Math 5707 Graph theory
Spring 2013, Vic Reiner
Acyclic and totally cyclic orientation exercises

Our goal here is to develop deletion-contraction recurrences that let one compute two interesting quantities for an undirected graph: its number of *acyclic* orientations, and of its *totally cyclic orientations*.

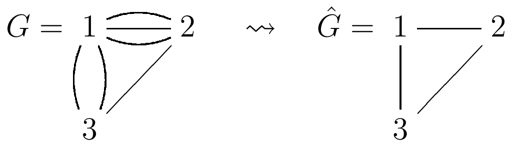
1. For an undirected multigraph $G = (V, E)$, an *orientation* ω of G is a choice of one of the two possible directions for each edge[1] of E, making them all directed arcs.

(a) Explain why the number of orientations of G is $2^{|E|}$.

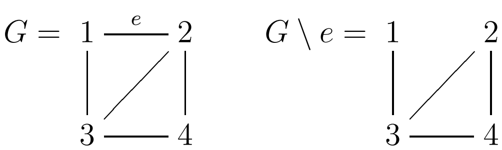
Say that the orientation ω of G is an *acyclic orientation* if it contains no directed cycles; in particular, this requires that G have no self-loops. Let $\text{ac}(G)$ denote the number of acyclic orientations of G.

(b) Show the complete graph K_3 has $\text{ac}(K_3) = 6$ by drawing all 6 of its acyclic orientations.

(c) Explain why $\text{ac}(G) = \text{ac}(\hat{G})$ if $\hat{G}$ is obtained from G by replacing multiple (parallel) copies of edges $\{x, y\}$ with a single copy of $\{x, y\}$:

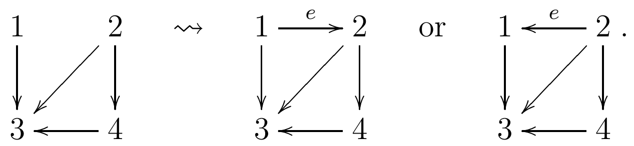


We now work on developing the deletion-contraction recurrence for $\text{ac}(G)$. Given an undirected multigraph $G = (V, E)$ and a non-loop edge e, fix some acyclic orientation of the *deletion* $G \setminus e$, and then consider the two possible orientations of e, some of which may make G acyclic. For example, if G and e and $G \setminus e$ are as shown here

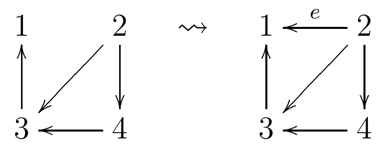


[1]By convention, we even consider self-loops to have two possible orientations!

then this acyclic orientation of $G \setminus e$ shown on the left can be extended in the *two* ways shown to an acyclic orientation of G:



However, the following acyclic orientation of $G \setminus e$, on the left below, can be extended *only one* way (shown) to an acyclic orientation of G:



Let a_0, a_1, a_2, respectively, denote the number of acyclic orientations of $G \setminus e$ for which $0, 1$, or 2, respectively, out of these possible orientations of e extend it acyclically to all of G. Thus the first example above contributed toward a_2, and the second example contributed toward a_1.

(d) Prove $a_0 = 0$ and $a_1 + a_2 = \mathrm{ac}(G \setminus e)$.

(e) Prove $a_1 + 2a_2 = \mathrm{ac}(G)$.

(f) Prove $a_2 = \mathrm{ac}(G/e)$, where G/e is the *contraction* of e in G, and therefore why

$$\mathrm{ac}(G) = \mathrm{ac}(G \setminus e) + \mathrm{ac}(G/e) \tag{1}$$

for any non-loop edge e of G.

(g) Use these initial conditions

$$\mathrm{ac}(G) = 0 \text{ if there are any self-loops in } G,$$
$$\mathrm{ac}(G) = 1 \text{ if there are no edges at all in } G.$$

together with equation (1) to illustrate how you can compute $\mathrm{ac}(K_3)$ via recursion on the number of edges.

(h) Use this method to prove more generally that $\mathrm{ac}(K_n) = n!$.
Optional: can you also give a second proof that $\mathrm{ac}(K_n) = n!$?

2. Say that an orientation ω of G is *totally cyclic* if every directed arc lies in at least one directed cycle. One can show that this is equivalent to the condition that the orientation on each connected component of G is *strongly connected*: for every pair x, y in V in the same connected component of G, there existed directed paths both x to y and y to x.

Let $\text{tc}(G)$ denote the number of totally cyclic orientations of G. For example, the cycle C_n for $n \geq 1$ has $\text{tc}(C_n) = 2$; even the loop C_1 has $\text{tc}(C_1) = 2$!

Given an undirected multigraph $G = (V, E)$ and a non-bridge edge e, fix some totally cyclic orientation ω of the contraction G/e, and then consider the two possible orientations of e one could use to extend ω to an orientation of G, some of which may make G totally cyclic. We adopt here the convention for contracting on a loop edge e which says that G/e is the same as the deletion $G \setminus e$ if e is a loop.

Let t_0, t_1, t_2, respectively, denote the number of totally cyclic orientations ω of G/e for which $0, 1$, or 2, respectively, out of these possible orientations of e extend it totally cyclically to all of G.

(a) Prove $t_0 = 0$ and $t_1 + t_2 = \text{tc}(G/e)$.

(b) Prove $t_1 + 2t_2 = \text{tc}(G)$.

(c) Prove $t_2 = \text{tc}(G \setminus e)$, where $G \setminus e$ is the deletion of e in G, and therefore why

$$\text{tc}(G) = \text{tc}(G \setminus e) + \text{tc}(G/e) \tag{2}$$

for any non-bridge edge e of G.

(d) Explain why

$$\begin{aligned} \text{tc}(G) &= 0 \text{ if there are any bridges in } G, \\ \text{tc}(G) &= 1 \text{ if there are no edges at all in } G. \end{aligned}$$

and show how one can use these together with equation (2) to compute $\text{tc}(C_n)$ via recursion on the number of edges.