Math 8680 Jan 29, 2021

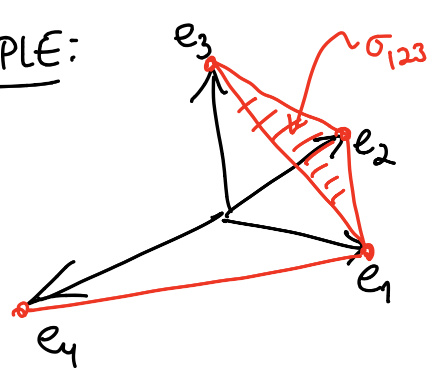
Simplicial complexes and Stanley-Reisner rings

DEF'N: An (abstract) simplicial complex Δ on a vertex set $V(\Delta) = \{1,2,\ldots,n\}$ is a collection
$$\Delta \subset 2^{V(\Delta)} := \{\text{all subsets of } V(\Delta)\}$$
called faces $F \in \Delta$ (or simplices), such that $\forall F \in \Delta$ and $G \subseteq F \Rightarrow G \in \Delta$

EXAMPLES:
$$\Delta = \{\emptyset,\ 1,\ 2,\ 3,\ 4,\ 12,\ 13,\ 14,\ 23,\ 123\} \subset V(\Delta) = \{1,2,3,4\}$$

DEF'N: Every such Δ has a topological space $\|\Delta\|$ called its geometric realization that we can embed in $\mathbb{R}^n$ with standard basis vectors $\{e_1,\ldots,e_n\}$ as
$$\|\Delta\| := \bigcup_{F \in \Delta} \sigma_F \quad \text{where } \sigma_F = \text{convex hull of } \{e_i\}_{i \in F} = \left\{ \sum_{i\in F} c_i e_i : \begin{array}{l} c_i \geq 0 \\ \sum_{i \in F} c_i = 1 \end{array} \right\}$$

EXAMPLE:



$$\Delta = \{\emptyset,\ 1,\ 2,\ 3,\ 4,\ 12,\ 13,\ 14,\ 23,\ 123\}$$

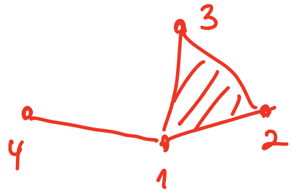
$= \|\Delta\| \subset$ "$\mathbb{R}^4$"

But, of course, we can project $\mathbb{R}^4 \xrightarrow{\pi} \mathbb{R}^2$ linearly and still embed $\|\Delta\|$ as $\pi(\|\Delta\|)$ in $\mathbb{R}^2$

$\|\Delta\|$

$\cong \|\pi(\Delta)\|$

homeomorphic to

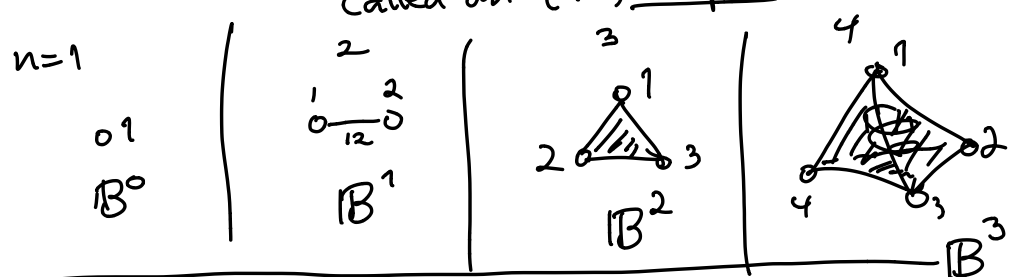


$\subset \mathbb{R}^2$

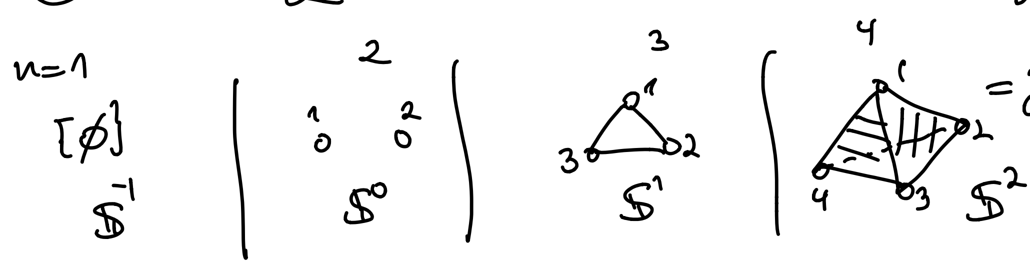
Terminology: $\|\Delta\|$ is triangulated by Δ

EXAMPLES: (1) $\Delta = 2^{\{1,2,\ldots,n\}}$ triangulates an $(n-1)$-dimensional ball $\mathbb{B}^{n-1}$

called an $(n-1)$-simplex



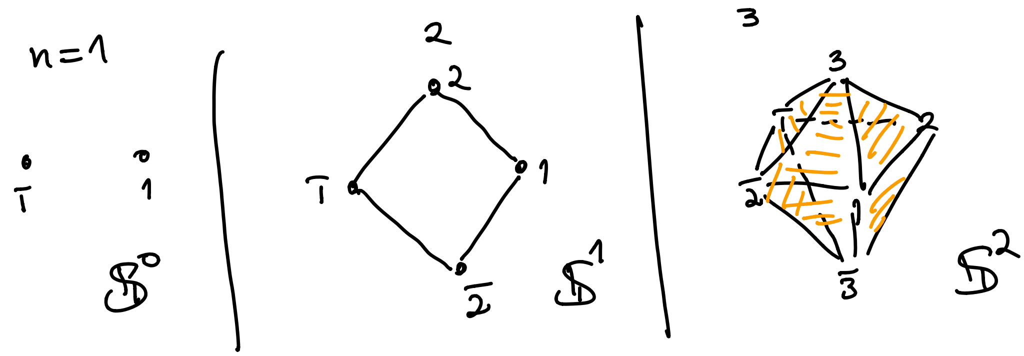
(2) $\Delta = 2^{\{1,2,\ldots,n\}} - \{123\ldots n\}$ triangulates an $(n-2)$-dim'l sphere $\mathbb{S}^{n-2}$



$= \partial \mathbb{B}^{n-1}$

③ $\Delta = \partial\Big(\underbrace{\text{n-dim'l cross polytope/hyperoctahedron}}_{\text{convex hull of } \{+e_1,-e_1,+e_2,-e_2,\dots,+e_n,-e_n\} \subset \mathbb{R}^n}\Big) \subset 2^{\{1,2,\dots,n,\bar{1},\bar{2},\dots,\bar{n}\}}$

$= \{$ subsets F containing no pairs $\{i,\bar{i}\}\}$



DEF'N: A face $F \in \Delta$ has dimension $\dim(F) = \#F - 1$ and it's called a d-face if $\dim(F) = d$

$\dim(\Delta) := \max\{\dim(F) : F \in \Delta\}$

vertices = 0-faces

edges = 1-faces

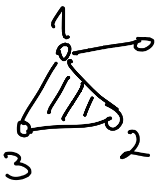
facets = faces that are maximal under inclusion

$f_k(\Delta) = \#$ k-dim'l faces of Δ

$\underline{f}(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$ if $\dim \Delta = d-1$

Say Δ is pure if all facets have dimension $\dim(\Delta)$

EXAMPLES:

① $\Delta =$  has $\underline{f}(\Delta) = (f_{-1}, f_0, f_1, f_2)$

$= (1, 4, 4, 1)$

$\emptyset$	1	12	123
	2	13	
	3	14	
	4	23	

$\dim(\Delta) = 2$

Δ is <u>not</u> <u>pure</u>; it has two facets, 123, 14

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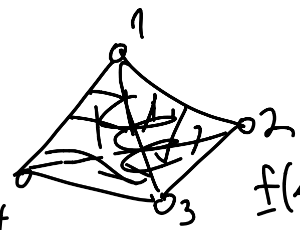
② $\Delta = 2^{\{1,2,\dots,n\}} = (n-1)\text{-simplex}$ is <u>pure</u>, $(n-1)$-dimensional

with $\underline{f}(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{n-1})$

$$= \left(\underset{1}{\underset{\shortparallel}{\binom{n}{0}}}, \underset{n}{\underset{\shortparallel}{\binom{n}{1}}}, \binom{n}{2}, \dots, \binom{n}{n-1}, \underset{1}{\underset{\shortparallel}{\binom{n}{n}}} \right)$$

$n = 4$

$\mathbb{B}^3 \cong$  $\underline{f}(\Delta) = (1, 4, 6, 4, 1)$

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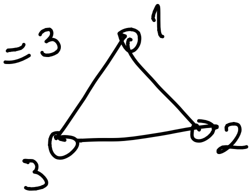
③ $\Delta = \partial\big(\underbrace{2^{\{1,2,\dots,n\}}}_{(n-1)\text{-simplex}}\big) = 2^{\{1,2,\dots,n\}} - \{12\cdots n\}$ is pure, $(n-2)$-dimensional

with $\underline{f}(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{n-2})$

$$= \left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n-1} \right)$$

$n = 3$

$\mathbb{S}^1 \cong$  $\underline{f}(\Delta) = (f_{-1}, f_0, f_1)$

$= (1, 3, 3)$

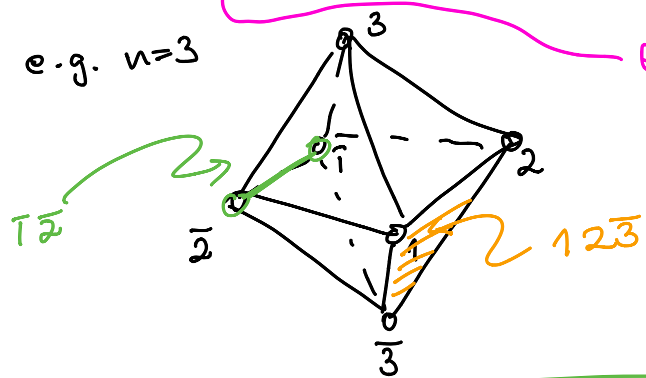
④ $\Delta = \partial\begin{pmatrix} n\text{-dim'l} \\ \text{cross-polytope}\end{pmatrix}$ is pure $(n-1)$-dim'l

with $\underline{f}(\Delta) = \left(2^0\binom{n}{0}, 2^1\binom{n}{1}, 2^2\binom{n}{2}, \ldots, 2^n\binom{n}{n}\right)$

i.e. $f_{k-1}(\Delta) = 2^k\binom{n}{k}$ $\quad = (f_{-1}, f_0, f_1, \ldots, f_{n-1})$

EXERCISE: Check this!

e.g. $n=3$



⑤ DEF'N/EXAMPLE:

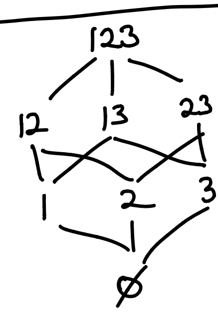
A **poset** P is a set with binary relation $x \le y$
(partially ordered set)

satisfying
$$\begin{cases} x \le x & \text{reflexive} \\ x\le y,\ y \le x \Rightarrow x = y & \text{anti-symmetric} \\ x \le y,\ y \le z \Rightarrow x \le z & \text{transitive}\end{cases}$$

and it gives rise to a simplicial complex called the **order complex** ΔP having P as its vertex set and faces $F \subseteq P$ being the totally/linearly ordered subsets (chains in P)

EXAMPLE $P = 2^{\{1,2,3\}}$ ordered via inclusion
$S \le T$ if $S \subseteq T$

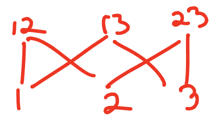
$P =$

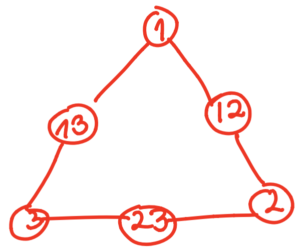


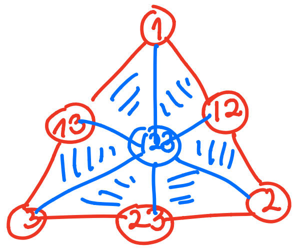
Hasse diagram with edge $\begin{matrix} y \\ | \\ x \end{matrix}$ if $x \lessdot y$ ("x covered by y"
$x < y$ and $\nexists z$
$x \lneq z \lneq y$)

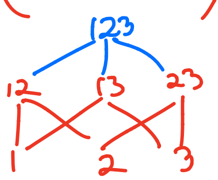
Let's draw ΔP by starting with

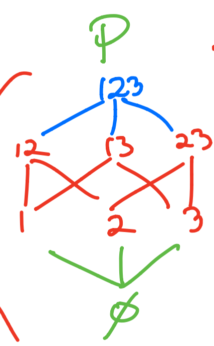
$\Delta(\underbrace{P - \{\emptyset, 123\}})$

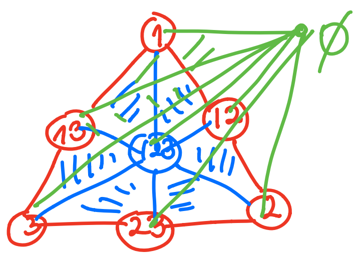
$\Delta\Big($ 

$\Big) =$ 

$\Delta(P - \{\emptyset\}) =$ 



$\Delta\Big($ 

$\Big) =$ 

Math 8680 Feb 1, 2021

Stanley - Reisner rings

Harvard 1971 PhD Thesis under G.-C. Rota

U. Minn. PhD thesis 1974 under Mel Hochster

DEF'N: Let K be a commutative ring (assume K is a field, unless I specifically say $K=\mathbb{Z}$)

and a simplicial complex Δ on vertices $\{1,2,\ldots,n\}$

the Stanley-Reisner ring

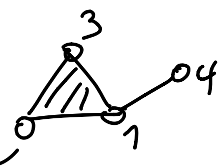
$$K[\Delta] := K[x_1,x_2,\ldots,x_n]/I_\Delta$$

where the S-R ideal $I_\Delta = \left(\underset{\prod_{i\in G} x_i}{\underset{\shortparallel}{\underline{x}^G}} \right)_{G\notin\Delta}$

It is a free K-module with K-basis

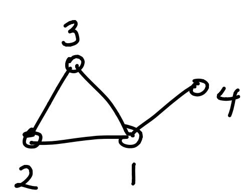
$$\{ \underset{x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}}{\underset{\shortparallel}{\underline{x}^a}} \}_{\mathrm{supp}(\underline{x}^a)\in\Delta} \quad \text{where } \mathrm{supp}(\underline{x}^a) := \{i : a_i>0\}$$

EXAMPLES:

① $\Delta_1 =$ 

$$K[\Delta_1] = K[x_1,x_2,x_3,x_4] \Big/ \begin{pmatrix} x_2x_4, x_3x_4, \\ x_1x_3x_4, x_2x_3x_4 \end{pmatrix}$$

$$= K[x_1,x_2,x_3,x_4]/(x_2x_4, x_3x_4)$$

$\Delta_2 =$ 

$$K[\Delta_2] = K[x_1,x_2,x_3,x_4] \Big/ \begin{pmatrix} x_2x_4, x_3x_4, \\ x_1x_2x_3 \end{pmatrix}$$

$\Delta_2 \subset \Delta_1$ is a <u>subcomplex</u>, so $I_{\Delta_2} \supseteq I_{\Delta_1}$

so $k[\Delta_1] \overset{\pi}{\twoheadrightarrow} k[\Delta_2]$ surjects

induced by $x_i \longmapsto x_i$

and in fact, get a <u>short exact sequence</u>

$$0 \to \underbrace{(x_1x_2x_3)}_{\substack{\text{ideal inside} \\ k[\Delta_1] \\ \text{generated by } \{x_1x_2x_3\}}} \xrightarrow{i} k[\Delta_1] \xrightarrow{\pi} k[\Delta_2] \to 0$$

i.e. $\ker(\pi) = \operatorname{im}(i)$

i.e. $k[\Delta_2] \cong k[\Delta_1]/(x_1x_2x_3)$

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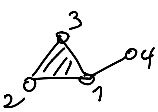
e.g. in $k[\Delta_1]$, $(x_1^2x_2)(x_2^{10}x_3^{100}) = x_1^2x_2^{11}x_3^{100}$

since $\operatorname{supp}(x_1^2x_2^{11}x_3^{100}) = \{1,2,3\} \in \Delta_1$

$\Delta_1 =$ 

$(x_1^2x_2^5)(x_1x_4) = x_1^3x_2^5x_4 = 0$

since $\operatorname{supp}(x_1^3x_2^5x_4) = \{1,2,4\} \notin \Delta_1$

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② $K[\underbrace{(n-1)\text{-simplex}}_{=2^{\{1,2,\dots,n\}}}] = K[x_1,x_2,\dots,x_n]$ since $I_\Delta = \{0\}$

$K[\partial(\underbrace{(n-1)\text{-simplex}}_{2^{\{1,2,\dots,n\}} - \{12\dots n\}})] = K[x_1,x_2,\dots,x_n]/(x_1x_2\cdots x_n)$

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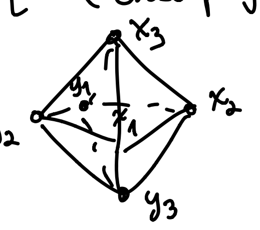
③ $K\left[\partial\left(\begin{matrix}n\text{-dim'l} \\ \text{cross-polytope}\end{matrix}\right)\right] = K[x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n] / (x_1y_1, x_2y_2, \dots, x_ny_n)$

$n=3$ 

DEF'N: $K[\underline{x}] := K[x_1, x_2, \ldots, x_n]$ is an example of an **$\mathbb{N}$-graded** **K-algebra**

$\deg(\underline{x}^{\underline{a}}) = a_1 + \ldots + a_n$
$\deg(x_i) = 1$

$\mathbb{N} = \{0, 1, 2, \ldots\}$

A **$\mathbb{N}$-graded ring** R is one with $R = \bigoplus_{d=0}^{\infty} R_d$ ← direct sum of abelian groups

with $R_d \cdot R_e \subset R_{d+e}$ ($x \in R_d$ means "x is homogeneous of degree d")

$$\underset{x}{R_d} \cdot \underset{y}{R_e} \Rightarrow \underset{xy}{R_{d+e}}$$

A K-algebra R has a ring map $K \hookrightarrow R$
(if $K \hookrightarrow R_0$ if R is graded)

The S-R ideal $I_\Delta \subset K[\underline{x}]$ is a **homogeneous** ideal I

meaning $I = \bigoplus_{d=0}^{\infty} I_d$ where $I_d := I \cap (K[\underline{x}]_d)$

(easy) **EXERCISE:** Show that if R is a graded ring $R = \bigoplus_{d=0}^{\infty} R_d$ then an ideal $I \subset R$ has a set of generators $I = (f_\alpha)$ with each f_α homogeneous (i.e. $f_\alpha \in R_{d_\alpha}$)
$\Leftrightarrow$ I is a homogeneous ideal i.e. $I = \bigoplus_{d=0}^{\infty} I_d$

Hence $K[\Delta] = K[\underline{x}]/I_\Delta$ has inherited an $\mathbb{N}$-graded K-algebra structure

$$= \bigoplus_{d=0}^{\infty} \underbrace{K[\Delta]_d}$$

← K-span of $\{\underline{x}^{\underline{a}}\}_{\substack{\mathrm{supp}(\underline{x}^{\underline{a}}) \in \Delta, \\ a_1 + \ldots + a_n = d}}$

DEF'N: For K a field, R a graded K-algebra

$$\underbrace{Hilb(R,t)}_{\text{Hilbert series of } R} := \sum_{d=0}^{\infty} \dim_K(R_d)\cdot t^d \in \mathbb{Z}[[t]]$$

$$= \dim_K R_0 + \dim_K R_1 \cdot t + \dim_K R_2 \cdot t^2 + \dots$$

EXAMPLES:

① $$Hilb(\underbrace{K[(n-1)\text{-simplex}]}_{K[x_1,x_2,\dots,x_n]}, t) = \sum_{d=0}^{\infty} \dim_K K[\underline{x}]_d \cdot t^d$$

$$= \sum_{\substack{\text{all monomials} \\ \underline{x}^{\underline{a}} = x_1^{a_1}\cdots x_n^{a_n}}} t^{a_1+a_2+\dots+a_n}$$

$$= \left(\sum_{a_1=0}^{\infty} t^{a_1}\right)\left(\sum_{a_2=0}^{\infty} t^{a_2}\right)\cdots\left(\sum_{a_n=0}^{\infty} t^{a_n}\right)$$

$$= (1+t+t^2+\dots)(1+t+t^2+\dots)\cdots(1+t+t^2+\dots)$$

$$= \frac{1}{1-t}\cdot\frac{1}{1-t}\cdots\frac{1}{1-t}$$

$$= \frac{1}{(1-t)^n} = (1-t)^{-n}$$

general binomial theorem

$$= \sum_{d=0}^{\infty}\binom{-n}{d}(-t)^d$$

$$= \sum_{d=0}^{\infty}\frac{(-n)(-n-1)\cdots(-n-d+1)}{d!}(-1)^d\cdot t^d$$

$$= \sum_{d=0}^{\infty}\underbrace{\binom{n+d-1}{d}}_{\substack{\#\text{ of ways to choose a} \\ d\text{-element multiset from } \{1,2,\dots,n\}}} t^d$$

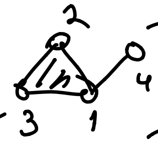
$$\boxed{\#\text{ ways to write } d \text{ stars, } n-1 \text{ bars} \quad \underbrace{**}_{a_1}|\underbrace{}_{a_2}|\underbrace{*}_{a_3}|***|*|\underbrace{**}_{a_n}} = \#\text{ of ways to choose a } d\text{-element multiset from } \{1,2,\dots,n\}$$

② **PROPOSITION:** For K a field, Δ a simplicial complex of dimension $d-1$, one has

$$\mathrm{Hilb}(K[\Delta],t) = \sum_{i=0}^{d} f_{i-1}(\Delta)\cdot\left(\frac{t}{1-t}\right)^i$$

$$= \sum_{\text{faces } F\in\Delta}\left(\frac{t}{1-t}\right)^{\#F}$$

EXAMPLE:



$$\mathrm{Hilb}\left(K\left[\ \Delta\ \right],t\right) = \underbrace{1}_{f_{-1}} + \underbrace{4\frac{t}{1-t}}_{f_0} + \underbrace{4\left(\frac{t}{1-t}\right)^2}_{f_1} + \underbrace{1\cdot\left(\frac{t}{1-t}\right)^3}_{f_2}$$

counts monomials supported on $\varnothing$ $\{1\}$

counts monomials supported on vertices
$x_1, x_1^2, x_1^3, \ldots$
$x_2, x_2^2, x_2^3, \ldots$
$x_3, x_3^2, \ldots$
$x_4, x_4^2, \ldots$

supported on edges
$x_1x_2,\ x_1^2x_2 \ldots$
$x_1x_2^2,\ x_1^2x_2^2 \ldots$
$\vdots$

supported on triangles

Feb 3, 2021

proof of PROPOSITION:

$$\mathrm{Hilb}(K[\Delta],t) = \sum_{F\in\Delta}\ \sum_{\substack{\text{monomials } \underline{x}^\alpha:\\ \mathrm{supp}(\underline{x}^\alpha)=F}} t^{a_1+a_2+\cdots+a_n}$$

$$= \sum_{\substack{F\in\Delta\\ \| \\ \{i_1,i_2,\ldots,i_k\}}} \underbrace{(t+t^2+\cdots)}_{\substack{\text{picking a term}\\ \text{chooses power}\\ \text{on } x_{i_1}}}\ \underbrace{(t+t^2+\cdots)}_{\text{on } x_{i_2}}\cdots\underbrace{(t+t^2+\cdots)}_{\text{on } x_{i_k}}$$

$$= \sum_{F\in\Delta} \underbrace{\left(\frac{t}{1-t}\right)\left(\frac{t}{1-t}\right)\cdots\left(\frac{t}{1-t}\right)}_{k=\#F \text{ times}}$$

$$= \sum_{F\in\Delta} \left(\frac{t}{1-t}\right)^{\#F} \quad \blacksquare$$

<u>EXAMPLE:</u> $\Delta = \partial\left(\begin{smallmatrix}\text{n-dim'l}\\ \text{cross-polytope}\end{smallmatrix}\right)$ has $f_{k-1} = 2^k\binom{n}{k}$ for $k=0,1,2,\ldots,n$

so $$\mathrm{Hilb}(K[\Delta],t) = \sum_{k=0}^{n} f_{k-1}\left(\frac{t}{1-t}\right)^k$$

$$= \sum_{k=0}^{n} 2^k\binom{n}{k}\left(\frac{t}{1-t}\right)^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{2t}{1-t}\right)^k$$

binomial theorem

$$= \left(1+\frac{2t}{1-t}\right)^n = \left(\frac{1+t}{1-t}\right)^n$$

$$= \frac{\sum_{k=0}^{n}\binom{n}{k}t^k}{(1-t)^n}$$

<u>DEF'N/PROP:</u> K a field, Δ a $(d-1)$-dim'l simplicial complex. Then if Δ has $\underline{f}(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, one has

$$\mathrm{Hilb}(K[\Delta],t) = \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d}{(1-t)^d}$$

where $\underline{h}(\Delta) \underset{\text{DEF'N}}{:=} (h_0, h_1, h_2, \ldots, h_d)$ is called the $\underline{h}$-vector of Δ

DEF'N/PROP: K a field, Δ a $(d-1)$-dim'l simplicial complex

Then if Δ has $\underline{f}(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, one has

$$\mathrm{Hilb}(K[\Delta], t) = \frac{h_0 + h_1 t + h_2 t^2 + \ldots + h_d t^d}{(1-t)^d}$$

where $\underline{h}(\Delta) \underset{\text{DEF'N}}{:=} (h_0, h_1, h_2, \ldots, h_d)$ is called the $\underline{h}$-vector of Δ

and is related to $\underline{f}(\Delta)$ via a unitriangular relation having $\mathbb{Z}$ coefficients:

$$\boxed{\begin{aligned} f_{i-1} &= \sum_{k=0}^{i} h_k \binom{d-k}{d-i} \\ h_k &= \sum_{i=0}^{k} f_{i-1} \binom{d-i}{d-k} (-1)^{k-i} \end{aligned}}$$

$i=d$

$k=d$

In particular, $\boxed{f_{d-1} = h_0 + h_1 + \ldots + h_d}$

$$\boxed{h_d = \sum_{i=0}^{k} (-1)^{d-i} f_{i-1}} = (-1)^{d-1}\left(-f_{-1} + f_0 - f_1 + \ldots + (-1)^{d-1} f_{d-1}\right)$$

$$= (-1)^{d-1} \tilde{\chi}(\Delta)$$

(reduced) Euler characteristic of Δ

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i \underbrace{\tilde{\beta}_i(\Delta; K)}_{\dim_K \tilde{H}_i(\Delta; K)}$$

of i-dim'l "holes" in $\|\Delta\|$.

proof: Know $\mathrm{Hilb}(K[\Delta],t) = \sum_{i=0}^{d} f_{i-1}\left(\frac{t}{1-t}\right)^i = \frac{\sum_{k=0}^{d} h_k t^k}{(1-t)^d}$

TO BE USED LATER!

In fact, even if we had ∞ sequences/vectors

$$(f_{-1}, f_0, f_1, \ldots)$$
$$(h_0, h_1, h_2, \ldots)$$

related in this way, one has the same unitriangular relations:

$$\sum_{i=0}^{\infty} f_{i-1}\left(\frac{t}{1-t}\right)^i = \frac{1}{(1-t)^d}\sum_{k=0}^{\infty} h_k t^k \quad \text{in } \mathbb{Z}[[t]]$$

$$u = \frac{t}{1-t}$$
$$t = \frac{u}{1+u}$$
$$1-t = \frac{1}{1+u}$$

$$\sum_{i=0}^{\infty} f_{i-1} u^i = (1+u)^d \sum_{k=0}^{\infty} h_k \left(\frac{u}{1+u}\right)^k$$
$$= \sum_{k=0}^{\infty} h_k u^k (1+u)^{d-k}$$
$$= \sum_{k=0}^{\infty} h_k u^k \sum_{j=0}^{\infty} \binom{d-k}{j} u^j$$

extract coeff of u^i

$$f_{i-1} = \sum_{k=0}^{i} h_k \underbrace{\binom{d-k}{i-k}}_{\binom{d-k}{d-i}}$$

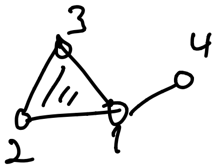
$$\sum_{i=0}^{\infty} f_{i-1} t^i (1-t)^{d-i} = \sum_{k=0}^{\infty} h_k t^k$$
$$\sum_{i=0}^{\infty} f_{i-1} t^i \sum_{j=0}^{\infty} \binom{d-i}{j} (-t)^j$$

extract coeff. of t^k

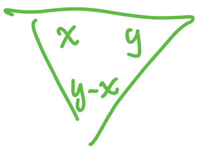
$$\sum_{i=0}^{k} f_{i-1} \underbrace{\binom{d-i}{k-i}}_{\binom{d-i}{d-k}} (-1)^{k-i} = h_k$$

$\blacksquare$

REMARK: EXERCISE 1 asks you to check Stanley's triangle shortcut also works to compute $\underline{h}(\Delta)$ from $\underline{f}(\Delta)$ (or vice-versa):

$\Delta =$ 

has $\underline{f}(\Delta) = (\underset{f_{-1}}{1}, \underset{f_0}{4}, \underset{f_1}{4}, \underset{f_2}{1})$



$$\begin{array}{ccccccccc}
 & & & & 1 & & & & = f_{-1} \\
 & & & 1 & & 4 & & & = f_0 \\
 & & 1 & & 3 & & 4 & & = f_1 \\
 & 1 & & 2 & & 1 & & 1 & = f_2 \\
\hline
\end{array}$$

$(\underset{h_0}{1}, \underset{h_1}{1}, \underset{h_2}{-1}, \underset{h_3}{0}) = \underline{h}(\Delta)$

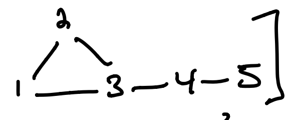
REMARK: If $\underline{h}(\Delta) \geq 0$ then note

$$f_{i-1} = \sum_{k=0}^{i} h_k \underbrace{\binom{d-k}{d-i}}_{\in \mathbb{N}} \quad \underline{\text{nonnegative}},$$

then $h_k \leq f_k \ \forall k$, so they're $\underline{\text{smaller}}$ than $\underline{f}(\Delta)$.

Math 8680 Feb 4, 2021

Recall via EXAMPLE



$$\mathrm{Hilb}\Big(K\big[\underbrace{\;1,2,3,4,5\;}_{\Delta}\big],\, t\Big) = 1 + 5\frac{t}{1-t} + 5\left(\frac{t}{1-t}\right)^2$$

$$= \frac{1\cdot(1-t)^2 + 5t(1-t) + 5t^2}{(1-t)^2}$$

$$= \frac{1+3t+t^2}{(1-t)^2}$$

$$\underline{f}(\Delta) = (\underset{f_{-1}}{1}, \underset{f_0}{5}, \underset{f_1}{5})$$

$$\begin{array}{r} 1-2t+t^2 \\ +5t-5t^2 \\ +5t^2 \\ \hline 1+3t+t^2 = h(\Delta,t) \end{array}$$

$$\begin{array}{ccccccc} & & & 1^{\,f_{-1}} & & & \\ & & 1 & & 5^{\,f_0} & & \\ & 1 & & 4 & & 5^{\,f_1} & \\ \hline (1, & & 3, & & 1) & = \underline{h}(\Delta) & \end{array}$$

MINI OVERVIEW

$\Delta = \partial\left(\begin{smallmatrix}\text{simplicial}\\ \text{polytope}\end{smallmatrix}\right) \overset{①}{\Rightarrow} \Delta$ shellable

Δ shellable $\Rightarrow$ Δ partitionable $\Rightarrow$ $\underline{h}(\Delta) \geq 0$

Δ shellable $\Rightarrow$ $K[\Delta]$ Cohen-Macaulayness $\Rightarrow$ $\underline{h}(\Delta) \geq 0$

$\dim(\Delta) := d-1$
$\#\text{vertices} = n$ $\overset{②}{\Rightarrow}$ $0 \leq h_k(\Delta) \leq \binom{n-d+k-1}{k} = h_k(\partial C(n,d))$, $0 \leq k \leq \lfloor d/2 \rfloor$

①, ② $\Rightarrow$ Motzkin's U.B.C.

DEF'N: A simplicial complex Δ is <u>partitionable</u> if it is <u>pure</u> and it has at least one <u>partitioning</u>

$$\Delta := \underset{\text{disjoint}}{\bigsqcup_{i=1}^{s}} \underbrace{[G_i, F_i]}_{\text{interval in the face poset of } \Delta} \quad \text{with } F_1, F_2, \ldots, F_s \text{ the facets of } \Delta$$

$$= \{ F \in \Delta : G_i \subseteq F \subseteq F_i \}$$

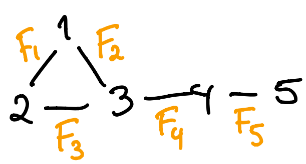
PROP: Δ $(d-1)$-dim'l partitionable

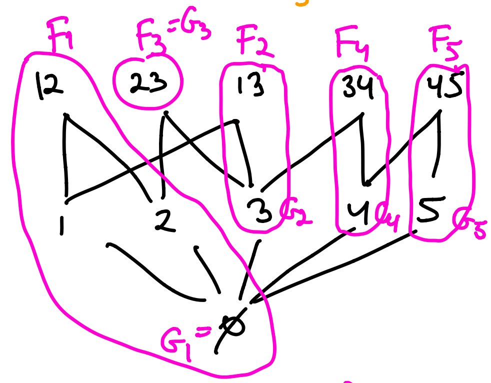
$$\Rightarrow \mathrm{Hilb}(K[\Delta], t) = \frac{\sum_{i=1}^{s} t^{\#G_i}}{(1-t)^d}$$

i.e. the h-polynomial $h(\Delta, t) = \sum h_k t^k$
$$= \sum_{i=1}^{s} t^{\#G_i}$$

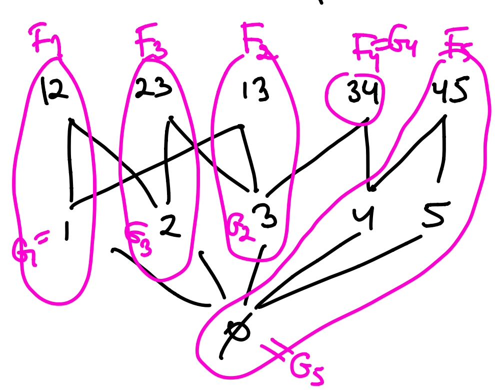
so $h_k(\Delta) = \#\{ i : \#G_i = k \} \geq 0 \;\; \forall i$

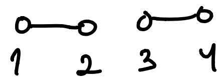
EXAMPLES:

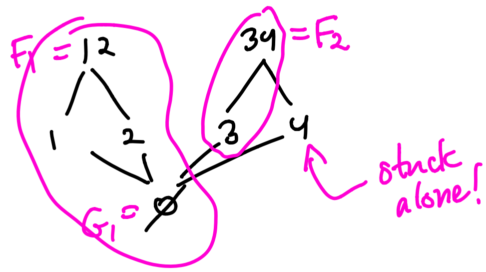
① $\Delta =$  is pure 1-dim'l $(d=2)$, and is partitionable with ≥ 2 very different partitionings:

$\Delta =$ 

$h(\Delta) = (1, 3, 1)$
$\quad\quad\quad\; h_0 \;\; h_1 \;\; h_2$



(2) $\Delta =$  is pure 1-dim'l but not partitionable

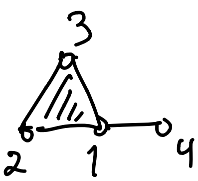


$h(\Delta) =$

$$f(\Delta) = (1, 4, 2)$$
$$f_{-1} \quad f_0 \quad f_1$$

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 4 & \\ 1 & & 3 & & 2 \\ \hline \end{array}$$

$h(\Delta) = (1, 2, -1)$ negative!

(3) $\Delta =$  is not pure, so not partitionable.

Rather than just proving...

PROP: Δ $(d-1)$-dim'l partitionable

$$\Rightarrow \mathrm{Hilb}(k[\Delta], t) = \frac{\sum_{i=1}^{s} t^{\#G_i}}{(1-t)^d}$$

... let's do it better with a finer grading!

DEF'N: A ring R has an $\mathbb{N}^n$-grading (multi-grading) if $R = \bigoplus_{\underline{a} \in \mathbb{N}^n} R_{\underline{a}}$

with $R_{\underline{a}} \cdot R_{\underline{b}} = R_{\underline{a}+\underline{b}}$

EXAMPLES: ① $K[\underline{x}] = K[x_1, \ldots, x_n]$ has an $\mathbb{N}^n$-grading

$$= \bigoplus_{\underline{a} \in \mathbb{N}^n} \underbrace{K[\underline{x}]_{\underline{a}}}_{\substack{= K\text{-span of } \{\underline{x}^{\underline{a}}\}, \\ \text{a 1-dim'l } K\text{-vector space}}}$$

② $K[\Delta] = K[x_1, \ldots, x_n] / I_\Delta$ inherits the $\mathbb{N}^n$-grading

$$= \bigoplus_{\underline{a} \in \mathbb{N}^n} \underbrace{K[\Delta]_{\underline{a}}}_{= \begin{cases} K\text{-span of } \{\underline{x}^{\underline{a}}\} & \text{if } \mathrm{supp}(\underline{a}) = \{i : a_i > 0\} \in \Delta \\ 0 & \text{else} \end{cases}}$$

DEF'N: The $\mathbb{N}^n$-graded Hilbert series of such an R is

$$\mathrm{Hilb}(R; t_1, \ldots, t_n) = \sum_{\underline{a} \in \mathbb{N}^n} \dim_K(R_{\underline{a}}) \cdot \underbrace{\underline{t}^{\underline{a}}}_{t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}} \quad \in \mathbb{Z}[[t_1, \ldots, t_n]]$$

can always specialize to $\mathbb{N}$-grading

$$\Big\downarrow \; t_1 = t_2 = \cdots = t$$

$$\mathrm{Hilb}(R, t) = \sum_{k=0}^{\infty} \dim_K(R_k) t^k$$

PROP: (a) For any simplicial complex Δ,

$$\mathrm{Hilb}(K[\Delta]; t_1,\dots,t_n) = \sum_{F\in\Delta} \prod_{i\in F} \frac{t_i}{1-t_i}$$

$$\left(\underset{\substack{\text{specialize} \\ t_i = t\ \forall i}}{\rightsquigarrow} \quad \mathrm{Hilb}(K[\Delta], t) = \sum_{F\in\Delta} \left(\frac{t}{1-t}\right)^{\#F} \right)$$

(b) For Δ partitionable as $\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$,

$$\mathrm{Hilb}(K[\Delta]; t_1,\dots,t_n) = \sum_{i=1}^{s} \frac{\prod_{j\in G_i} t_j}{\prod_{j\in F_i}(1-t_j)}$$

specialize $t_i = t$

$$\mathrm{Hilb}(K[\Delta], t) = \frac{\sum_{i=1}^{s} t^{\#G_i}}{(1-t)^d}$$

---

proof:

(a) $$\mathrm{Hilb}(K[\Delta]; t_1,\dots,t_n) = \sum_{\underline{a}\in\mathbb{N}^n} \dim_K K[\Delta]_{\underline{a}} \cdot \underline{t}^{\underline{a}}$$

$$= \sum_{\substack{\underline{a}\in\mathbb{N}^n: \\ \mathrm{supp}(\underline{a})\in\Delta}} \underline{t}^{\underline{a}} = \sum_{F\in\Delta} \sum_{\substack{\underline{a}\in\mathbb{N}^n: \\ \mathrm{supp}(\underline{a}) = F}} \underline{t}^{\underline{a}}$$

$$= \sum_{F \in \Delta} \prod_{i \in F} \left(t_i + t_i^2 + t_i^3 + \dots \right)$$

$$= \sum_{F \in \Delta} \prod_{i \in F} \frac{t_i}{1 - t_i}$$

A shelling of the boundary of the octahedron