Math 8680 Spring 2021 (Online)
Topics in Combinatorics – Combinatorial rings & the Kähler package

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Jan 20, 2021

Course structure:

- Notes, videos (YouTube) to be posted
- Registered students do either
  8 HW problems (4 from each "half")
  or
  give a 20-minute talk
- U of M grad students should skip class to attend occasional MW prof. development and community-building dept. meetings
- Office hours TBD
- Study group? Discord channel?

OVERVIEW (Wed- Fri- Mon?)
1/20/21 1/22 1/25

Finite sets of points, vectors in space $\mathbb{R}^d$ or K^d for K a field

$\rightsquigarrow$ discrete geometric sets that we might count by their cardinality or dimension ---

Q: What can we say about these sequences of numbers?

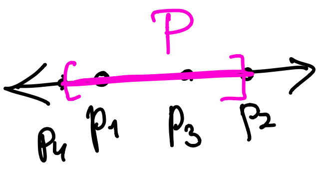
EXAMPLE: Points $\{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^d$

⤳ a convex polytope $P \underset{\text{DEF'N}}{:=} \text{conv}\{p_i\}_{i=1}^{n} = \left\{ \sum_{i=1}^{n} c_i p_i : \begin{array}{l} c_i \geq 0 \\ \sum_i c_i = 1 \end{array} \right\}$

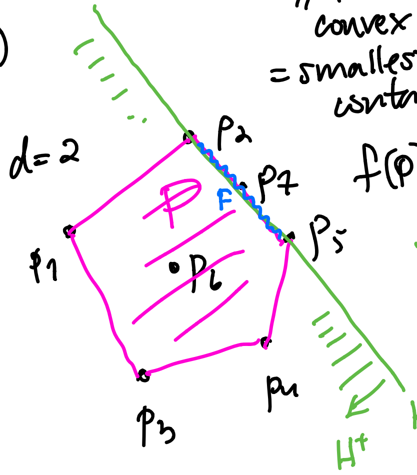
"convex hull of" = smallest convex set containing them

$\underline{f}(P) = (f_{-1}, f_0, f_1) = (1, 2, 1)$

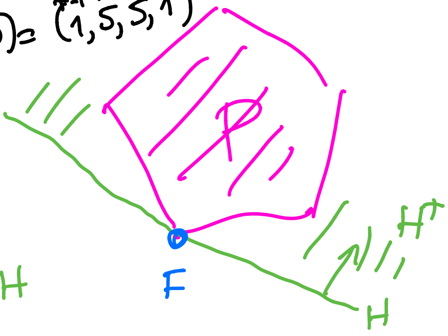
$d=1$



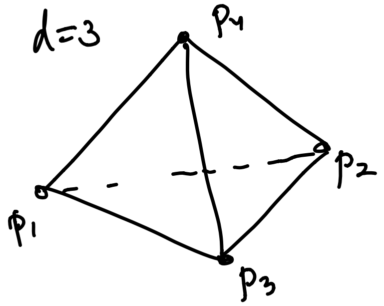
$d=2$



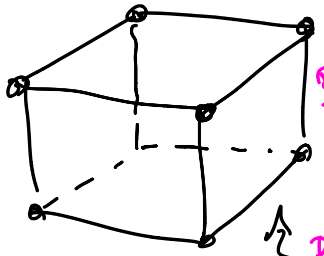
$f(P) = (\overset{f_{-1}}{1}, \overset{f_0}{5}, \overset{f_1}{5}, \overset{f_2}{1})$



$d=3$



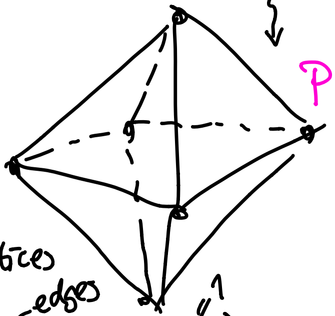
$\underline{f} = (1, 4, 6, 4, 1)$
$= (f_{-1}, f_0, f_1, f_2, f_3)$
$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$



$P = P^{\Delta\Delta}$

$\underline{f} = (\overset{f_{-1}}{1}, \overset{f_0}{8}, \overset{f_1}{12}, \overset{f_2}{6}, \overset{f_3}{1})$

polar dual



P^{Δ}

$\underline{f} = (\overset{f_1}{1}, \overset{f_0}{6}, \overset{f_1}{12}, \overset{f_2}{8}, \overset{f_3}{1})$

vertices

edges

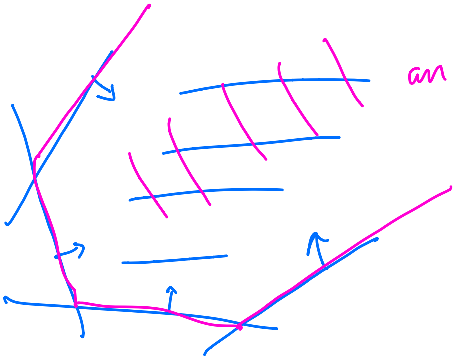
Let $\underline{f}(P) := (\overset{1}{\overset{\|}{f_{-1}}}, f_0, f_1, \ldots, f_{d-1}, \overset{1}{\overset{\|}{f_d}})$ where f_i = # of i-dimensional faces F of P

f-vector

empty face $\emptyset$

P itself, if P is full-dimensional

A face of P is $H \cap P$ where H^+ is a half-space containing P



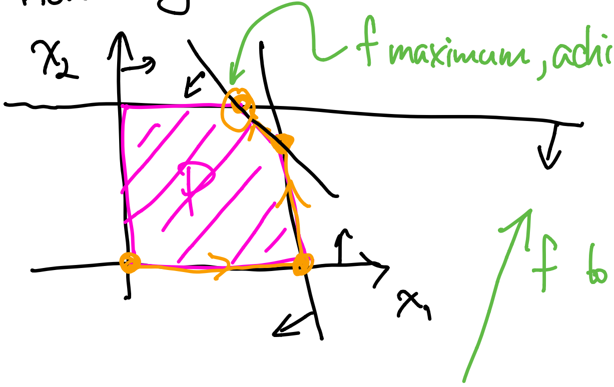
an unbounded polyhedron $= \bigcap$ half-spaces (finitely many)

$\{$bounded polyhedra$\} = \{$convex polytopes$\}$

Q: (1950's) How large can $f_i(P)$ be if we fix $f_0(P)=n$?

Motivated by Simplex Method in Linear Programming as the polar dual question:

How large can $f_0(P), f_1(P)$ be if we fix $f_{d-1}(P)=n$?



f maximum, achieved at a vertex

f to be maximized

We will prove using a certain ring...

CONJ (Motzkin's upper-bound conjecture) 1957 UBC

THM (P. McMullen) 1970

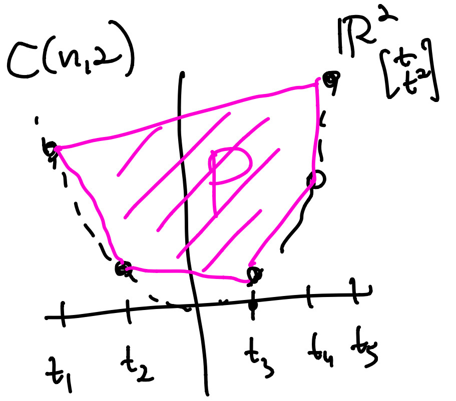
Any d-dimensional cyclic polytope

$$C(n,d) := \mathrm{conv}\{\underline{x}(t_1),\dots,\underline{x}(t_n)\} \subset \mathbb{R}^d$$

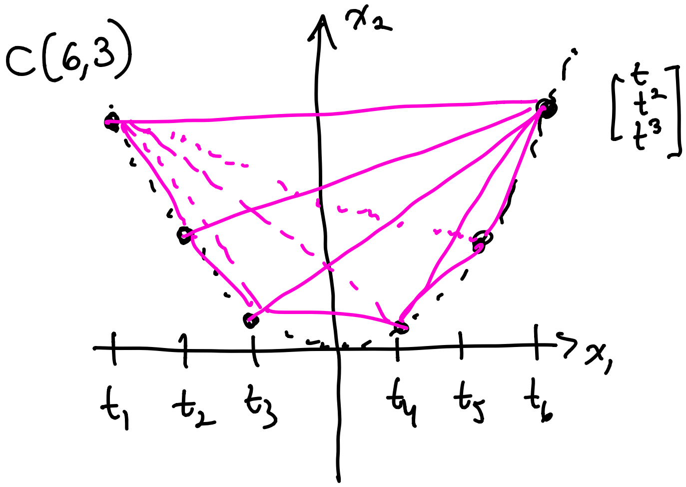
$t_1 < \dots < t_n$

and $\underline{x}(t) = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$

simultaneously maximizes all $f_i(P)$ among d-dim'l polytopes P with $n = f_0(P)$

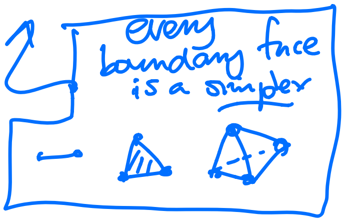


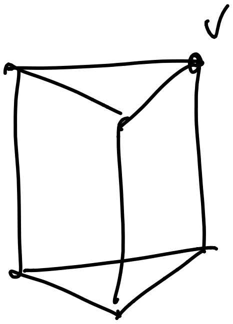
$$f(C(n,2)) = (\underset{f_{-1}}{1}, \underset{f_0}{n}, \underset{f_1}{n}, \underset{f_2}{1})$$



$$f(C(6,3)) = (1, 6, 12, 8)$$

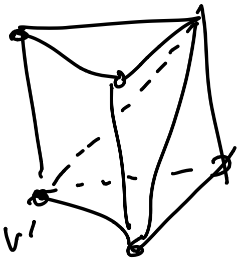
Not hard to see, using "pulling vertices", that $f_i(P)$ is maximized for $n = f_0(P)$ fixed, by simplicial polytopes



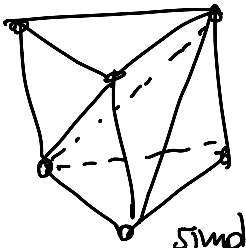


not simplicial

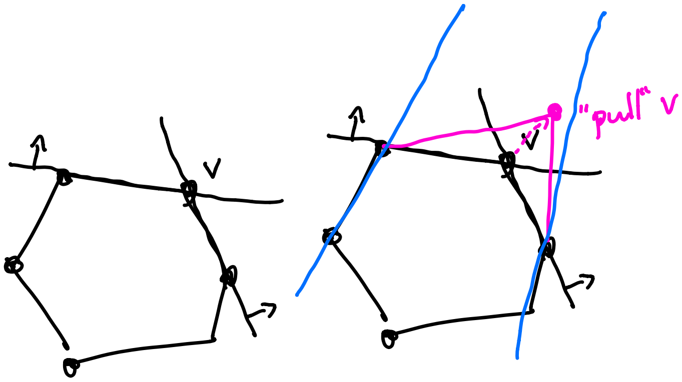
"pull" v ⇝



⇝ "pull" v'



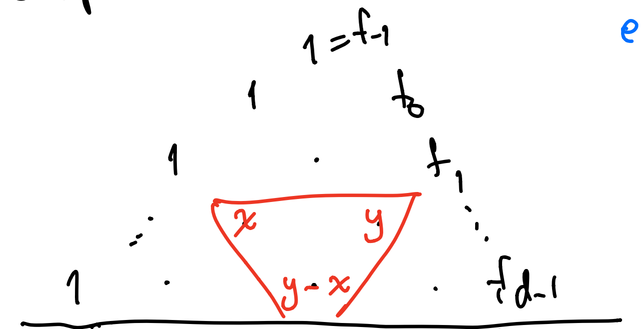
simplicial



Math 8680 Jan 22, 2021

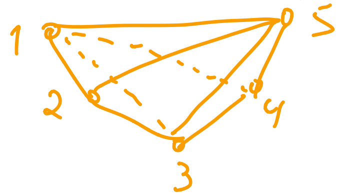
A key idea of McMullen:
$\underline{f}$-vector of simplicial polytope is T.M.I. –
it's redundant, too large.

Better to use the $\underline{h}$-vector $(h_0, h_1, h_2, .., h_d) = h(P) = \underline{h}(P)$
defined as an invertible linear transformation of $\underline{f}(P)$,
computable, e.g., via Stanley's triangle trick

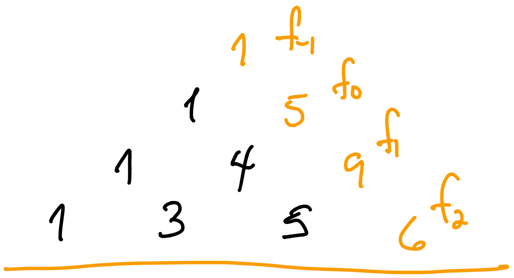


$(1, h_0, h_1, \ldots, h_d)$

e.g. $d=3$ $C(5,3)$



$\underline{f}(P) = (f_{-1}, f_0, f_1, f_2)$
$= (1, 5, 9, 6)$



$\underline{h} = (1, 2, 2, 1)$
$h_0 \quad h_1 \quad h_2 \quad h_3$

① THM (Dehn- Sommerville) $\overset{d\le5}{1905}$ 1927

For simplicial d-polytopes $h_i(P) = h_{d-i}(P)$ $\forall i \le \frac{d}{2}$

② Motzkin's UBC follows from McMullen showing for simplicial d-polytopes with n vertices, $h_i(P) \le \binom{(n-d)+i-1}{i} \xlongequal{\forall i} \dim_{\mathbb{R}} \mathbb{R}[x_1, x_2, \ldots, x_{n-d}]_i$

③ In fact, the $h_i(P)$ are nonnegative and even (symmetric and) <u>unimodal</u> $h_0 \leq h_1 \leq \dots \leq h_{\lfloor \frac{d}{2} \rfloor} \geq \dots \geq h_d$

(but this is <u>false</u> for $\underline{f}(P)$; not unimodal)

... an McMullen started putting it all together in...

<u>CONJ</u> (McMullen's "g-conjecture" 1974) The face numbers $\underline{f}(P)$ for simplicial d-polytopes are characterized by

$$\left\{\begin{array}{l} h_i \in \mathbb{Z} \\ h_i = h_{d-i} \\ (\underset{\overset{\parallel}{g_0=1}}{h_0}, \underbrace{h_1-h_0}_{g_1}, \underbrace{h_2-h_1}_{g_2}, \dots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1}) = \underline{g}(P) \end{array}\right.$$

is the Hilbert function (= dimensions of homog. components) of a graded $\mathbb{R}$-algebra generated in degree 1

In particular, $h_0 = 1 = g_0$ and $h_0 \leq h_1 \leq \dots \leq h_{\lfloor \frac{d}{2} \rfloor}$

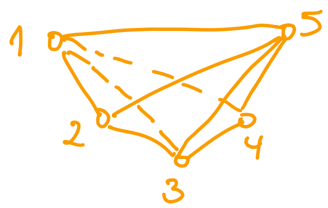
Around 1980, Billera & Lee constructed (cleverly!) a simplicial polytope P given $\underline{f}$ satisfying McMullen's conditions.

and 1980 Stanley constructed the rings interpreting the $\underline{h}(P)$ and $\underline{g}(P)$ from P to show $\underline{g}$ satisfies McMullen's conditions.

These rings are well-motivated, come from a ring that satisfies the Kähler package, a quotient of a Stanley-Reisner ring by a certain linear system of parameters.

EXAMPLE

$d=3$, $P = C(5,3)$



$$\underline{f} = \begin{array}{cccc} f_{-1} & f_0 & f_1 & f_2 \\ (1, & 5, & 9, & 6) \end{array}$$

$$\underline{h} = \begin{array}{cccc} (1, & 2, & 2, & 1) \\ h_0 & h_1 & h_2 & h_3 \end{array}$$

$$\underline{g} = \begin{array}{cc} (1, & 1) \\ g_0 & g_1 \end{array}$$

Let $\Delta = \partial P =$ boundary (simplicial) complex of P

with ~~faces~~ simplices

$$\left\{ \begin{array}{cccc} ① & ⑤ & ⑨ & ⑥ \\ \emptyset, & 1 & 12 & 123 \\ & 2, & 13, & \cancel{124} \\ & 3 & 14 & 125 \\ & 4 & 15 & 134 \\ & 5 & 23 & \cancel{135} \\ & & \cancel{24} & 145 \\ & & 25 & \cancel{234} \\ & & 34 & 235 \\ & & 35 & \cancel{245} \\ & & 45 & 345 \end{array} \right\}$$

2 minimal non-faces

$\{24, 135\}$

and Stanley-Reisner ring

$$\mathbb{R}[\Delta] = \mathbb{R}[\underbrace{x_1, x_2, x_3, x_4, x_5}_{\text{vertex variables}}] / (x_2 x_4, x_1 x_3 x_5)$$

$$= \mathbb{R}\text{-span of } \left\{ \begin{array}{cccc} 1, & x_1, & x_1^2 & , \; x_1^3 \quad , \; \dots \\ & x_2, & \vdots & \vdots \\ & x_3, & x_5^2 & \\ & x_4, & x_1 x_2 & x_1 x_2 x_3 \\ & x_5, & \vdots & \vdots \\ & & \cancel{x_2 x_4} & \cancel{x_1 x_3 x_5} \\ & & \vdots & \vdots \\ & & x_4 x_5 & \end{array} \right\}$$

having Hilbert series

$$\mathrm{Hilb}(\mathbb{R}[\Delta], t) := \sum_{k=0}^{\infty} \dim_{\mathbb{R}} \underbrace{\mathbb{R}[\Delta]_k}_{\substack{\text{degree k homogenous} \\ \text{component}}} \cdot t^k = 1 + 5t + 14t^2 + \dots \in \mathbb{R}[[t]]$$

In fact, $\mathrm{Hilb}(\mathbb{R}[\Delta],t)$ encodes $\underline{f}(P)$ and $\underline{h}(P)$:

$$\mathrm{Hilb}(\mathbb{R}[\Delta],t) = \underset{\substack{\shortparallel \\ f_{-1}}}{1} + \underset{\substack{\shortparallel \\ f_0}}{5}\,\frac{t}{1-t} + \underset{\substack{\shortparallel \\ f_1}}{9}\,\frac{t^2}{(1-t)^2} + \underset{\substack{\shortparallel \\ f_2}}{6}\,\frac{t^3}{(1-t)^3}$$

$$= \frac{1\cdot(1-t)^3 + 5t(1-t)^2 + 9t^2(1-t) + 6t^3}{(1-t)^3} = \frac{\overset{\substack{h_0 \\ \shortparallel}}{1} + \overset{\substack{h_1 \\ \shortparallel}}{2}t + \overset{\substack{h_2 \\ \shortparallel}}{2}t^2 + \overset{\substack{h_3 \\ \shortparallel}}{1}\cdot t^3}{(1-t)^3}$$

Better yet, $\mathbb{R}[\Delta]$ turns out to be a free module over the subring

$$\mathbb{R}[\underbrace{\underset{\substack{\shortparallel \\ x_2-x_4}}{\theta_1},\ \underset{\substack{\shortparallel \\ x_1-x_5}}{\theta_2},\ \underset{\substack{\shortparallel \\ x_3-x_5}}{\theta_3}}_{\substack{\text{a linear system of parameters} \\ \text{for } \mathbb{R}[\Delta]}}] =: \mathbb{R}[\underline{\theta}]$$

and the quotient ring

$$\mathbb{R}[\Delta]/(\underline{\theta}) = \mathbb{R}[\Delta]/(\theta_1,\theta_2,\theta_3)$$

$$= \mathbb{R}[x_1,x_2,x_3,x_4,x_5]/(x_2x_4,\ x_1x_3x_5,\ x_2-\underline{\underline{x_4}},\ \underline{\underline{x_1}}-x_5,\ \underline{\underline{x_3}}-x_5)$$

$\swarrow$ $x_4=x_2,\ x_1=x_3=x_5$

$$\cong \mathbb{R}[x_2,x_5]/(x_2^2,\ x_5^3)$$

$$= \mathbb{R}\text{-span of } \left\{ \begin{array}{c|c|c|c} 1, & x_2, & x_2x_5, & x_2x_5^2 \\ & x_5, & x_5^2 & \end{array} \right\}$$

has Hilbert series $\mathrm{Hilb}(\mathbb{R}[\Delta]/(\underline{\theta}),t) = 1 + 2t + 2t^2 + 1\cdot t^3$

$$= \sum_i h_i t^i$$

And then inside this ring $H := \mathbb{R}[\Delta]/(\underline{\Theta})$,

the degree one element $\ell := x_1+x_2+x_3+x_4+x_5 \equiv 2x_2+3x_5$

gives a Lefschetz element, meaning it satisfies the

Hard Lefschetz Theorem:

The multiplication maps $H_0 \xrightarrow{\cdot \ell^3} H_3$

$H_1 \xrightarrow{\cdot \ell^1} H_2$

are $\mathbb{R}$-vector space isomorphisms:

$$\underbrace{1}_{\substack{\mathbb{R}\text{- basis}\\ \text{for } H_0}} \cdot \ell^3 = (2x_2+3x_5)^3 = \cancelto{0}{8x_2^3} + \cancelto{0}{12x_2^2x_5} + 18\underbrace{x_2x_5^2}_{\substack{\mathbb{R}\text{- basis}\\ \text{for } H_3}} + \cancelto{0}{27x_5^3}$$

$$\left.\begin{array}{l} \boxed{\begin{array}{c} x_2 \\ x_5 \end{array}} \begin{array}{l} \cdot \ell^1 = \cancelto{0}{2x_2^2} + 3x_2x_5 \\ \cdot \ell^1 = 2x_2x_5 + 3x_5^2 \end{array} \end{array}\right\} \leftarrow \text{an } \mathbb{R}\text{-basis for } H_2$$

$\mathbb{R}$- basis for H_1

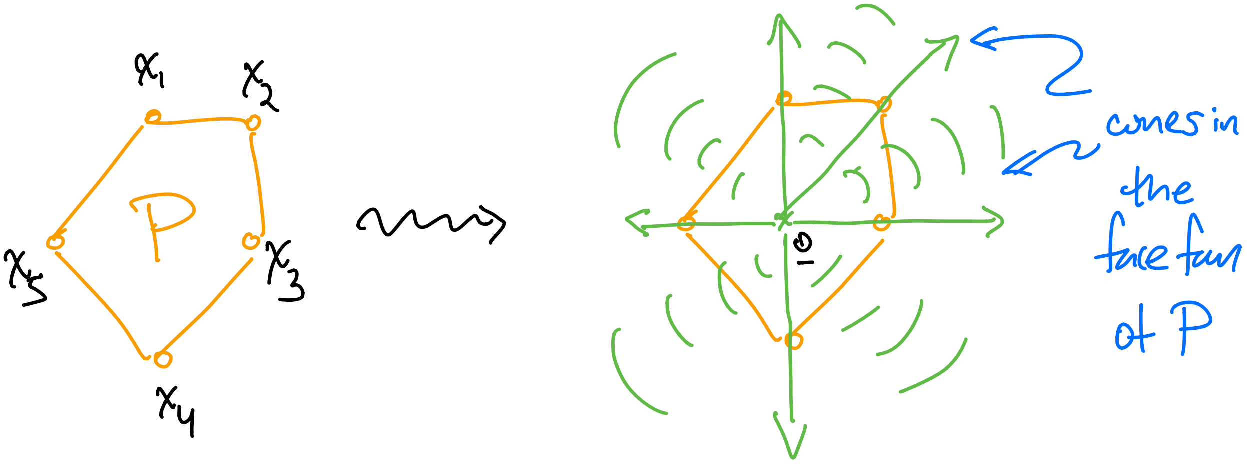
Consequently, $H/(\ell) = \mathbb{R}[\Delta]/(\underline{\Theta}, \ell)$ has Hilbert function $\underline{g} = (h_0, h_1 - h_0) = (1, 1)$:

$$\begin{aligned} H/(\ell) &= \mathbb{R}[x_2, x_5]/(x_2^2, x_5^3, 2x_2+3x_5) \\ &\cong \mathbb{R}[x_2]/(x_2^2, x_2^3) \\ &= \mathbb{R}[x_2]/(x_2^2) \\ &= \mathbb{R}\text{-span of } \{1, \mid x_2\} \end{aligned}$$

$$\underline{g} = (1, 1)$$

Where will $(\Theta_1, \Theta_2, \Theta_3) = (\underline{\Theta})$ come from?

From the coordinates/geometry of the face fan of the simplicial polytope P

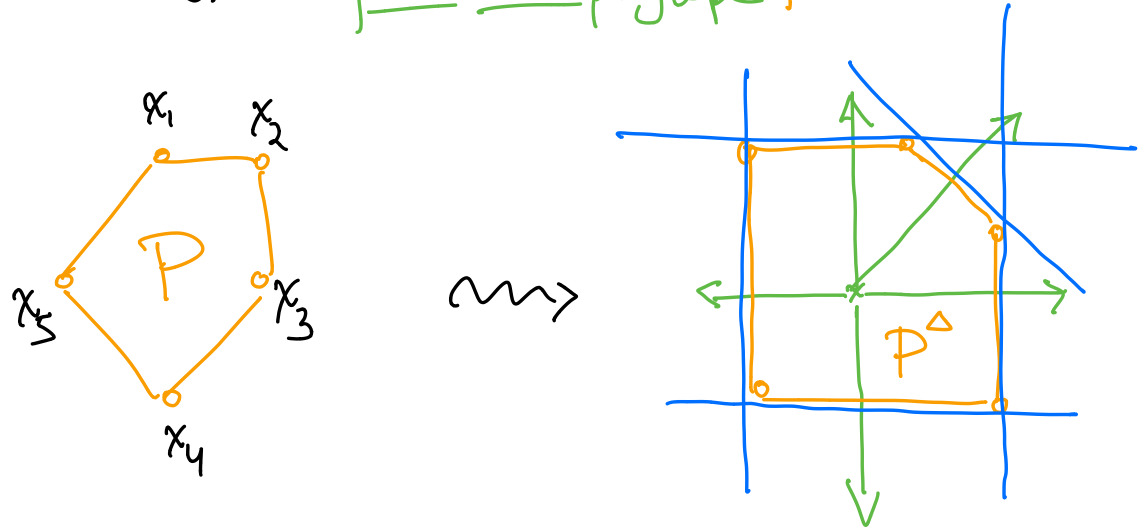


Where will ℓ come from?

From the level sets of the facet hyperplanes

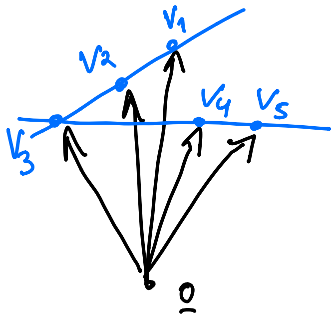
↖ $(d-1)$-dim'l face

of the polar dual polytope $P^{\triangle}$

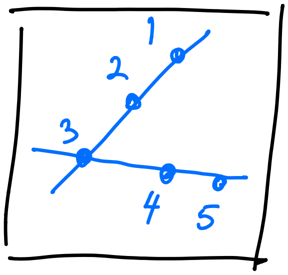


Our other examples start from a set $\{v_1,\dots,v_n\}$ of vectors in K^r for some field K

EXAMPLE: These $\{v_1,v_2,v_3,v_4,v_5\} \subset \mathbb{R}^3$



⇝



affine picture

Let's consider three sequences $\{v_i\}_{i=1,\dots,n}$ give rise to, based not so much on their actual coordinates, but on which of them are linearly independent or dependent, which lie in each other's span; called the matroid data M for $\{v_1,\dots,v_n\}$.

① i_k := # of linearly independent subsets of size k

⇝ $(i_0, i_1, i_2, \dots, i_r)$

$\emptyset$	1	12	~~123~~
	2	13	124
	3	14	125
	4	15	134
	5	23	135
		24	145
		25	234
		34	235
		35	245
		45	~~345~~

$(1, 5, 10, 8)$
$i_0 \quad i_1 \quad i_2 \quad i_3$

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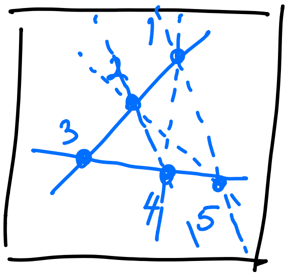
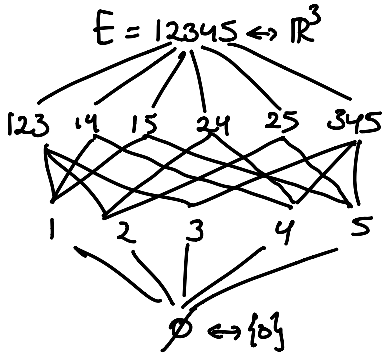
② W_k := # k-dim'l linear subspaces they span ← called flats



The poset L of flats (lattice) ordered by inclusion



$W_3 = 1$

$W_2 = 6$

$W_1 = 5$

$W_0 = 1$

W_k = kth Whitney number of 2nd kind for L

$W_0 \; W_1 \; W_2 \; W_3$
$(1, 5, 6, 1)$

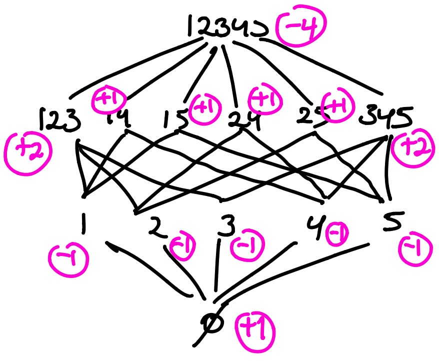
---

③ ω_k := kth Whitney number of 1st kind for L

$$:= \sum_{\substack{k\text{-dim'l} \\ \text{flats } F}} |\mu(\emptyset, F)|$$

← Möbius function of L

$$\begin{cases} \mu(\emptyset, \emptyset) = +1 \\ \mu(\emptyset, F) = -\sum_{G:\, G<F} \mu(\emptyset, G) \end{cases}$$



$\omega_3 = 4$

$\omega_2 = 8$

$\omega_1 = 5$

$\omega_0 = 1$

$\omega_0 \; \omega_1 \; \omega_2 \; \omega_3$
$(1, 5, 8, 4)$

Several long-standing conjectures, proven recently:

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① CONJ$\left(\overset{1971}{\text{Welsh}}, \overset{1972}{\text{Mason}}\right)$ $(i_0, i_1, \dots, i_r)$ is not only <u>unimodal</u>

(i.e. $i_0 \le i_1 \le \dots \le i_p \ge \dots \ge i_r$ for some p)

but also <u>log-concave</u>: $i_k^2 \ge i_{k-1} \cdot i_{k+1}$ for $2 \le k \le r-1$

(easy) EXERCISE: When $i_k > 0\ \forall k$, log-concave ⇒ unimodal.

e.g. $\left(\overset{i_0}{1}, \overset{i_1}{5}, \overset{i_2}{10}, \overset{i_3}{8}\right)$

$5^2 \ge 1 \cdot 10, \quad 10^2 \ge 5 \cdot 8$

---

② CONJ$\left(\overset{1968}{\text{Read}}, \overset{1974}{\text{Hoggar}}, \overset{1971}{\text{Rota}}, \overset{1972}{\text{Heron}}, \overset{1976}{\text{Welsh}}\right)$ $(w_0, w_1, \dots, w_r)$ is <u>log-concave</u>.

e.g. $\left(\overset{w_0}{1}, \overset{w_1}{5}, \overset{w_2}{8}, \overset{w_3}{4}\right)$

$5^2 \ge 1 \cdot 8, \quad 8^2 \ge 5 \cdot 4$

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③ CONJ(Dowling-Wilson^{1974} "Top-heavy conjecture") $(W_0, W_1, \dots, W_r)$ has

THM (Braden-Huh-Matherne-Proudfoot-Wang 2020) $W_k \le W_m$ if $k \le m \le r-k$

e.g. $\left(\overset{W_0}{1}, \overset{W_1}{5}, \overset{W_2}{6}, \overset{W_3}{1}\right)$

$1 \le 5$

$1 \le 6$

$1 \le 1$

$5 \le 6$

<u>REMARK</u>: In particular $W_0 \le W_1 \le \dots \le W_{\lfloor r/2 \rfloor}$

(but <u>CONJ</u> (Rota) (W_i) are <u>unimodal</u> is still <u>open</u>)

① CONJ ($\overset{1971}{\text{Welsh}}$, $\overset{1972}{\text{Mason}}$) $(i_0, i_1, \ldots, i_r)$ is <u>log-concave</u>:

② CONJ ($\overset{1968}{\text{Read}}$, $\overset{1974}{\text{Hoggar}}$, $\overset{1971}{\text{Rota}}$, $\overset{1972}{\text{Heron}}$, $\overset{1976}{\text{Welsh}}$) $(w_0, w_1, \ldots, w_r)$ is <u>log-concave</u>.

These two were proven via a remarkable <u>common generalization</u>...

The generating function $\sum_{i=0}^{r} w_i t^i$ (sometimes called the <u>Poincaré polynomial</u>) for $\{v_1, \ldots, v_n\}$

will always factor, by construction, as $(1+t) \sum_{i=0}^{r-1} \bar{w}_i t^i$ ↖ <u>called reduced Whitney numbers of 1st kind</u>

e.g. $(w_0, w_1, w_2, w_3) = (1,5,8,4) \rightsquigarrow 1 + 5t + 8t^2 + 4t^3 = (1+t)(1+4t+4t^2)$

$\Rightarrow (\bar{w}_0, \bar{w}_1, \bar{w}_2) = (1,4,4)$

THEOREM ($\overset{2010}{\text{Huh}}$, $\overset{2011}{\text{Huh-Katz}}$, $\overset{2015}{\text{Adiprasito-Huh-Katz}}$) $(\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{r-1})$ is <u>log-concave</u>.

$\swarrow$ EXERCISE 10 on HW list

$(w_0, w_1, \ldots, w_r)$ is log-concave

$\searrow$ (Brylawski 1977, Lenz 2011) ↖ Given $\{v_i\}_{i=1,\ldots,n} \subset K^{r-1}$ can construct $\{\hat{v}_i\}_{i=1,\ldots,n+1} \subset K^r$ with

$(i_0, i_1, \ldots, i_r)$ is log-concave

$$(\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{r-1}) \text{ for } \{\hat{v}_i\}$$
$$\|$$
$$(i_0, i_1, \ldots, i_{r-1}) \text{ for } \{v_i\}$$

One only really needs to prove log-concavity <u>at the right end</u> $\bar{w}_{r-2}^2 \geq \bar{w}_{r-3} \cdot \bar{w}_{r-1}$,

because we'll show that generic linear projection $K^r \xrightarrow{\pi} K^{r-1}$ sends $\{v_i\} \mapsto \{\pi(v_i)\}$

and sends $(\bar{w}_0, \ldots, \bar{w}_{r-2}, \bar{w}_{r-1}) \mapsto (\bar{w}_0, \ldots, \bar{w}_{r-2})$.

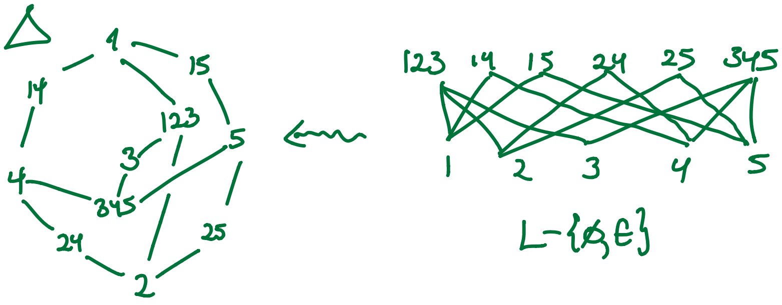
But how do A-H-K prove $\bar{\omega}_{r-2}^2 \geq \bar{\omega}_{r-3} \cdot \bar{\omega}_{r-1}$?

They revisit Feichtner & Yuzvinsky's (2003) Chow ring for $\{v_i\}_{i=1,\dots,n}$ or M:

$$A(M) := \underbrace{\left(\mathbb{R}[x_F]_{\substack{\text{flats } F \\ \emptyset \neq F \neq E \\ \overset{\shortparallel}{\{1,2,\dots,n\}}}} \Big/ \left(x_F x_G \right)_{\substack{F \not\subseteq G, \\ G \not\subseteq F}} \right)}_{\substack{\text{a Stanley-Reisner ring} \\ \mathbb{R}[\Delta] \text{ for} \\ \Delta := \text{order complex of } L - \{\emptyset, E\}}} \Big/ \left(\underline{\Theta} \right)$$

degree one elements $(\alpha_i - \alpha_j)$ where $\alpha_i := \sum_{\substack{\text{flats } F: \\ i \in F}} x_F$

These linear forms again arise from embedding Δ as a $\underline{\underline{\text{fan}}}$ $(\subset \mathbb{R}^{n-1})$



They show...

(1) $A(M) = \underset{\underset{\mathbb{R}}{\shortparallel}}{A_0} \oplus A_1 \oplus \dots \oplus A_{r-1} \cong \mathbb{R}$ call this isomorphism deg

(2) Hard Lefschetz Thm: $A_i \xrightarrow{\cdot \ell^{r-1-2i}} A_{r-1-i}$ is an isomorphism for certain $\ell \in A_1$.

(3) $(\bar{\omega}_{r-3}, \bar{\omega}_{r-2}, \bar{\omega}_{r-1})$
$= \left(\deg(\alpha^2 \cdot \beta^{r-3}), \deg(\alpha \cdot \beta^{r-2}), \deg(\beta^{r-1}) \right)$

where $\alpha := \alpha_i$ (for $\underline{\text{any}}$ $i \in E$)
$\beta := \sum_F x_F - \alpha$

(4) This symmetric matrix

$$\begin{array}{ccc} & \alpha & \beta \\ \alpha & \underbrace{\deg(\alpha^2 \beta^{r-3})}_{\bar{\omega}_{r-3}} & \underbrace{\deg(\alpha\beta\, \beta^{r-3})}_{\bar{\omega}_{r-2}} \\ \beta & \underbrace{\deg(\beta\alpha\, \beta^{r-3})}_{\bar{\omega}_{r-2}} & \underbrace{\deg(\beta^2 \beta^{r-3})}_{\bar{\omega}_{r-1}} \end{array}$$

must correspond to (a limit of) indefinite quadratic forms on $\mathrm{span}_{\mathbb{R}}\{\alpha, \beta\}$,

and hence have determinant ≤ 0.

$= \bar{\omega}_{r-3}\bar{\omega}_{r-1} - \bar{\omega}_{r-2}^2$

This last part comes from them proving the Hodge-Riemann relations, which predict the signature of all the quadratic forms Q_ℓ on A_i

\# of $\oplus$, $\ominus$, ~~zero~~ none! eigenvalues

defined by

$Q_\ell(x) := \deg(x^2 \cdot \ell^{r-1-2i})$

An interesting feature of many of these proofs:

It's difficult to prove Hard Lefschetz Theorem without simultaneously proving Hodge-Riemann relations!

How do BHMPW prove $(W_0, W_1, \ldots, W_r)$ is top-heavy,

meaning $W_k \leq W_m$ if $k \leq m \leq r-k$?

They start with a (commutative, graded) ring $H(M)$

that certainly has $\dim_{\mathbb{R}} H(M)_k = W_k$,

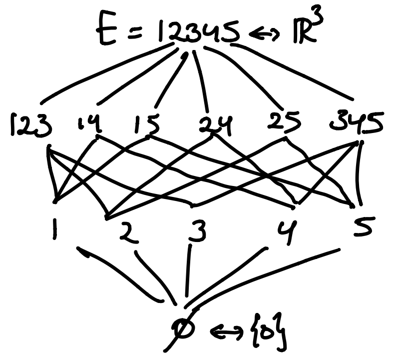
namely the graded Möbius algebra of L:

it has $\mathbb{R}$-basis $\{y_F\}_{\text{flats } F \in L}$ and $y_F \cdot y_G = \begin{cases} y_{F \vee G} & \text{if } \dim(F \vee G) = \dim F + \dim G, \\ 0 & \text{otherwise} \end{cases}$

$F \vee G := F + G = \operatorname{span}_K(F \cup G)$

EXAMPLE

$L=$



$W_3 = 1$

$W_2 = 6$

$W_1 = 5$

$W_0 = 1$

$H(M)$ has $\mathbb{R}$-basis $\{y_\emptyset, y_1, y_2, \ldots, y_{123}, y_{14}, \ldots, y_{12345}\}$

and

$y_1 y_3 = y_{123}$

$y_1 y_{123} = 0$

$y_5 y_{123} = y_{12345}$

$y_{14} y_{123} = 0$

$\vdots$

This $H(M)$ does not satisfy the Kähler package,

e.g. no Poincaré duality, $H(M)_k \ncong H(M)_{r-k}$

no Hard Lefschetz Thm.

no Hodge-Riemann Relations.

But $H(M)$ naturally lives inside another ring that does, the augmented Chow ring

$$\mathrm{CH}(M) := \left(\underbrace{\mathbb{R}\left[\{x_F\}_{\substack{\text{flats}\\ F\neq E}},\ y_1, y_2, \dots, y_n\right] \Big/ (x_F x_G)_{\substack{F\not\subseteq G\\ G\not\subseteq F}} + (y_i x_F)_{i\notin F}}_{\substack{\text{a Stanley-Reisner } \mathbb{R}[\Delta]\\ \text{for } \Delta=\text{augmented Bergman complex}}} \right) \Big/ \underset{\substack{\| \\ \left(y_i - \sum_{F:\, F\not\ni i} x_F\right)}}{(\Theta)}$$

again, degree one elements, that come from a fan embedding $\Delta \subset \mathbb{R}^n$

with $H(M) \hookrightarrow \mathrm{CH}(M)$

$$y_F \longmapsto \prod_{i\in I} y_i \quad \text{for \underline{any} basis } I \text{ of the flat } F$$

In fact, they introduce an intermediate ring $IH(M)$

having $\quad H(M) \hookrightarrow IH(M) \hookrightarrow \mathrm{CH}(M)$

such that $IH(M)$ satisfies the Kähler package with Lefschetz element

$$\ell := \sum_{\substack{\text{flats } F:\\ \dim F = 1}} y_F$$ lying inside the smaller ring $H(M)$, which does it:

If $k \leq m \leq r-k$, then $\quad IH_k \xrightarrow{\cdot \ell^{r-2k}} IH_{r-k}$ being an isomorphism

$\Rightarrow \quad IH_k \xhookrightarrow{\cdot \ell^{m-k}} IH_m$ is injective

$\Rightarrow \quad H(M)_k \xhookrightarrow{\cdot \ell^{m-k}} H(M)_m$ is injective

$\Rightarrow \quad W_k \leq W_m$

PLAN:

- Simplicial complexes
  & Stanley-Reisner rings
- Simplicial fans
  & piecewise polynomials
- Fleming-Karu (2018) proof of Kähler package for simplicial polytopes
- Matroids
- Bergman fans & Chow rings
- Sketch of Adiprasito-Huh-Katz proof of Kähler package
- Augmented Bergman fans, Chow rings, graded Möbius algebra

If time allows...

- Conormal fan of Ardila-Denham-Huh (2020)
- Kazhdan-Lusztig-Stanley polynomials
- More f-vectors, cd-indices, γ-vectors