# Math 8680 Spring 2021 HW Solutions

#1 Let's show parts (a), (b) simultaneously, by checking that the d-skeleton $\Delta^{(d)}$ of a simplicial complex $\Delta$ having f-vector $\underline{f}=(f_{-1}, f_0, f_1, f_2, \ldots)$ satisfies $h_k(\Delta^{(d)}) \overset{(*)}{=} h_k(\Delta^{(d-1)}) - h_{k-1}(\Delta^{(d-1)})$

if we define (as in class)

$$h_k(\Delta^{(d)}) := \sum_{i=0}^{k} f_{i-1} \binom{d-i}{d-k}(-1)^{k-i}.$$

In the desired equality $(*)$, one can rewrite the left side as

$$h_k(\Delta^{(d)}) = f_k + \sum_{i=0}^{k-1} f_{i-1}\cdot(-1)^{k-i}\binom{d-i}{d-k}.$$

One can rewrite the right side

$$h_k(\Delta^{(d-1)}) - h_{k-1}(\Delta^{(d-1)}) = f_k + \sum_{i=0}^{k-1} f_{i-1}(-1)^{k-i}\binom{d-1-i}{d-1-k} - \sum_{i=0}^{k-1} f_{i-1}(-1)^{k-1-i}\binom{d-1-i}{d-1-(k-1)}$$

$$= f_k + \sum_{i=0}^{k-1} f_{i-1}\cdot(-1)^{k-i}\underbrace{\left[\binom{d-i-1}{d-k-1}+\binom{d-i-1}{d-k}\right]}_{=\binom{d-i}{d-k} \text{ by Pascal recurrence}}$$

This proves both (a), (b) via induction on d.

#2 (a) Given $\Delta$ a simplicial complex on vertex set $\{1,2,\dots,n\}$ with $n \geq 1$, to see the sum $\Theta := x_1 + \dots + x_n$ in $K[\Delta]$ is a nonzerodivisor (NZD), assume some $f \in K[\Delta] - \{0\}$ has $0 = \Theta \cdot f$. Writing $f = \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_n) \\ \mathrm{supp}(\underline{x}^\alpha) \in \Delta}} c_\alpha \underline{x}^\alpha$ where $c_\alpha \in K$ and $\underline{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$,

one can pick some variable $x_{i_0}$ appearing in some monomial $\underline{x}^\alpha$ with $c_\alpha \neq 0$, and choose $\alpha_0$ to be such a monomial achieving the maximum exponent on $x_{i_0}$.

Then
$$\Theta \cdot f = (x_1 + \dots + x_n) f = c_{\alpha_0} \underbrace{x_{i_0} \cdot \underline{x}^{\alpha_0}}_{\substack{\text{still supported on} \\ \text{a face of } \Delta \text{, since } \underline{x}^{\alpha_0} \text{ was,} \\ \text{and } x_{i_0} \text{ divides } \underline{x}^{\alpha_0}.}} + \left( \begin{array}{c} \text{sum of monomials} \\ c_\beta \underline{x}^\beta \text{ with } \underline{x}^\beta \neq x_{i_0} \underline{x}^{\alpha_0} \end{array} \right)$$

So $\Theta \cdot f \neq 0$, a contradiction.

(b) If the connected components of $\Delta$ decompose the vertex set as $\{1,2,\dots,n\} = V_1 \sqcup V_2 \sqcup \dots \sqcup V_k$ with $k \geq 2$, then we claim $g := \sum_{i \in V_1} x_i$ has $x_j g = 0$ in $K[\Delta]/(\Theta)$ $\forall j$, and hence $f \cdot g = 0$ $\forall f \in K[\Delta]$ of positive degree:

$$x_j \cdot g = 0 \quad \text{for } j \notin V_1$$

and
$$x_j g \equiv x_j \Big( - \sum_{i \notin V_1} x_i \Big) = 0 \quad \text{for } j \in V_1$$

#3 Assume $\Delta$ is a pure $d$-dim'l complex, so its facets $F_1, F_2, \ldots, F_s$ are all $d$-dim'l, and assume this is a shelling order, meaning that $\forall\, 1 \le i < j$ $\exists\, k$ having $1 \le k < j$ with

$$F_i \cap F_j \subseteq F_k \cap F_j \text{ and } F_k \cap F_j \text{ is } (d-1)\text{-dim'l}.$$

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Then the facets of $\mathrm{star}_\Delta(F)$ are simply the subset $F_{j_1}, F_{j_2}, \ldots, F_{j_s}$ of the facets of $\Delta$ that contain $F$, and hence all $d$-dim'l, so $\mathrm{star}_\Delta(F)$. Also if this order is the restriction of the above shelling order to these facets, meaning $j_1 < j_2 < \ldots < j_s$, then for any $i_1 \le i_p < i_q$ $\exists\, k$ having $1 \le k < i_q$ with

$$F_{i_p} \cap F_{i_q} \subseteq F_k \cap F_{i_q} \text{ and } F_k \cap F_{i_q} \text{ is } (d-1)\text{-dim'l}.$$

But then $F \subset F_{i_p} \cap F_{i_q} \subseteq F_k \cap F_{i_q} \Rightarrow F \subset F_k$

$\Rightarrow F_k$ is a facet of $\mathrm{star}_\Delta(F)$, i.e. $F_k = F_{i_r}$ for some $r$.

So $\mathrm{star}_\Delta(F)$ is shellable.

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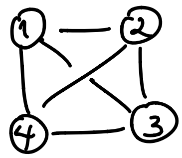
The facets of $\mathrm{link}_\Delta(F)$ biject with those of $\mathrm{star}_\Delta(F)$:

$$\underset{\text{facet of } \mathrm{star}_\Delta(F)}{F_{j_t}} \longmapsto \underset{\text{facet of } \mathrm{link}_\Delta(F)}{F_{j_t} \setminus F}$$

So $\mathrm{link}_\Delta(F)$ is pure $(d-\#F)$-dim'l and inherits the same shelling order $F_{j_1} \setminus F, F_{j_2} \setminus F, \ldots, F_{j_s} \setminus F$ from $\mathrm{star}_\Delta(F)$.

#4 (a) For a field $K$, a criterion proven in class says that

$\Delta =$  has $\Theta_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4,$
$\Theta_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \in K[\Delta]_1$

for which $K[\Delta]$ is a finitely gen'd $K[\Theta_1,\Theta_2]$-module

$\iff A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \in K^{2\times 4}$ has all $2\times 2$ submatrices invertible.

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When $K = \mathbb{F}_2$ this is impossible, since it requires the 4 columns of $A$ to be 4 distinct, nonzero vectors in $\mathbb{F}_2^2$, which has only 3 such vectors total: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$.

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When $K \neq \mathbb{F}_2$, it can always be achieved for example by

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & c \end{bmatrix} \text{ with } c \in K - \{0,1\}$$

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(b) (Solution suggested by Dania Morales)
If one had $\Theta_1, \Theta_2 \in \mathbb{Z}[\Delta]_1$ for which $\mathbb{Z}[\Delta]$ is a finitely gen'd $\mathbb{Z}[\Theta_1,\Theta_2]$-module, with generators $\{f_i\}_{i\in I}$, get a contradiction to part (a) as follows. The map $\mathbb{Z} \xrightarrow{\pi} \mathbb{F}_2$ reduces coefficients mod 2 induces ring surjections

$$\mathbb{Z}[x_1,x_2,x_3,x_4] \xrightarrow{\pi} \mathbb{F}_2[x_1,x_2,x_3,x_4]$$
$$\mathbb{Z}[\Delta] \xrightarrow{\pi} \mathbb{F}_2[\Delta]$$
$$\mathbb{Z}[\Theta_1,\Theta_2] \xrightarrow{\pi} \mathbb{F}_2[\bar{\Theta}_1,\bar{\Theta}_2]$$

that show $\mathbb{F}_2[\Delta]$ is a finitely gen'd $\mathbb{F}_2[\bar{\Theta}_1,\bar{\Theta}_2]$-module; apply $\pi$ to the generators for $\mathbb{Z}[\Delta]$ as a $\mathbb{Z}[\Theta_1,\Theta_2]$-module, giving $\{\pi(f_i)\}_{i\in I}$ that contradict part (a) for $K = \mathbb{F}_2$.

#5 Recall that the cyclic polytope $C_d(n)$ for $d \geq 2$ is the convex hull in $\mathbb{R}^d$ of points $\{\underline{x}(t_1), \ldots, \underline{x}(t_n)\}$ where $\underline{x}(t) = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$

and $t_1 < t_2 < \ldots < t_n$ in $\mathbb{R}$

Thus a $d$-subset $F = \{i_1 < i_2 < \ldots < i_d\} \subset \{1, 2, \ldots, n\}$ corresponds to vertices $\{\underline{x}(t_{i_1}), \ldots, \underline{x}(t_{i_d})\}$ of a $(d-1)$-dimensional facet of $C_d(n)$

$\iff$ $\exists$ an affine functional $f(\underline{x}) = c_0 + c_1 x_1 + \ldots + c_d x_d$ on $\mathbb{R}^d$ that has $f(\underline{x}(t)) = 0$ for $t \in \{t_{i_1}, \ldots, t_{i_d}\}$

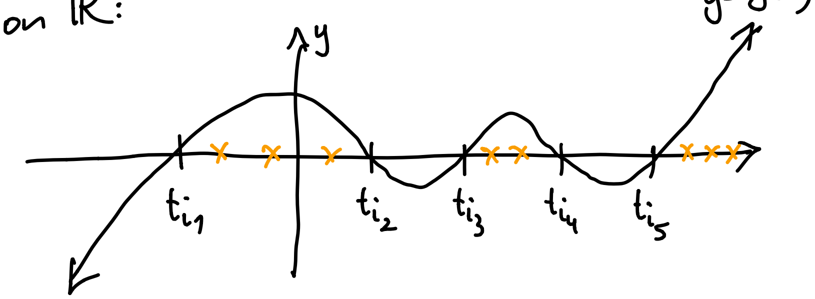
$f(\underline{x}(t)) > 0$ for $t \in \{t_1, \ldots, t_n\} \setminus \{t_{i_1}, \ldots, t_{i_d}\}$

$\iff$ $\exists$ a polynomial $g(t)\ (= f(\underline{x}(t))) = c_0 + c_1 t + c_2 t^2 + \ldots + c_d t^d \in \mathbb{R}[t]$ of degree $d$ having $g(t) = 0$ for $t \in \{t_{i_1}, \ldots, t_{i_d}\}$

$g(t) > 0$ for $t \in \{t_1, \ldots, t_n\} \setminus \{t_{i_1}, \ldots, t_{i_d}\}$

$\iff$ the polynomial $g(t) = (t - t_{i_1})(t - t_{i_2}) \cdots (t - t_{i_d})$ has $g(t) > 0$ for $t \in \{t_1, \ldots, t_n\} \setminus \{t_{i_1}, \ldots, t_{i_d}\}$

$\iff$ between every $t_i < t_j$ in $\{t_1, \ldots, t_n\} \setminus \{t_{i_1}, \ldots, t_{i_d}\}$ there are evenly many $\{t_{i_1}, \ldots, t_{i_d}\}$ in the usual order on $\mathbb{R}$:



#6 (a) The definition of $\Delta_1 * \Delta_2$ shows that

$$f_{i-1}(\Delta_1 * \Delta_2) = \sum_{a+b=i} f_{a-1}(\Delta_1) f_{b-1}(\Delta_2)$$

so $$f(\Delta_1 * \Delta_2, t) = \sum_{i \geq 0} f_{i-1}(\Delta_1 * \Delta_2) t^i = \sum_{i \geq 0} t^i \sum_{a+b=i} f_{a-1}(\Delta_1) f_{b-1}(\Delta_2)$$

$$= \sum_{a \geq 0} f_{a-1}(\Delta_1) t^a \cdot \sum_{b \geq 0} f_{b-1}(\Delta_2) t^b$$

$$= f(\Delta_1, t)\, f(\Delta_2, t)$$

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Now recall the definition of the h-vector $\underline{h} = (h_0, h_1, \ldots, h_d)$ for a $(d-1)$-dim'l simplicial complex:

$$\sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t}\right)^i \left( = \mathrm{Hilb}(K[\Delta], t)\right) = \frac{\sum_{k=0}^{d} h_k t^k}{(1-t)^d}$$

or in other words,

$$f\left(\Delta, \frac{t}{1-t}\right) \overset{(*)}{=} \frac{h(\Delta, t)}{(1-t)^d} \quad \text{where } \dim \Delta = d-1$$

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Note that if $\dim \Delta_1 = d_1 - 1$, $\dim \Delta_2 = d_2 - 1$ then $\dim \Delta_1 * \Delta_2 = d_1 + d_2 - 1$

and here

$$\frac{h(\Delta_1 * \Delta_2, t)}{(1-t)^{d_1+d_2}} \overset{(*)}{=} f\left(\Delta_1 * \Delta_2, \frac{t}{1-t}\right)$$

shown above

$$= f\left(\Delta_1, \frac{t}{1-t}\right) f\left(\Delta_2, \frac{t}{1-t}\right)$$

$$\overset{(*)}{=} \frac{h(\Delta_1, t)}{(1-t)^{d_1}} \cdot \frac{h(\Delta_2, t)}{(1-t)^{d_2}}$$

$$\Rightarrow h(\Delta_1 * \Delta_2, t) = h(\Delta_1, t)\, h(\Delta_2, t)$$

(b) Since $\Delta_0 = \{\emptyset, \{v\}\}$ has h-vector $\underline{h} = (1,0)$
$h(\Delta_0, t) = 1$,

and $\Delta_1 = \{\emptyset, \{v\}, \{v'\}\}$ has h-vector $\underline{h} = (1,1)$,
$h(\Delta_1, t) = 1+t$,

the cone $\Delta_0 * \Delta$ has $h(\Delta_0 * \Delta, t) = 1 \cdot h(\Delta, t)$

and hence $\underline{h}(\Delta_0 * \Delta) = \underline{h}(\Delta)$ (with an extra 0 at the end)

i.e. if $h(\Delta) = (h_0, h_1, \ldots, h_{d-1})$ then
$h(\Delta_0 * \Delta) = (h_0, h_1, \ldots, h_{d-1}, 0)$

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Similarly the suspension $\Delta_1 * \Delta$ has
$h(\Delta_1 * \Delta, t) = (1+t) h(\Delta, t)$

so if $h(\Delta) = (h_0, h_1, \ldots, h_{d-1})$ then

$$h(\Delta_1 * \Delta) = (h_0, h_1 + h_0, h_2 + h_1, \ldots, h_{d-1} + h_{d-2}, h_{d-1})$$

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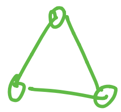
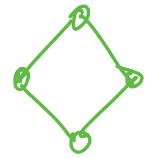
(c) Using suspension, one can create new simplicial polytopes $P$ in each dimension $d$ that have affinely independent $\underline{h}$-vectors $h(\partial P)$, along with the simplices in each dimension having $\underline{h}$-vector of their boundary being $(1, 1, \ldots, 1)$.

$d=1$

$\underline{h} = (1,1)$ bipyramid

$d=2$



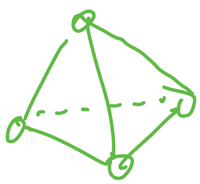


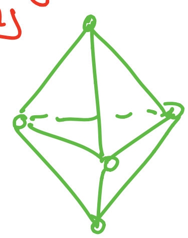
$\underline{h} = (1,1,1)$ $\underline{h} = (1,2,1)$

bipyramid

1st half entries $(h_0, h_1, \dots, h_{\lfloor \frac{d+1}{2} \rfloor})$ underlined

$d=3$





$\underline{h} = (1,1,1,1)$ $\underline{h} = (1,2,2,1)$

bipyramid bipyramid

$d=4$ boundary of 4-simplex

$\underline{h} = (1,1,1,1,1)$ $\underline{h} = (1,2,2,2,1)$ $\underline{h} = (1,3,4,3,1)$

bipyramid bipyramid

$d=5$ boundary of 5-simplex

$\underline{h} = (1,1,1,1,1,1)$ $\underline{h} = (1,2,2,2,2,1)$ $\underline{h} = (1,3,4,4,3,1)$

bipyramid bipyramid bipyramid

$d=6$ boundary of 6-simplex

$\underline{h} = (1,1,1,1,1,1,1)$ $\underline{h} = (1,2,2,2,2,2,1)$ $\underline{h} = (1,3,4,4,4,3,1)$ $\underline{h} = (1,4,7,8,7,4,1)$

$\vdots$ etc

Note this produces recursively $\lfloor \frac{d+1}{2} \rfloor$ different $\underline{h}$-vectors. Then since the Dehn-Sommerville equations show the $\underline{h}$-vector entries are affinely dependent on the 1st $\lfloor \frac{d+1}{2} \rfloor$ entries $(h_0, h_1, \ldots, h_{\lfloor \frac{d+1}{2} \rfloor})$, it suffices to show the $\lfloor \frac{d+1}{2} \rfloor$ different $\underline{h}$-vectors produced are affinely independent.

Let's write these $\lfloor \frac{d+1}{2} \rfloor$ different vectors $(h_0, h_1, \ldots, h_{\lfloor \frac{d+1}{2} \rfloor})$ as the columns of a matrix for various values of $d$:

$\underline{d=1}$: $\begin{bmatrix} 1 \end{bmatrix}$

$\underline{d=2,3}$: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\underline{d=4,5}$: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

$\underline{d=6,7}$: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 7 \\ 1 & 2 & 4 & 8 \end{bmatrix}$

Since each successive column comes from the previous using the bipyramid operation's effect on $\underline{h}$-vector, one sees that subtracting column $i$ from column $i+1$ yields a recursive structure:

$d=6,7$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 7 \\ 1 & 2 & 4 & 8 \end{bmatrix} \xrightarrow[\substack{\text{col 3 from col 4} \\ \text{col 2 from col 3} \\ \text{col 1 from col 2}}]{\text{subtract}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 \end{bmatrix}$$

the matrix for $d=4,5$

Hence by induction, the columns are linearly independent, and these $\underline{h}$-vectors are affinely independent (since $h_0 = 1$ for all).

**#7** (a) If $n_1, n_2, \dots, n_{d+1} \in \mathbb{R}^d$ are minimally dependent and if $\sum_{i=1}^{d+1} a_i n_i = 0 = \sum_{i=1}^{d+1} b_i n_i$ are two nontrivial dependences among them, we claim they differ by a scalar: both $a_{d+1} \neq 0$ and $b_{d+1} \neq 0$ since otherwise $n_1, n_2, \dots, n_d$ are dependent (contradicting minimality), and so

$$0 = \sum_{i=1}^{d+1} a_i n_i - \frac{a_{d+1}}{b_{d+1}} \sum_{i=1}^{d+1} b_i n_i = \sum_{i=1}^{d} \left(a_i - \frac{a_{d+1}}{b_{d+1}} b_i\right) n_i$$

is a dependence among $n_1, n_2, \dots, n_d$, which must have all zero coefficients, i.e. $a_i = \frac{a_{d+1}}{b_{d+1}} b_i$ for $i = 1, 2, \dots, d+1$.

Segregating the $a_1, a_2, \dots, a_{d+1}$ into positive, negative and calling the negative $a_i$ by $-b_i$ gives $\sum_{i \in G} a_i n_i \overset{(*)}{=} \sum_{j \in G^c} b_j n_j$ with $a_i, b_j > 0$ and $G \sqcup G^c = \{1, 2, \dots, n\}$

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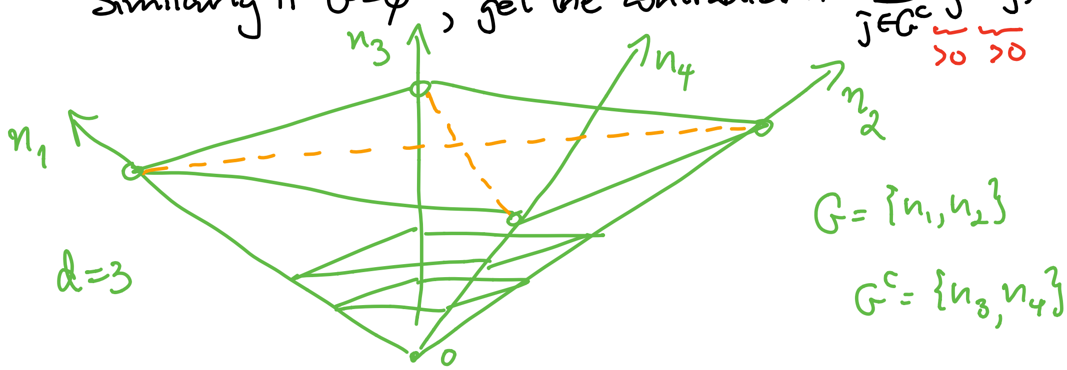
(b) If $f \in (\mathbb{R}^d)^*$ has $f(n_i) > 0$ $\forall i$, then one cannot have $G^c = \emptyset$, else applying $f$ to $(*)$ gives $\underbrace{\sum_{i \in G} \underbrace{a_i}_{>0} \underbrace{f(n_i)}_{>0} = f(0) = 0}_{\text{contradiction.}}$

Similarly if $G = \emptyset$, get the contradiction $\sum_{j \in G^c} \underbrace{b_j}_{>0} \underbrace{f(n_j)}_{>0} = f(0) = 0$.



(c) Given a triangulation of $\mathbb{R}_{\geq 0} n_1 + \dots + \mathbb{R}_{\geq} n_{d+1}$ into simplicial cones, note that the maximal ($d$-dim'l) cones are all of the form $\sigma_i := \mathbb{R}_{\geq 0} n_1 + \dots + \widehat{\mathbb{R}_{\geq 0} n_i} + \dots + \mathbb{R}_{\geq 0} n_{d+1}$. If the triangulation contains a cone $\sigma_i$ with $i \in G$, then it contains the subcone $\sigma_{G^c} := \sum_{j \in G^c} \mathbb{R}_{\geq 0} n_j$.

This means that the triangulation cannot contain the subcone $\sigma_G := \sum_{i \in G} \mathbb{R}_{\geq 0} n_i$, since the two cones $\sigma_G, \sigma_{G^c}$ intersect badly, in a nonface of either one, by equation (*):

$$\sum_{i \in G} a_i n_i = \sum_{j \in G^c} b_j n_j$$

Hence the triangulation cannot contain any $d$-cones $\sigma_j$ for $j \in G^c$, and thus can only contain $d$-cones $\sigma_i$ for $i \in G$. But then it needs all such $\sigma_i$ for $i \in G$ because one can check that their interiors are disjoint in $\mathbb{R}^d$.

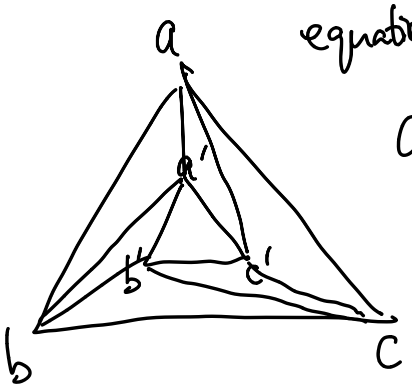
(#8) (I'm stealing a calculational approach from Aaron Li here)

Let's assume using the action of $Gl_3(\mathbb{R})$ that $\begin{array}{l} a = e_1 \\ b = e_2 \\ c = e_3 \end{array}$

and pick some particular directions for the rays $\mathbb{R}a'$, $\mathbb{R}b'$, $\mathbb{R}c'$ so that

$$a' = \alpha \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}, \ b' = \beta \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}, \ c' = \gamma \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} \quad \text{for some } \alpha, \beta, \gamma > 0$$

One can check that the plane containing $a, a', b$ has equation $\alpha x + \alpha y + (2 - 3\alpha) z = \alpha$.




Convexity of the fan and the quadrangle $a a' b' b$ triangulated as shown would require $b' = \begin{bmatrix} \beta/2 \\ \beta \\ \beta/2 \end{bmatrix}$ to lie below this plane, so

$$\frac{\alpha\beta}{2} + \alpha\beta + (2-3\alpha)\frac{\beta}{2} < \alpha$$

which one can check implies $\beta < \alpha$.

Similarly the quadrangle $a a' c' c$ triangulated as shown requires $\gamma > \alpha$,

while the quadrangle $b b' c' c$ triangulated as shown requires $\beta > \gamma$.

This gives the contradiction $\beta > \gamma > \alpha > \gamma$.

#9 Assume $\Delta$ is a pure 1-dim'l simplicial complex, that is, a simple graph with at least one edge, and no isolated vertices.

(a) To show $\Delta$ is shellable $\Longleftrightarrow$ connected, the forward implication is easy by induction on the number of facets $F_1, F_2, \ldots, F_s$ in the shelling: the first edge $F_1$ is connected, and as new edges $F_j$ for $j \geq 2$ are added in, they always share at least one endpoint with some previous edge $F_i$ for $1 \leq i < j$, and hence adding $F_j$ it stays connected

For the reverse implication, show a connected graph is shellable by induction on the number of edges:

- if there are any edges $e = \{i,j\}$ contained in a cycle, remove $e$, leaving its endvertices, and make $e = F_s$ be the last shelling step after one produced by induction, since it is still connected.

- if no edges are contained in a cycle, it is a connected acyclic graph, so a tree, and has a leaf vertex $i$ in some unique edge $\{i,j\} = e$; remove vertex $i$ and the edge $e$, leaving vertex $j$, and make $e = F_s$ be the last shelling step after one produced by induction, since it is still connected

(b) To show $\Delta$ is partitionable $\Longleftrightarrow$ $\Delta$ has at most one connected component that is a nontrivial tree

first show the forward implication via contradiction.

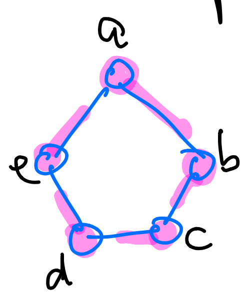
If $\Delta$ has two connected components $\Delta_1, \Delta_2$ that are trees, then WLOG $\Delta_1$ does not have the unique interval of the partitioning $\Delta = \dot{\bigcup}_{i=1}^{s} [G_i, F_i]$ of the form $[G_i, F_i] = [\emptyset, \{i, j\}]$. This means $\Delta_1$ is partitioned into intervals of form $[\{i\}, \{i,j\}]$ and $[\{i,j\}, \{i,j\}]$. This forces $\Delta_1$ to have at least as many edges as vertices, contradicting that it is a tree, so has #edges = #vertices - 1.

If $\Delta$ has connected components $\Delta_1, \Delta_2, \ldots, \Delta_s$ and $\Delta_2, \ldots, \Delta_s$ are all non-trees, then we can partition it as follows. First use (a) to shell $\Delta_1$, since it is connected. Then it suffices to show for each non-tree component $\Delta_j$ with $j = 2, \ldots, s$ how to partition it into intervals $[\{i\}, \{i,j\}]$ and $[\{i,j\}, \{i,j\}]$. Assume $j=2$, and let $C$ be a cycle of edges inside $\Delta_2$.

We show how to partition $\Delta_2$ via induction on the number of edges of $\Delta_2 \setminus C$

BASE CASE: $\Delta_2 = C$

Then partition it like this:




$= [a, ab] \sqcup [b, bc] \sqcup [c, cd] \sqcup [d, de] \sqcup [e, ae]$

INDUCTIVE STEP: If there is an edge $e = \{i,j\} \in \Delta_2 \setminus C$ that lies in some cycle, then remove $e$ and add $[\{i,j\}, \{i,j\}]$ to the partitioning of what is left, which is still a non-tree component.

If there is no edge $e = \{i,j\} \in \Delta_2 \setminus C$ that lies in a cycle, then $\exists$ a "leaf" vertex $i$ of degree 1, in a unique edge $\{i,j\}$. Remove this vertex $i$ and edge $\{i,j\}$, leaving vertex $j$, and add $[\{i\}, \{i,j\}]$ to the partitioning of what is left, which is still a non-tree component.