $|\lambda| = n$.

Ref. [7] shows how to use such elements to define 2D Toda τ-functions that serve as generating functions for combinatorial invariants enumerating certain paths in the Cayley graph of S_n generated by transpositions. The associated 2D Toda τ-function $\tau^G(\mathbf{t}, \mathbf{s})$ is of hypergeometric type [18], and may be expressed as a diagonal sum over products of Schur functions

$$\tau^G(\mathbf{t}, \mathbf{s}) = \sum_\lambda r_\lambda^{G(z)} S_\lambda(\mathbf{t})S_\lambda(\mathbf{s}), \tag{2.6}$$

where

$$\mathbf{t} = (t_1, t_2, \dots), \quad \mathbf{t} = (s_1, s_2, \dots) \tag{2.7}$$

are the 2D Toda flow variables, which may be identified in this notation with the power sum symmetric functions

$$t_i = ip_i, \quad s_i = ip_i'. \tag{2.8}$$

(See [14] for notation and definitions involving symmetric functions.)

Remark 2.1. No lattice site $N \in \mathbf{Z}$ is indicated in (2.6), since in the examples considered below, only $N = 0$ is required. The N dependence can be introduced by replacing the factor $G(z(j-i))$ in the content product formula (2.5) by $G(z(N+j-i))$, and multiplying by an overall λ-independent factor $r_0^G(z)$ in a standard way [18, 8]. This produces a lattice of τ-functions $\tau^G(N, \mathbf{t}, \mathbf{s})$ which, for all the cases considered below, may be explicitly expressed in terms of $\tau^G(0, \mathbf{t}, \mathbf{s}) =: \tau^G(\mathbf{t}, \mathbf{s})$ by applying a suitable transformation of the parameters involved [17, 9], and a multiplicative factor depending on N.

Using the Frobenius character formula, and the series expansions (2.1), we can obtain an equivalent expansion in terms of products of power sum symmetric functions and a power series in z

$$\tau^G(\mathbf{t}, \mathbf{s}) = \sum_{k=0}^{\infty} \sum_{\substack{\mu,\nu,\\ |\mu|=|\nu|}} F_G^k(\mu, \nu)P_\mu(\mathbf{t})P_\nu(\mathbf{s})z^k. \tag{2.9}$$

The coefficients $F_G^k(\mu,\nu)$ will be interpreted as weighted enumerations of paths in the Cayley graph starting at an element in the conjugacy class of cycle type μ to the class of type ν. A geometrical interpretation of $F_G^k(\mu,\nu)$ will also be given as weighted Hurwitz numbers, which give weighted enumerations of branched covers of the Riemann sphere. Various examples of such generating functions have been studied in [5, 7, 9], including more complicated cases that involve dependence on families of auxiliary parameters constructed in a multiplicative way.

2.2 Fermionic representation

Double KP τ-functions of the form (2.6) have the fermionic representation [18, 7, 8, 9]

$$\tau^G(\mathbf{t},\mathbf{s}) = \langle 0|\hat{\gamma}_+(\mathbf{t})\hat{C}_G\hat{\gamma}_-(\mathbf{t})|0\rangle \tag{2.10}$$

where, in terms of the fermionic creation and annihilation operators $\{\psi_i, \psi_i^\dagger\}_{i\in\mathbf{Z}}$ satisfying the usual anticommutation relations

$$[\psi_i,\, \psi_j^\dagger]_+ = \delta_{ij} \tag{2.11}$$

and vacuum state $|0\rangle$ vanishing conditions

$$\psi_i|0\rangle = 0, \quad \text{for } i<0, \quad \psi_i^\dagger|0\rangle = 0, \quad \text{for } i\geq 0, \tag{2.12}$$

we have

$$\hat{C}_G = e^{\sum_{j=-\infty}^{\infty} T_j^G :\psi_j\psi_j^\dagger:} \tag{2.13}$$

$$T_j^G = \sum_{k=1}^{j} \ln G(zk), \quad T_0^G = 0, \quad T_{-j}^G = -\sum_{k=0}^{j-1} \ln G(-zk) \quad \text{for } j>0, \tag{2.14}$$

$$\hat{\gamma}_+(\mathbf{t}) = e^{\sum_{i=1}^{\infty} t_i J_i}, \quad \hat{\gamma}_-(\mathbf{s}) = e^{\sum_{i=1}^{\infty} s_i J_i}, \quad J_i = \sum_{k\in\mathbf{Z}} \psi_k\psi_{k+i}^\dagger, \quad i\in\mathbf{Z}. \tag{2.15}$$

More generally, using the charge N vacuum state

$$|N\rangle = \psi_{N-1}\cdots\psi_0|0\rangle, \quad |-N\rangle = \psi_{-N}^\dagger\cdots\psi_{-1}^\dagger|0\rangle, \quad N\in\mathbf{N}^+, \tag{2.16}$$

we may define a 2D Toda lattice of τ-functions by

$$\tau^G(N,\mathbf{t},\mathbf{s}) = \langle N|\hat{\gamma}_+(\mathbf{t})\hat{C}_G\hat{\gamma}_-(\mathbf{t})|N\rangle \tag{2.17}$$

$$= \sum_\lambda r_\lambda^{G(z)}(N) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}), \tag{2.18}$$

where

$$r_\lambda^{G(z)}(N) := \prod_{(i,j)\in\lambda} G(z(N+j-i)), \tag{2.19}$$

and these satisfy the infinite set of Hirota bilinear equations for the 2D Toda lattice hierarchy [21, 19, 20].

2.3 Examples

The following lists some examples that were studied previously in [17, 5, 7, 9], as well as some further examples that will be considered in the present work, together with their interpretation as generating functions for the weighted enumeration of various configurations of branched covers of the Riemann sphere and, correspondingly, weighted paths in the Cayley graph of the symmetric group.

Example 2.1. Double Hurwitz numbers [17].

$$
\begin{aligned}
&G(z) = \exp(z) := e^z, \quad \exp(z, \mathcal{J}) = e^{z\sum_{b=1}^n \mathcal{J}_b}, \quad \exp_i = \frac{1}{i!} \\
&r_j^{\exp} = e^{zj}, \quad r_\lambda^{\exp}(z) = e^{\frac{z}{2}\sum_{i=1}^{\ell(\lambda)} \lambda_i(\lambda_i - 2i+1)}, \quad T_j^{\exp} = \frac{1}{2} j(j+1)z.
\end{aligned} \tag{2.20}
$$

The coefficients $F_G^k(\mu,\nu)$ in this case are Okounkov's double Hurwitz numbers [17], which enumerate branched coverings of the Riemann sphere with ramification types μ and ν at 0 and ∞, and k additional simple branch points. Equivalently, they enumerate d-step paths in the Cayley graph of S_n from an element in the conjugacy class of cycle type μ to the class of cycle type ν. The genus is given by the Riemann-Hurwitz formula

$$
2 - 2g = \ell(\mu) + \ell(\nu) - k. \tag{2.21}
$$

Example 2.2. Monotone double Hurwitz numbers [7, 9].

$$
\begin{aligned}
&G(z) = E(z) := 1 + z, \quad E(z, \mathcal{J}) = \prod_{a=1}^n (1 + z\mathcal{J}_a) \\
&E_1 = 1, \quad G_j = E_j = 0 \text{ for } j > 1,
\end{aligned}
$$

$$
r_j^E = 1 + zj, \quad r_\lambda^E(z) = \prod_{((ij)\in\lambda} (1 + z(j-i)), \tag{2.22}
$$

$$
T_j^E = \sum_{k=1}^{j} \ln(1+kz), \quad T_{-j}^E = -\sum_{k=1}^{j-1} \ln(1-kz), \quad j > 0. \tag{2.23}
$$

The coefficients $F_G^k(\mu,\nu)$ are double Hurwitz numbers for Belyi curves, [22, 11, 2], which enumerate n-sheeted branched coverings of the Riemann sphere having three ramification points, with ramification profile types μ and ν at 0 and ∞, and a single additional branch point, with $n-k$ preimages. The genus is again given by the Riemann-Hurwitz formula (2.21).

Equivalently, they enumerate k-step paths in the Cayley graph of S_n from an element in the conjugacy class of cycle type μ to the class of cycle type ν, that are strictly monotonically increasing in their second elements [7].

Example 2.3. Multimonotone double Hurwitz numbers [9].

$$
G(z) = E^l(z) := (1+z)^l, \quad E^l(z, \mathcal{J}) = \prod_{a=1}^n (1 + z\mathcal{J}_a)^l, \quad E_i^l = \binom{l}{i}
$$

$$
\begin{aligned}
r_j^{E^l} &= (1+zj)^l, \quad r_\lambda^{E^l}(z) = \prod_{(ij)\in\lambda} (1+z(j-i))^l, \\
T_j^{E^l} &= l\sum_{k=1}^{j} \ln(1+kz), \quad T_{-j}^{E} = -l\sum_{k=1}^{j-1} \ln(1-kz), \quad j>0.
\end{aligned} \tag{2.24}
$$

The coefficients $F^k_{E^l}(\mu,\nu)$ are double Hurwitz numbers that enumerate branched coverings of the Riemann sphere with ramification profile types μ and ν at 0 and ∞, and l additional branch points [22, 11, 2], such that the sum of the colengths of their ramification profile type is equal to k.

Equivalently, they enumerate k-step paths in the Cayley graph of S_n, formed from consecutive transpositions, from an element in the conjugacy class of cycle type μ to the class of cycle type ν, that consist of a sequence of l strictly monotonically increasing subsequences in their second elements [7, 9].

Example 2.4. Weakly monotone double Hurwitz numbers [5, 7].

$$
\begin{aligned}
G(z) = H(z) &:= \frac{1}{1-z}, \quad H(z,\mathcal{J}) = \prod_{a=1}^{n} (1-z\mathcal{J}_a)^{-1}, \quad H_i = 1,\ i\in \mathbf{N}^+ \\
r_j^{H} &= (1-zj)^{-1}, \quad r_\lambda^{H}(z) = \prod_{(ij)\in\lambda} (1-z(j-i))^{-1},
\end{aligned} \tag{2.25}
$$

$$
T_j^{H} = -\sum_{k=1}^{j} \ln(1-kz), \quad T_{-j}^{E} = \sum_{k=1}^{j-1} \ln(1+kz), \quad j>0. \tag{2.26}
$$

The coefficients $F^k_H(\mu,\nu)$ are double Hurwitz numbers n-sheeted branched coverings of the Riemann sphere curves with branch points at 0 and ∞ having ramification profile types μ and ν, and an arbitrary number of further branch points, such that the sum of the complements of their ramification profile lengths (i.e., the "defect" in the Riemann Hurwitz formula) is equal to k. The latter are counted with a sign, which is $(-1)^{n+d}$ times the parity of the number of branch points [9]. The genus is again given by (2.21).

Equivalently, they enumerate k-step paths in the Cayley graph of S_n from an element in the conjugacy class of cycle type μ to the class cycle type ν, that are weakly monotonically increasing in their second elements [7].

Example 2.5. Quantum Hurwitz numbers (i).

$$G(z) = E(q,z) := \prod_{k=0}^{\infty}(1+q^k z) = \sum_{k=0}^{\infty} E_k(q) z^k, \tag{2.27}$$

$$E_i(q) := \prod_{j=0}^{i} \frac{q^j}{1-q^j}, \tag{2.28}$$

$$E(q,\mathcal{J}) = \prod_{a=1}^{n}\prod_{i=0}^{\infty}(1+q^i z \mathcal{J}_a), \tag{2.29}$$

$$r_j^{E(q)} = \prod_{k=0}^{\infty}(1+q^k zj), \quad r_\lambda^{E(q)}(z) = \prod_{k=0}^{\infty}\prod_{(ij)\in\lambda}^{n}(1+q^k z(j-i)), \tag{2.30}$$

$$T_j^{E(q)} = -\sum_{i=1}^{j} \mathrm{Li}_2(q,-zi), \quad T_{-j}^{E(q)} = \sum_{i=0}^{j} \mathrm{Li}_2(q,zi), \quad j>0. \tag{2.31}$$

This generating function for weights is related to the quantum dilogarithm function by

$$E(q,z) = e^{-\mathrm{Li}_2(q,-z)}, \quad \mathrm{Li}_2(q,z) := \sum_{k=1}^{\infty}\frac{z^k}{k(1-q^k)}. \tag{2.32}$$

The combinatorial meaning of $E_k(q)$ is: these are generating functions for the number of partitions having at most k parts. A slight modification of this is obtained by removing the q^0 factor, which will be seen below to correspond to removing the zero energy level, giving the generating function

$$E'(q,z) := \prod_{k=1}^{\infty}(1+q^k z). \tag{2.33}$$

This will be seen to correspond to a Bose gas with energy levels proportional to the integers.

Geometrically, the weighted Hurwitz numbers these generate may be viewed as the expectation values for weighted covers, in which the weight coincides with that for the Bose gas state whose energy levels are identified as proportional to the total ramification defect over each branch point. In the classical limit $q \to 1$, we recover example 2.1

Example 2.6. Quantum Hurwitz numbers (ii).

$$G(z) = H(q,z) := \prod_{i=0}^{\infty}(1-q^i z)^{-1} = e^{\mathrm{Li}_2(q,z)} = \sum_{k=0}^{\infty} H_k(q) z^k, \tag{2.34}$$

$$H_k(q) := \prod_{j=1}^{k} \frac{q}{1-q^j}, \tag{2.35}$$

$$H(q,\mathcal{J}) = \prod_{k=0}^{\infty}\prod_{a=1}^{n}(1-q^k z\mathcal{J}_a)^{-1}, \tag{2.36}$$

$$r_j^{H(q)} = \prod_{k=0}^{\infty}(1-q^k zj)^{-1}, \quad r_\lambda^{H(q)}(z) = \prod_{k=0}^{\infty}\prod_{(ij)\in\lambda}(1-q^k z(j-i))^{-1}. \tag{2.37}$$

$$T_j^{H(q)} = \sum_{i=1}^{j}\mathrm{Li}_2(q,zi), \quad T_{-j}^{H(q)} = -\sum_{i=1}^{j-1}\mathrm{Li}_2(q,-zi), \quad j>0. \tag{2.38}$$

The modification corresponding to removing the zero energy level state is based similarly on the generating function

$$H'(q,z) := \prod_{k=1}^{\infty}(1-q^k z)^{-1}. \tag{2.39}$$

The combinatorial meaning of $H_k(q)$ is: these are generating functions for the number of partitions having exactly k parts.

Geometrically, these generate the signed version of the weighted Hurwitz numbers associated to the Bose gas interpretation, with the sign determined by the parity of the number of branch points. In the classical limit $q \to 1$, we again recover example 2.1.

Example 2.7. Double Quantum Hurwitz numbers.

$$G(z) = Q(q,p,z) := E(q,z)H(p,z) = \prod_{i=0}^{\infty}(1+q^i z)(1-p^i z)^{-1} = \sum_{k=0}^{\infty} Q_k(q,p) z^k, \tag{2.40}$$

$$Q_k(q,p) := \sum_{m=0}^{k}\left(\prod_{j=1}^{m}\frac{q^j}{1-q^j}\right)\left(\prod_{j=1}^{k-m}\frac{p}{1-p^j}\right), \tag{2.41}$$

$$Q(q,p,\mathcal{J}) = E(q,z,\mathcal{J})H(p,z,\mathcal{J}), \tag{2.42}$$

$$r_j^{Q(q,p)} = \prod_{k=0}^{\infty}\frac{1+q^k zj}{1-p^k zj}, \quad r_\lambda^{Q(q,p)}(z) = \prod_{k=0}^{\infty}\prod_{(ij)\in\lambda}\frac{1+q^k z(j-i)}{1-p^k z(j-i)}. \tag{2.43}$$

$$\begin{aligned} T_j^{Q(q,p)} &= \sum_{i=1}^{j}\mathrm{Li}_2(p,zi) - \sum_{i=1}^{j}\mathrm{Li}_2(q,-zi), \\ T_{-j}^{Q(q,p)} &= -\sum_{i=1}^{j-1}\mathrm{Li}_2(p,-zi) + \sum_{i=1}^{j-1}\mathrm{Li}_2(q,zi), \quad j>0. \end{aligned} \tag{2.44}$$

Geometrically, these generate a hybrid of two types of counting; i.e., two species of branch points, one of which ("uncoloured") are counted with the weight corresponding to a Bose gas, the other ("coloured") are counted with signs, as in the previous example, determined by the parity of the number of such branch points. In the classical limit $q \to 1, p \to 1$, we again recover example 2.1. The generating partition function $Q(q, p, z)$ for the general case may be identified as the measure used in defining the orthogonality of Macdonald polynomials.

2.4 Weighted expansions

Lemma 2.1. *For any weight generating function $G(z)$, we have the following expansion for $G(z, \mathcal{J})$*

$$G(z, \mathcal{J}) = \sum_{\lambda} G_{\lambda}\ M_{\lambda}(\mathcal{J}) z^{|\lambda|}, \tag{2.45}$$

where

$$G_{\lambda} := \prod_{i=1}^{|\lambda|} (G_i)^{j_i} = \prod_{j=1}^{\ell(\lambda)} G_{\lambda_j}, \tag{2.46}$$

with j_i the number of parts of λ equal to i, and

$$M_{\lambda}(\mathcal{J}) = \sum_{b_1, \dots, b_{\ell(\lambda)} \subset \{1, \dots, n\}} \mathcal{J}_{b_1} \cdots \mathcal{J}_{b_{\ell(\lambda)}} \tag{2.47}$$

the monomial sum symmetric function evaluated on the Jucys-Murphy elements.

Proof.

$$\begin{aligned} G(z, \mathcal{J}) &= \prod_{a=1}^{n} \left(\sum_{k=0}^{\infty} G_k z^k \mathcal{J}_a^k \right) \\ &= \left(\sum_{k_1=0}^{\infty} G_{k_1} z^{k_1} \mathcal{J}_1^{k_1} \right) \cdots \left(\sum_{k_n=0}^{\infty} G_{k_n} z^{b_k} \mathcal{J}_n^{k_n} \right) \\ &= \sum_{k=0}^{\infty} z^k \sum_{\lambda,\ |\lambda|=k} \ \sum_{(b_1, \dots, b_k) \subset (1, \dots, n)} \left(\prod_{i=1}^{\ell(\lambda)} G_{\lambda_i} J_{b_i}^{\lambda_i} \right) \\ &= \sum_{\lambda} G_{\lambda}\ M_{\lambda}(\mathcal{J}) z^{|\lambda|}. \end{aligned} \tag{2.48}$$

□

Let $\{C_{\mu}\}$ denote the basis of the center $\mathbf{Z}(\mathbf{C}[S_n])$ of the group algebra $\mathbf{C}[S_n]$ consisting of the cycle sums

$$C_{\mu} = \sum_{g \in cyc(\mu)} g \tag{2.49}$$

where $cyc(\mu)$ is the conjugacy class of elements whose cycle lengths are given by the partition μ.

Lemma 2.2. *Multiplication by $M_\lambda(\mathcal{J})$ defines an endomorphism of $\mathbf{Z}(\mathbf{C}[S_n])$ which, expressed in the $\{C_\mu\}$ basis is given by*

$$M_\lambda(\mathcal{J})C_\mu = \sum_{\nu,\,|\nu|=|\mu|} |cyc(\nu)|^{-1} m^\lambda_{\mu\nu} C_\nu, \tag{2.50}$$

where

$$|cyc(\nu)| = \frac{|\nu|!}{Z_\nu}, \quad Z_\nu = \prod_{i=1}^{|\nu|} i^{j_i}(j_i)! \tag{2.51}$$

with j_i the number of parts of ν equal to i and $m^\lambda_{\mu\nu}$ the number of $|\lambda|$-step paths $(a_1b_1)\cdots(a_{|\lambda|}b_{|\lambda|})$ in the Cayley graph of S_n generated by transpositions, starting at an element in the conjugacy class with cycle type μ and ending in the class ν, that are weakly monotonically increasing, with λ_i elements in the successive bands in which the second elements b_i remains constant.

Remark 2.2. The enumerative constants $m^\lambda_{\mu\nu}$ may be interpreted in another way, that is perhaps more natural, since it puts no restrictions on the monotonicity of the path. If we identify the partition λ as a *signature* for a path in the sense that, without any ordering restriction, we count how many times each particular second element appears in the sequence, and these, in a weakly descending sequence, are equated to the parts of the partition λ, it is clear that for every sequence that is weakly increasing in the above sense, with constancy of the second elements in successive subsequences of lengths equal to the parts of λ, the number of unordered sequences in which such the second elements are repeated these numbers of times is

$$\tilde{m}^\lambda_{\mu\nu} = \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda} \lambda_i!} m^\lambda_{\mu\nu}. \tag{2.52}$$

It follows that

Proposition 2.3.

$$G(z,\mathcal{J})C_\mu = \sum_{k=1}^{\infty} Z_\nu F^k_G(\mu,\nu) C_\nu z^k, \tag{2.53}$$

where

$$F^k_G(\mu,\nu) = \frac{1}{n!} \sum_{\lambda,\ |\lambda|=k} G_\lambda m^\lambda_{\mu\nu} = \sum_{\lambda,\ |\lambda|=k} (\prod_{i=1}^{\ell(\lambda} \lambda_i!) G_\lambda \tilde{m}^\lambda_{\mu\nu} \tag{2.54}$$

is the weighted sum over all such k-step paths, with weight G_λ.

In particular, if $G(z)$ is chosen to be $\exp := e^z$, $E = 1+z$ and $H = (1-z)^{-1}$, we have

$$\exp_\lambda = \frac{1}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!}, \quad E_\lambda = \delta_{\lambda,(1)^{|\lambda|}} \quad H_\lambda = 1, \tag{2.55}$$

and the corresponding weighted combinatorial Hurwitz numbers are

$$F^k_{\exp}(\mu,\nu) = \sum_{\lambda,\ |\lambda|=k} \frac{1}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} m^\lambda_{\mu\nu} \tag{2.56}$$

$$F^k_E(\mu,\nu) = m^{(1)^k}_{\mu\nu} \tag{2.57}$$

$$F^k_H(\mu,\nu) = \sum_{\lambda,\ |\lambda|=k} m^\lambda_{\mu\nu}. \tag{2.58}$$

This means that $F^k_{\exp}(\mu,\nu)$ is the number of (unordered) sequences of k transpositions leading from an element in the class of type μ to the class of type ν; $F^k_E(\mu,\nu)$ is the number of strictly monotonically increasing such paths, and $F^k_H(\mu,\nu)$ is the number of weakly monotonically increasing such paths.

If $G(z)$ is chosen to be $E(q,z)$, $H(q,z)$ or $Q(q,p,z)$, respectively, we have

$$E_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} E_{\lambda_i}(q), \quad H_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} H_{\lambda_i}(q) \quad Q_\lambda(q,p) = \prod_{i=1}^{\ell(\lambda)} Q_{\lambda_i}(q,p) \tag{2.59}$$

and the corresponding weighted combinatorial Hurwitz numbers are

$$F^k_{E(q)}(\mu,\nu) = \frac{1}{n!} \sum_{\lambda,\ |\lambda|=k} E_\lambda(q) m^\lambda_{\mu\nu} \tag{2.60}$$

$$F^k_{H(q)}(\mu,\nu) = \frac{1}{n!} \sum_{\lambda,\ |\lambda|=k} H_\lambda(q) m^\lambda_{\mu\nu} \tag{2.61}$$

$$F^k_{Q(q,p)}(\mu,\nu) = \frac{1}{n!} \sum_{\lambda,\ |\lambda|=k} Q_\lambda(q,p) m^\lambda_{\mu\nu} \tag{2.62}$$

Their interpretation in terms of weighted branched covers will be given below.

3 The 2D Toda τ-functions $\tau^{G(z)}(\mathbf{t},\mathbf{s})$

3.1 Generating functions for weighted paths

For each choice of $G(z)$, we define a corresponding 2D Toda τ-function of generalized hypergeometric type (for $N=0$) by the formal series

$$\tau^{G(z)}(\mathbf{t},\mathbf{s}) := \sum_\lambda r^G_\lambda(z) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}). \tag{3.1}$$

It follows from general considerations [20, 8] that this is indeed a double KP τ-function, that, when extended suitably to a lattice $\tau^{G(z)}(N,\mathbf{t},\mathbf{s})$ of such τ functions, satisfies the corresponding system of Hirota bilinear equations of the 2D Toda hierarchy [19, 21, 20].

Substituting the Frobenius character formula

$$S_\lambda = \sum_{\mu,\, |\mu|=|\lambda|} Z_\mu^{-1} \chi_\lambda(\mu) P_\mu \tag{3.2}$$

for each of the factors $S_\lambda(\mathbf{t})S_\lambda(\mathbf{s})$ into (3.1) and substituting the corresponding relation between the bases $\{C_\mu\}$ and $\{F_\lambda\}$

$$F_\lambda = h_\lambda^{-1} \sum_{\mu,\, |\mu|=|\lambda} \chi_\lambda(\mu) C_\mu \tag{3.3}$$

where h_λ is the product of the hook lengths of the partitions λ

$$h_\lambda^{-1} = \det\left(\frac{1}{(\lambda_i - i + j)!}\right), \tag{3.4}$$

into eqs. (2.4) and (2.53), equating coefficients in the C_μ basis, and using the orthogonality relation for the characters

$$\sum_{\mu,\, |\mu|=|\lambda|} \chi_\lambda(\mu)\chi_\rho(\mu) = Z_\mu \delta_{\lambda\rho}, \tag{3.5}$$

we obtain the expansion

$$\tau^{G(z)}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^\infty \sum_{\substack{\mu,\nu,\\ |\mu|=|\mu|}} z^k F_G^k(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}). \tag{3.6}$$

This proves the following result.

Theorem 3.1. *$\tau^{G(z)}(\mathbf{t},\mathbf{s})$ is the generating function for the numbers $F_G^k(\mu,\nu)$ of weighted k-step paths in the Cayley graph, starting at an element in the conjugacy class of cycle type μ and ending at the conjugacy class of type ν, with weights of all weakly monotonic paths of type λ given by G_λ.*

3.2 The special cases corresponding to examples 2.1 - 2.4

Example 3.2.1. (Sec. 2, example 2.1) In this case $G(z) = \exp(z)$,

$$\exp_i = \frac{1}{i!}, \quad \exp_\lambda = \exp_\lambda = \left(\prod_{i=1}^{|\lambda|} (i!)^{j_i}\right)^{-1} = \left(\prod_{j=1}^{\ell(\lambda)} (\lambda_i)!\right)^{-1}, \tag{3.7}$$

where j_i is the number of parts of λ equal to i,

$$\exp(z,\mathcal{J}) = \sum_{k=0}^\infty \frac{z^k}{k!}\left(\sum_{b=1}^n \mathcal{J}_b\right)^k \tag{3.8}$$

$$= \sum_{k=0}^\infty z^k \sum_{\lambda,\, |\lambda|=k} \left(\prod_{j=1}^{\ell(\lambda)} (\lambda_i)!\right)^{-1} \sum_{\{b_1,\dots b_k\}\subset\{1,\dots,n\}} \mathcal{J}_{b_1}^{\lambda_1} \cdots \mathcal{J}_{b_k}^{\lambda_{b_k}} \tag{3.9}$$

and

$$F^k_{\exp}(\mu,\nu) = \frac{1}{n!}\sum_{\lambda,\ |\lambda|=k}\left(\prod_{j=1}^{\ell(\lambda)}(\lambda!)\right)^{-1} m^{\lambda}_{\mu\nu} \tag{3.10}$$

is the number of k-term products $(a_1b_1)\dots(a_kb_k)$ such that, if $g \in \mathrm{cyc}(\mu)$, $(a_1b_1)\dots(a_kb_k)g \in \mathrm{cyc}(\nu)$.

Example 3.2.2. (Sec. 2, example 2.2) In this case $G(z) = E(z) = 1 + z$,

$$E_1 = 1, \quad E_j = 0 \ \text{ for } j > 1, \quad E_\lambda = \delta_{\lambda,(1)^{|\lambda|}}. \tag{3.11}$$

Therefore

$$\sum_{\lambda,|\lambda|=k} E_\lambda M_\lambda(\mathcal{J}) = \sum_{b_1<\cdots<b_k} \mathcal{J}_{b_1}\cdots\mathcal{J}_{b_k}. \tag{3.12}$$

and

$$F^k_E(\mu,\nu) = m^{(1)^k}_{\mu\nu} \tag{3.13}$$

is the number of k-term products $(a_1b_1)\dots(a_kb_k)$ that are strictly monotonically increasing, such that, if $g \in \mathrm{cyc}(\mu)$, $(a_1b_1)\dots(a_kb_k)g \in \mathrm{cyc}(\nu)$.

Example 3.2.3. (Sec. 2, example 2.3) In this case $G(z) = E^l(z) = (1+z)^l$, so

$$E^l_i = \binom{l}{i}, \quad E^l_\lambda = \prod_{i=1}^{\ell(\lambda)}\binom{l}{\lambda_i}, \quad \lambda_i \le l. \tag{3.14}$$

Therefore

$$\begin{aligned}\sum_{\lambda,|\lambda|=k} E^l_\lambda M_\lambda(\mathcal{J}) &= \sum_{\lambda,|\lambda|=k}\ \sum_{(b_1,\dots,b_{\ell(\lambda)})\subset(1,\dots,n)}\ \prod_{i=1}^{\ell(\lambda)}\binom{l}{\lambda_i}\mathcal{J}^{\lambda_i}_{b_i}\\ &= [z^k]\prod_{a=1}^{n}(1+zJ_a)^l\end{aligned} \tag{3.15}$$

where $[z^k]$ means the coefficient of z^k in the polynomial. and

$$F^k_{E^l}(\mu,\nu) = \sum_{\lambda,|\lambda|=k}\prod_{i=1}^{\ell(\lambda)}\binom{l}{\lambda_i} m^\lambda_{\mu\nu} \tag{3.16}$$

is the number of k-term products $(a_1b_1)\dots(a_kb_k)$ that are such that, if $g \in \mathrm{cyc}(\mu)$, $(a_1b_1)\dots(a_kb_k)g \in \mathrm{cyc}(\nu)$, consisting of l consecutive subsequences, each of which is strictly monotonically increasing in the second elements of each (a_ib_i).

Example 3.2.4. (Sec. 2, example 2.4) In this case $G(z) = H(z) = (1-z)^{-1}$, so

$$G_i = 1, \ \forall i \in \mathbf{N}^+, \quad G_\lambda = 1 \ \forall\lambda. \tag{3.17}$$

Therefore

$$\sum_{\lambda,|\lambda|=k} G_\lambda M_\lambda(\mathcal{J}) = \sum_{b_1\le\cdots\le b_k} \mathcal{J}_{b_1}\cdots\mathcal{J}_{b_k}. \tag{3.18}$$

and

$$F_H^k(\mu,\nu) = \sum_{\lambda,\ |\lambda|=k} m^\lambda_{\mu\nu} \tag{3.19}$$

is the number of number of k-term products $(a_1b_1)\dots(a_kb_k)$ that are weakly monotonically increasing, such that, if $g \in \mathrm{cyc}(\mu)$, $(a_1b_1)\dots(a_kb_k)g \in \mathrm{cyc}(\nu)$.

3.3 Generating functions for Hurwitz numbers: classical counting of covers

The following four examples were previously studied in ref. ([17, 7, 9]. The interpretation of the associated τ-functions as generating function with Hurwitz numbers of various groups of branched covers will just be recalled, without proof.

Example 3.3.1. For example 2.1, we have $G(z) = \exp(z) := e^z$ and the generating τ-function is [17]

$$\begin{aligned}\tau^{\exp(z)}(\mathbf{t},\mathbf{s}) &= \sum_\lambda e^{\frac{z}{2}\sum_{i=1}^{\ell(\lambda)}\lambda_i(\lambda_i-2i+1)} S_\lambda(\mathbf{t})S_\mu(\mathbf{s}) \\ &= \sum_{k=0}^\infty \sum_{\mu,\nu,\,|\mu|=|\nu|} F^k_{\exp}(\mu,\nu) z^k P_\mu(\mathbf{t})P_\nu(\mathbf{s}),\end{aligned} \tag{3.20}$$

where

$$F^k_{\exp}(\mu,\nu) = \frac{1}{k!} H(\mu^{(1)} = (2,(1)^{(n-2)}),\dots,\mu^{(k)} = (2,(1)^{(n-2)}),\mu,\nu) \tag{3.21}$$

is equal to $\frac{1}{k!}$ times Okounkov's double Hurwitz number; that is, the number of $n = |\mu| = |\nu|$ sheeted branched covers with branch points of ramification type μ and ν at the points 0 and ∞, and k further simple branch points.

Example 3.3.2. For example 2.2, we have $G(z) = E(z) := 1 + z$ and the generating τ-function is [7, 9]

$$\begin{aligned}\tau^{E(z)}(\mathbf{t},\mathbf{s}) &= \sum_\lambda z^{|\lambda|}(z^{-1})_\lambda S_\lambda(\mathbf{t})S_\mu(\mathbf{s}) \\ &= \sum_{k=0}^\infty z^k \sum_{\mu,\nu,\ \mu|=|\nu|} F^k_{\exp}(\mu,\nu) P_\mu(\mathbf{t})P_\nu(\mathbf{s}),\end{aligned} \tag{3.22}$$

where

$$F^k_E(\mu,\nu) = \sum_{\mu^{(1)},\ \ell^*(\mu_1)=k} H(\mu^{(1)},\mu,\nu) \tag{3.23}$$

is now interpreted as the number of $n = |\mu| = |\nu| = |\mu^{(1)}|$ sheeted branched covers with branch points of ramification type μ and ν at 0 and ∞, and one further branch point, with colength $\ell^*(\mu^{(1)} = 1$; i.e. the case of Belyi curves [22, 11, 9, 2] or *dessins d'enfants*.

Example 3.3.3. For example 2.3, we have $G(z) = E^l(z) := (1+z)^l$ and the generating τ-function is [9] is

$$\begin{aligned}\tau^{E^l(z)}(\mathbf{t},\mathbf{s}) &= \sum_{\lambda} z^{|\lambda|}(z^{-1})_\lambda S_\lambda(\mathbf{t})S_\mu(\mathbf{s}) \\ &= \sum_{k=0}^{\infty} z^k \sum_{\mu,\nu,\ |\mu|=|\nu|=n} F^k_{\exp}(\mu,\nu)P_\mu(\mathbf{t})P_\nu(\mathbf{s}),\end{aligned} \tag{3.24}$$

where

$$F^k_{E^l}(\mu,\nu) = \sum_{\substack{\mu^{(1)},\dots\mu^{(l)} \\ \sum_{i=1}^{l} \ell^*(\mu_i)=k}} H(\mu^{(1)},\dots\mu^{(l)},\mu,\nu) \tag{3.25}$$

is now interpreted [9] as the number of $n = |\mu| = \nu| = |\mu^{(i)}|$ sheeted branched covers with branch points of ramification type μ and ν at 0 and ∞, and l further branch points, the sums of whose colengths is k.

Example 3.3.4. For example 2.4, we have $G(z) = H(z) = (1-z)^{-1}$ and the generating τ-function is [7, 9]

$$\begin{aligned}\tau^{H(z)}(\mathbf{t},\mathbf{s}) &= \sum_{\lambda}(-z)^{|\lambda|}\left(-z^{-1}\right)_\lambda S_\lambda(\mathbf{t})S_\mu(\mathbf{s}) \\ &= \sum_{k=0}^{\infty} z^k \sum_{\mu,\nu,\ \mu|=|\nu|} F^k_H(\mu,\nu)P_\mu(\mathbf{t})P_\nu(\mathbf{s})\end{aligned} \tag{3.26}$$

where

$$F^k_H(\mu,\nu) = (-1)^{n+k}\sum_{j=1}^{\infty}(-1)^j \sum_{\substack{\mu^{(1)}\dots\mu^{(j)} \\ \sum_{i=1}^{j} \ell^*(\mu_i)=k}} H(\mu^{(1)},\dots\mu^{(j)},\mu,\nu) \tag{3.27}$$

is now interpreted as the signed counting of $n = |\mu| = |\nu|$ sheeted branched covers with branch points of ramification type μ and ν at 0 and ∞, and any number further branch points, the sum of whose colengths is k, with sign determined by the parity of the number of branch points [9].

3.4 The τ-functions $\tau^{E(q,z)}$, $\tau^{H(q,z)}$ and $\tau^{Q(q,p,z)}$ as generating functions for enumeration of q-weighted paths

The particular cases

$$\tau^{E(q,z)}(\mathbf{t},\mathbf{s}) := \sum_{\lambda} r_\lambda^{E(q)}(z) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) \tag{3.28}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{\mu,\nu, \\ |\mu|=|\nu|}} z^k F_{E(q)}^k(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \tag{3.29}$$

$$\tau^{H(q,z)}(\mathbf{t},\mathbf{s}) := \sum_{\lambda} r_\lambda^{H(q)}(z) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) \tag{3.30}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{\mu,\nu, \\ |\mu|=|\nu|}} z^k F_{H(q)}^k(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \tag{3.31}$$

$$\tau^{Q(q,p,z)}(\mathbf{t},\mathbf{s}) := \sum_{\lambda} r_\lambda^{Q(q,p)}(z) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) \tag{3.32}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{\mu,\nu, \\ |\mu|=|\nu|}} z^k F_{Q(q,p)}^k(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}). \tag{3.33}$$

may be viewed as special q-deformations of the generating functions associated to Examples 2.1, 2.2 , with $G(z) = 1+z$, $G(z) = (1-z)^{-1}$ respectively, and the hybrid combination generated by the product $\frac{1+w}{1-z}$. The latter were considered previously in [7, 9], and given both combinatorial and geometrical interpretations in terms of weakly or strictly monotonic paths in the Cayley graph.

Remark 3.1. Note that, for the special values of the flow parameters $(\mathbf{t},\mathbf{s})$ given by trace invariants of a pair of commuting $M \times M$ matrices, X and Y,

$$t_i = \frac{1}{i}\operatorname{tr}(X^i), \quad s_i = \frac{1}{i}\operatorname{tr}(Y^i) \tag{3.34}$$

with eigenvalues $(x_1,\dots,x_M)$, $(y_1,\dots,y_M)$, these may be viewed as special cases of the two types of basic hypergeometric functions of matrix arguments [6, 18].

3.5 The classical limits of examples 2.5, 2.6, 2.7

Setting $q = e^\epsilon$ for some small parameter, taking the leading term contribution in the limit $\epsilon \to 0$, we obtain

$$\lim_{\epsilon\to 0} E(q,\epsilon z) = e^z \tag{3.35}$$

and therefore, taking the scaled limit with $z \to \epsilon z$, we obtain

$$\lim_{\epsilon\to 0} \tau^{E(q,\epsilon z)}(\mathbf{t},\mathbf{s}) = \tau^{\exp(z)}(\mathbf{t},\mathbf{s}) \tag{3.36}$$

Similarly, we have

$$\lim_{\epsilon \to 0} H(q, \epsilon z) = e^z \tag{3.37}$$

and hence

$$\lim_{\epsilon \to 0} \tau^{H(q,\epsilon z)}(\mathbf{t}, \mathbf{s}) = \tau^{\exp(z)}(\mathbf{t}, \mathbf{s}) \tag{3.38}$$

And finally, for the double quantum Hurwitz case, example 2.7, setting

$$q = e^{\epsilon}, \quad p = e^{\epsilon'} \tag{3.39}$$

and replacing z by $z(\frac{1}{\epsilon} + \frac{1}{\epsilon'})$, we get

$$\lim_{\epsilon, \epsilon' \to 0} Q(q, p, \frac{z\epsilon\epsilon'}{\epsilon + \epsilon'}) = e^z \tag{3.40}$$

and hence

$$\lim_{\epsilon, \epsilon' \to 0} \tau^{Q(q,p,\frac{z\epsilon\epsilon'}{\epsilon+\epsilon'})}(\mathbf{t}, \mathbf{s}) = \tau^{\exp(z)}(\mathbf{t}, \mathbf{s}). \tag{3.41}$$

Thus, we recover Okounkov's classical double Hurwitz number generating function $\tau^{\exp(z)}(\mathbf{t}, \mathbf{s})$ as the classical limit in each case.

4 Quantum Hurwitz numbers

We now proceed to the interpretation of the quantities $F^k_{E(q,z)}(\mu, \nu)$, $F^k_{H(q,z)}(\mu, \nu)$ and $F^k_{Q(q,p,z)}(\mu, \nu)$ as weighted enumerations of branched coverings of the Riemann sphere. The key is the Frobenius-Schur-Burnside formula [13, Appendix A], [4], expressing the Hurwitz number $H(\mu^{(1)}, \dots, \mu^{(k)})$, giving the number of inequivalent n-sheeted branched coverings of the Riemann sphere, with k distinct branch points whose ramification profiles are given by the k partitions $(\mu^1, \dots, \mu^k)$, with weight n in terms of the characters of the symmetric group

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = \sum_{\lambda} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{Z_{\mu^{(i)}}}. \tag{4.1}$$

4.1 The case $E(q)$. Quantum Hurwitz numbers

It follows from the Frobenius character formula, that the Pochhammer symbol $(z)_\lambda$ may be written

$$(z)_\lambda = 1 + h_\lambda {\sum_{\mu,\, |\mu| = |\lambda|}}' \frac{\chi_\lambda(\mu)}{Z_\mu} z^{\ell^*(\mu)}. \tag{4.2}$$

where $\sum'$ means the sum over partitions, not including the cycle type of the identity element $(1)^{|\lambda|}$, and

$$\ell^*(\mu) = |\mu| - \ell(\mu) \tag{4.3}$$

is the colength of the partition μ. The content product formula (2.31) for this case may therefore be written as

$$r_\lambda^{H(q)}(z) = \prod_{k=0}^{\infty}\left(1 + \frac{h_\lambda \sum'_{\mu,\, |\mu|=|\lambda|} \chi_\lambda(\mu)}{Z_\mu}(zq^k)^{\ell^*(\mu)}\right) \tag{4.4}$$

$$= \sum_{d=0}^{\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_i=0}^{\infty}\sum'_{\substack{\mu^{(1)},\dots\mu^{(d)},\, |\mu^{(i)}|=|\lambda| \\ \sum_{i=1}^d \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d}\frac{h_\lambda \chi(\mu^{(i)})}{Z_{\mu^{(i)}}}(zq^{k_i})^{\ell^*(\mu^{(i)})} \tag{4.5}$$

$$= \sum_{k=0}^{\infty} z^k \sum_{d=0}^{\infty}\sum'_{\substack{\mu^{(1)},\dots\mu^{(d)},\, |\mu^{(i)}|=|\lambda| \\ \sum_{i=1}^d \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d}\frac{h_\lambda \chi(\mu^{(i)})}{Z_{\mu^{(i)}}}\left(\frac{1}{1-q^{\ell^*(\mu^{(i)})}}\right), \tag{4.6}$$

where $\sum'$ means the sum over partitions not including the cycle type of the identity element $(1)^{|\lambda|}$ and

$$\ell^*(\mu) = |\mu| - \ell(\mu) \tag{4.7}$$

is the complement of the length of the partitions μ, which we refer to as the *colength*.

Substituting this into (3.28) and using the Frobenius character formula (3.2) for each of the factors $S_\lambda(\mathbf{t})S_\lambda(\mathbf{s})$ gives

Theorem 4.1.

$$\tau^{E(q,z)}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} z^k \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^k_{E(q)}(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \tag{4.8}$$

where

$$H^k_{E(q)}(\mu,\nu) := \sum_{d=0}^{\infty}\sum'_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^d \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d}\left(\frac{1}{1-q^{\ell^*(\mu^{(i)})}}\right) H(\mu^{(1)},\dots,\mu^{(d)},\mu,\nu) \tag{4.9}$$

are the weighted (quantum) Hurwitz numbers that count the number of branched coverings with genus g given by (2.21) and sum of colengths k, with weight $\left(\frac{1}{1-q^{\ell^*(\mu^{(i)})}}\right)$ for every branch point with colength $\ell^*(\mu^{(i)})$.

From eq. (3.29) it follows that

Corollary 4.2. *The weighted (quantum) Hurwitz number for the branched coverings of the Riemann sphere with genus given by (2.21) is equal to the combinatorial Hurwitz number given by formula (2.60) enumerating weighted paths in the Cayley graph*

$$H^k_{E(q)}(\mu,\nu) = F^k_{E(q)}(\mu,\nu). \tag{4.10}$$

4.2 The case $H(q)$. Signed quantum Hurwitz numbers

We proceed similarly for this case. The content product formula (2.38) for this case may be written as

$$r_\lambda^{H(q)}(z) = \prod_{k=0}^{\infty} \left(1 + \frac{h_\lambda \sum'_{\mu,\,|\mu|=|\lambda|} \chi_\lambda(\mu)}{Z_\mu} (-zq^k)^{\ell^*(\mu)} \right)^{-1} \tag{4.11}$$

$$= \sum_{k=0}^{\infty} (-z)^k \sum_{d=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(d)},\,|\mu^{(i)}|=|\lambda| \\ \sum_{i=1}^{d} \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d} \frac{h_\lambda \chi(\mu^{(i)})}{Z_{\mu^{(i)}}} \left(\frac{1}{q^{\ell^*(\mu^{(i)})} - 1} \right). \tag{4.12}$$

Substituting this into (3.30) and using the Frobenius character formula (3.2) for each of the factors $S_\lambda(\mathbf{t})S_\lambda(\mathbf{s})$ gives

Theorem 4.3.

$$\tau^{H(q,z)}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} z^k \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^k_{H(q)}(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \tag{4.13}$$

where

$$H^k_{H(q)}(\mu,\nu) := (-1)^k \sum_{d=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^{d} \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d} \left(\frac{1}{q^{\ell^*(\mu^{(i)})} - 1} \right) H(\mu^{(1)},\dots,\mu^{(d)},\mu,\nu) \tag{4.14}$$

are the weighted (quantum) Hurwitz numbers that count the number of branched coverings with genus g given by (2.21) and sum of colengths k, with weight $\left(\frac{1}{q^{\ell^*(\mu^{(i)})}-1}\right)$ for every branch point with colength $\ell^*(\mu^{(i)})$.

From eq. (3.29) it follows that

Corollary 4.4. *The weighted (quantum) Hurwitz number for the branched coverings of the Riemann sphere with genus given by (2.21) is again equal to the combinatorial Hurwitz number given by formula (2.61) enumerating weighted paths in the Cayley graph*

$$H^k_{H(q)}(\mu,\nu) = F^k_{H(q)}(\mu,\nu). \tag{4.15}$$

4.3 The case $Q(q,p)$. Double Quantum Hurwitz numbers

This case can be understood by combining the results for the previous two multiplicatively. Since

$$r_\lambda^{Q(q,p)}(z) = r_\lambda^{E(q)}(z)\, r_\lambda^{H(p)}(z) \tag{4.16}$$

it follows that

Theorem 4.5.
$$\tau^{Q(q,p,z)}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} z^k \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^k_{Q(q,p)}(\mu,\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \tag{4.17}$$

where

$$\begin{aligned} H^k_{Q(q,p)}(\mu,\nu) := &\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}(-1)^{\sum_{j=1}^{e}\ell^*(\nu^{(j)})} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(d)},\nu^{(1)},\dots\nu^{(e)} \\ \sum_{i=1}^{d}\ell^*(\mu^{(i)})+\sum_{i=1}^{e}\ell^*(\nu^{(i)})=k}} \prod_{i=1}^{d}\left(\frac{1}{1-q^{\ell^*(\mu^{(i)})}}\right) \\ &\times \prod_{j=1}^{e}\left(\frac{1}{p^{\ell^*(\nu^{(j)})}-1}\right) H(\mu^{(1)},\dots,\mu^{(d)},\nu^{(1)},\dots,\nu^{(e)},\mu,\nu) \end{aligned} \tag{4.18}$$

are the weighted (quantum) Hurwitz numbers that count the number of branched coverings with genus g given by (2.21) and sum of colengths k, with weight $\left(\frac{1}{1-q^{\ell^*(\mu^{(i)})}}\right)$ for every branch point of type $\mu^{(i)}$ with colength $\ell^*(\mu^{(i)})$ and weight $\left(\frac{(-1)^{\ell^*(\nu^{(j)})}}{p^{\ell^*(\nu^{(j)})}-1}\right)$ for every branch point of type $\nu^{(j)}$ with colength $\ell^*(\nu^{(j)})$.

It follows from eq. (3.29) that

Corollary 4.6. *The weighted (quantum) Hurwitz number for the branched coverings of the Riemann sphere with genus given by (2.21) is again equal to the combinatorial Hurwitz number given by formula (2.62) enumerating weighted paths in the Cayley graph*

$$H^k_{Q(q,p)}(\mu,\nu) = F^k_{Q(q,p)}(\mu,\nu). \tag{4.19}$$

4.4 Bose gas model

A slight modification of example 2.4 consists of replacing the generating function $E(q,z)$ defined in eqs. (2.27) by $E'(q,z)$, as defined in (2.33) and (2.39). The effect of this is simply to replace the weighting factors $\frac{1}{1-q^{\ell^*(\mu)}}$ in eq. (4.9) by $\frac{1}{q^{-\ell^*(\mu)}-1}$.

If we identify
$$q := e^{-\beta\hbar\omega_0}, \quad \beta = kT, \tag{4.20}$$
where ω_0 is the lowest frequency excitation in a gas of identical bosonic particles and assume the energy spectrum of the particles consists of integer multiples of $\hbar\omega_0$

$$\epsilon_k = k\hbar\omega_0, \tag{4.21}$$

the relative probability of occupying the energy level ϵ_k is

$$\frac{q^k}{1-q^k} = \frac{e^{-\beta\epsilon_k}}{1-e^{-\beta\epsilon_k}}, \tag{4.22}$$

which is the energy distribution of a bosonic gas.

If we associate the branch points to the states of the gas and view the Hurwitz numbers $H(\mu^{(1)}, \dots \mu^{(l)})$ as independent, identically distributed random variables, with the state energies proportional to the sums over the colengths

$$\epsilon_{\ell^*(\mu)} = \hbar \ell^*(\mu) \beta \omega_0, \tag{4.23}$$

and weight $\frac{q^{\ell^*(\mu^{(i)})}}{1-q^{\ell^*(\mu^{(i)})}}$, the weighted Hurwitz numbers are given, as in eq. (4.9) by,

$$H^k_{E'(q)}(\mu,\nu) := \frac{1}{\mathbf{Z}^k_{E'(q)}} \sum_{d=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^d \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d} \left(\frac{q^{\ell^*(\mu^{(i)})}}{1-q^{\ell^*(\mu^{(i)})}} \right) H(\mu^{(1)}, \dots, \mu^{(d)}, \mu, \nu), \tag{4.24}$$

but normalized by the canonical partition function for fixed total energy $k\hbar\omega_0$,

$$\mathbf{Z}^k_{E'(q)} := \sum_{d=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^d \ell^*(\mu^{(i)})=k}} \prod_{i=1}^{d} \left(\frac{q^{\ell^*(\mu^{(i)})}}{1-q^{\ell^*(\mu^{(i)})}} \right). \tag{4.25}$$

We may therefore interpret these as expectation values of the Hurwitz numbers associated to the Fermi gas, and view the corresponding τ-function as a generating function for these expectation values.

4.5 Multiparameter extensions

By combining these cases multiplicatively, a multiparameter family of generating functions may be obtained, for which the underlying generator is the product

$$G(q, \mathbf{w}, \mathbf{z}) := \prod_{a=1}^{l} E(q, w_a) \prod_{b=1}^{m} H(q, z_b). \tag{4.26}$$

The resulting τ-functions may be interpreted as basic hypergeometric τ-functions of two matrix arguments [18]. The interpretation of these multiparametric quantum Hurwitz numbers, both in terms of weighted enumeration of branched covers, and weighted paths in the Cayley graph will be the subject of [10].

Acknowledgements. The authors would like to thank Gaetan Borot and Alexander Orlov for helpful discussions.

References

[1] A. Alexandrov, A.Mironov, A. Morozov and S.Natanzon‘, “On KP-integrable Hurwitz functions”, arXiv:1405.1395

[2] J. Ambjørn and L. Chekhov, “The matrix model for dessins d’enfants”, arXiv:1404.4240.

[3] P. Diaconis and C. Greene, “Applications of Murphy’s elements”, *Stanford Technical Report* 335 (1989).

[4] G. Frobenius, “Über die Charaktere der symmetrischen Gruppe”, *Sitzber. Pruess. Akad. Berlin*, 516-534 (1900).

[5] I. P. Goulden, M. Guay-Paquet and J. Novak, “Toda Equations and Piecewise Polynomiality for Mixed Double Hurwitz numbers”, arXiv:1307.2137.

[6] K.I. Gross and D.S. Richards, “Special functions of matrix arguments. I: Algebraic induction, zonal polynomials, and hypergeometric functions”, *Transactions Amer Math Soc.*, **301**, 781-811 (1987).

[7] Mathieu Guay-Paquet and J. Harnad, “2D Toda τ-functions as combinatorial generating functions”, arXiv:1405.6303

[8] J. Harnad, A. Yu. Orlov, “Convolution symmetries of integrable hierarchies, matrix models and tau functions”, in: *Integrable Systems, Random Matrices and Random Processes*, MSRI publications (2014, in press). See also arXiv:0901.0323.

[9] J. Harnad and A. Yu. Orlov, “Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths” arXiv:1407.7800.

[10] J. Harnad and A. Yu. Orlov, “Quantum Hurwitz numbers” (in preparation)

[11] M. Kazarian and P. Zograf, “‘Virasoro constraints and topological recursion for Grothendieck’s dessin counting”, arXiv:1406.5976

[12] A. A. Jucys, “Symmetric polynomials and the center of the symmetric group ring”, *Reports on Mathematical Physics* **5**(1) (1974), 107–112.

[13] S.K. Lando, A.K. Zvonkin, *Graphs on Surfaces and their Applications*, Encyclopaedia of Mathematical Sciences, Volume **141**, with appendix by D. Zagier, Springer, N.Y. (2004).

[14] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, (1995).

[15] G. E. Murphy, "A new construction of Young's seminormal representation of the symmetric groups", *Journal of Algebra* **69**, 287–297 (1981).

[16] S. M. Natanzon and A. Yu. Orlov, "Hurwitz numbers and BKP hierarchy" Hurwitz numbers and BKP hierarchy, arXiv:1407.8323 .

[17] A. Okounkov, "Toda equations for Hurwitz numbers", *Math. Res. Lett.* **7**, 447–453 (2000).

[18] A. Yu. Orlov and D. M. Scherbin, "Hypergeometric solutions of soliton equations", *Theoretical and Mathematical Physics* **128**, 906–926 (2001).

[19] K. Takasaki, "Initial value problem for the Toda lattice hierarchy", in: *Group Representation and Systems of Differential Equations*, 139–163, *Adv. Stud. in Pure Math.* **4**, (1984).

[20] T. Takebe, "Representation theoretical meaning of the initial value problem for the Toda lattice hierarchy I", *Lett. Math. Phys.* **21** 77–84, (1991).

[21] K. Ueno and K. Takasaki, "Toda Lattice Hierarchy", in:*Group Representation and Systems of Differential Equations, 1–95*, *Adv. Stud. in Pure Math.* **4**, (1984).

[22] P. Zograf, "Enumeration of Grothendieck's dessins and KP hierarchy", arXiv:1312.2538 (2013).