Variants of Bulgarian Solitaire

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Abstract

In this paper, we present recent findings about the games of Bulgarian Solitaire and Carolina Solitaire, specifically about their level sizes and the possible existence of a geometric constant c_p relating orbit sizes in Bulgarian Solitaire. We then introduce two non-deterministic extensions of Bulgarian Solitaire, Block Bulgarian Solitaire and Minnesota Solitaire, along with a method of making them deterministic. We prove that the sink of Block Bulgarian Solitaire contains exactly the Bulgarian Solitaire recurrent cycles. In Maximal Block Bulgarian Solitaire, the analogous structure to staircase partitions in Bulgarian Solitaire is a staircase of squares, while in Maximal Minnesota Solitaire, the number of analogous partitions is equal to the number of triangular numbers that divide the starting number.

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Acknowledgements

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1 Introduction

Bulgarian Solitaire is an interesting mathematical game invented in the second half of the 20th Century and popularized by Martin Gardner in 1983. The game follows a set of simple rules, dividing a deck of n cards into some number of piles, and then taking the top card from each pile in order to form a new pile. The game itself has inspired several variants that each add a twist to the original gameplay. Before diving into the games, we first provide a refresher on a few underlying concepts that we will use throughout this paper.

1.1 Integer Partitions and Young Diagrams

1.1.1 Integer Partitions

The game of Bulgarian Solitaire (and many of its variants) is closely tied to integer partitions.

Definition 1.1. An *integer partition* of n, denoted $\lambda \vdash n$, is an unordered list of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that

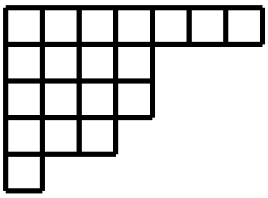
$$\sum_{i=1}^{l} \lambda_i = n$$

where each λ_i is a positive integer less than or equal to n.

For partitions, the order does not matter, but by convention they are usually written in weakly descending order. That is, $(2,3,6)$, $(3,2,6)$ and $(6,3,2)$ are all the same partition, but to avoid confusion, we write it as $(6,3,2)$. [Pha22] includes other interesting results that have been found regarding integer partitions. Finding a closed form for the number of partitions of n is an open question that still is of interest to mathematicians today.

1.1.2 Young Diagrams

A good visual representation of integer partitions is the Young Diagram. The Young Diagram assigns rows to be the elements of the partition in weakly descending order. The Young Diagram for the partition (7,4,4,3,1) is below.



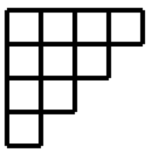
1.1.3 Special Partitions

There are a few special cases of partitions that come up when working with Bulgarian Solitiare and its variants.

Definition 1.2. When $n = \binom{b+1}{2}$ we say a partition is a *staircase partition*, $\Delta_b \vdash n$, if

$$\Delta_b = (b, b-1, \ldots, 2, 1)$$

If n is not a triangular number, no such staircase partition exists. The partition $\Delta_4 = (4,3,2,1)$ of $n = 10$ is a staircase partition.



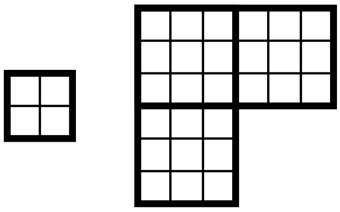
The second special partition we will work with is called a *square staircase partition*

Definition 1.3. When $n = a^2\binom{b+1}{2}$, a *square staircase partition* is a partition, $\Lambda_{(a,b)} \vdash n$, of the form

$$\Lambda_{(a,b)} = (ab, ab, \ldots, ab, a(b-1), a(b-1), \ldots, a, a, \ldots, a)$$

where each multiple of a has exactly a copies.

We can think of Δ_b as $\Lambda_{(1,b)}$. Some examples of square staircases are $\Lambda_{(2,1)} = (2,2)$ and $\Lambda_{(3,2)} = (6,6,6,3,3,3)$.



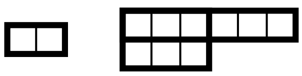
$\Lambda_{(2,1)}$ $\Lambda_{(3,2)}$

The final special partition we look at in this paper is the *stretched staircase partition.*

Definition 1.4. For $n = a\binom{b+1}{2}$, a partition is a *stretched staircase partition*, $\Gamma_{(a,b)}$, if

$$\Gamma_{(a,b)} = (ab, a(b-1), a(b-2), \ldots, a)$$

Again, we can view Δ_b as $\Gamma_{(1,b)}$. Some examples of stretched staircases are $\Gamma_{(2,1)} = (2)$ and $\Gamma_{(3,2)} = (6,3)$.



$\Gamma_{(2,1)}$ $\Gamma_{(3,2)}$

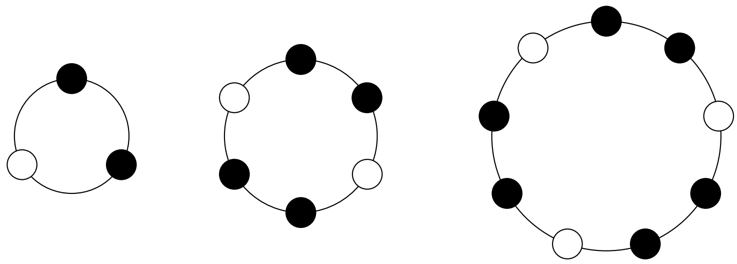
1.2 Black and White Necklaces

[Bra82] proved the existence of a bijection between Bulgarian Solitaire recurrent cycles and Black and White necklaces under rotation. Because of this, we will repeatedly refer to Bulgarian Solitaire recurrent cycles by their corresponding necklace. Pham ([Pha22, Definition 2.2.1] provides the following definition of a necklace as it is used in Bulgarian Solitaire:

Definition 1.5. Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a finite sequence of letters $\{B, W\}$. Define the cyclic rotation ω by

$$\omega(\alpha_j) = \alpha_{(j+1)} \mod n$$

A necklace N of black and white beads is an equivalence class of sequences of letters $\{B, W\}$ under cyclic rotation ω. We call N a primitive necklace if it cannot be written as a concatenation $N = P^k = PP \ldots P$ of copies of another necklace P. We will reserve P for primitive necklaces.



BBW $(BBW)^2$ $(BBW)^3$

1.3 Integer Compositions

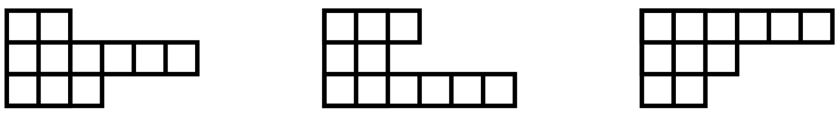
While nearly every variant of Bulgarian Solitaire we look at in this paper can be represented as integer partitions, when we discuss Carolina Solitaire, the order of the cards becomes relevant. As a result, we need a slightly different mathematical concept to work with Carolina Solitaire. That concept is *integer compositions.*

Definition 1.6. An *integer composition* of n, $C \models n$, is an ordered tuple of integers $C = (C_1, C_2, \ldots, C_l)$ such that

$$\sum_{i=1}^{l} C_i = n$$

where each C_i is a positive integer less than or equal to n.

An integer composition is essentially the same as an integer partition except that order matters. So $(2, 6, 3)$, $(3, 2, 6)$ and $(6, 3, 2)$ are the same *partition*, but are all distinct *compositions.*



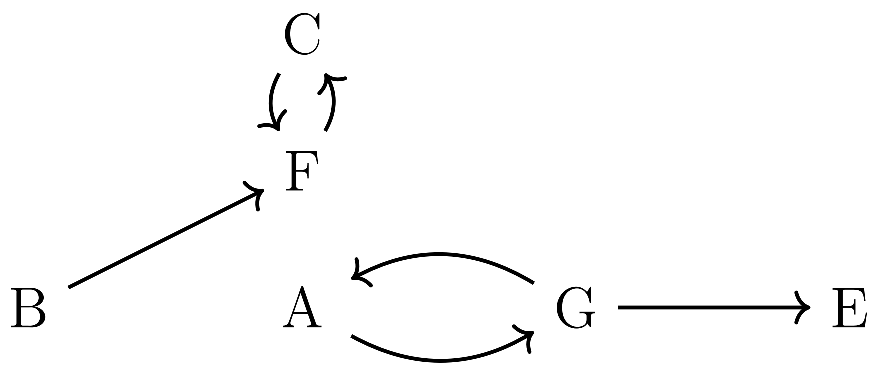
(2,6,3) (3,2,6) (6,3,2)

Remark 1.7. The composition analogue to a staircase partition is also a staircase, and a staircase composition must be in descending order.

The reason for Remark 1.7 will be clear when we discuss Carolina Solitaire.

1.4 Directed Graphs

Mapping possible outcomes of Bulgarian Solitaire and its variants involves using directed graphs, commonly referred to as digraphs. In graph theory, a digraph is a set of vertices connected by directed edges. An example of a digraph is below.

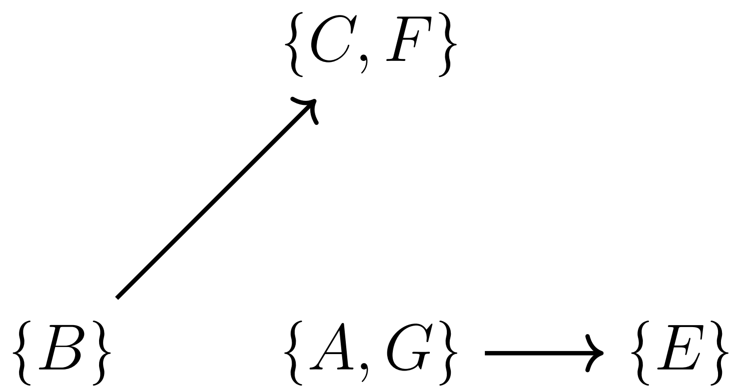


The first key concept for digraphs that we examine is the concept of *strongly connected components.*

Definition 1.8. In a digraph D, a *strongly connected component* is a maximal subgraph of $d \subseteq D$ such that there exists a path along the directed edges from any vertex v to any other vertex u in the subgraph.

In the graph above, we have 4 strongly connected components, $\{B\}$, $\{A, G\}$, $\{C, F\}$, and $\{E\}$.

It is possible to form a new digraph S from the strongly connected components of D, called the *strongly connected components digraph.* In the case above, the strongly connected components digraph is



This gives rise to the concept of a *sink.*

Definition 1.9. In a digraph, a *sink* is a vertex that has no edges leading out of it.

In the example above, we can see that E is a sink, while the other half of the graph has no sink. However, the strongly connected components digraph has both $\{E\}$ and $\{C, F\}$ as sinks. In general, for this paper we will use the term sink of a given game to refer to the sink of the strongly connected components digraph for that game's game graph.

2 Existing Variants

Now we are ready to get into our discussion of the main purpose of this thesis. The original game of Bulgarian Solitaire has been studied since the 1980s, and several key results regarding orbits and levels sizes have been proven.

2.1 Bulgarian Solitaire

Bulgarian Solitaire is played by starting with a deck of n cards (Gardner presented the game as starting with $n = 45$ cards), divided into a number of piles. Each turn, the player performs the Bulgarian Solitaire move, $\mathcal{B}$ by taking the top card from each pile and combining them to form a new pile. The original game would end when performing the Bulgarian Solitaire move does not change the sizes of the piles. For any triangular number T_k, Gardner conjectured in [Gar83] you will reach a stable configuration of $1, 2, 3, \ldots, k$ in at most $k(k-1)$ moves. While Gardner focused on the case where n is a triangular number, others began researching the behavior of Bulgarian Solitaire for non-triangular n.

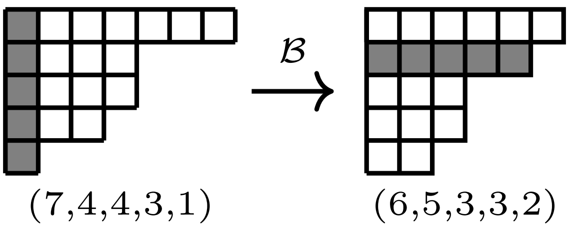
As mentioned above, one can use integer partitions to model Bulgarian Solitaire. In this case, we define the Bulgarian Solitaire move $\mathcal{B}$ to be

Definition 2.1. For a partition $\lambda \vdash n$ of length l, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we have the *Bulgarian Solitaire move*,

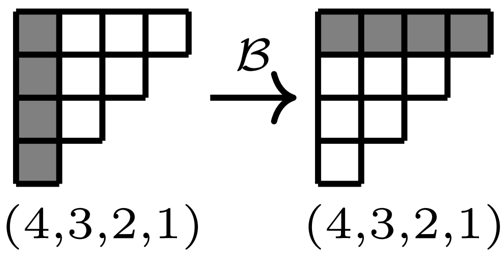
$$\mathcal{B}(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1, l)$$

(reordered as needed to be in nonincreasing order)

For Young Diagrams, the Bulgarian Solitaire move removes the leftmost column, and then adds it back into the Young Diagram as a row where it fits in the weakly decreasing order.



Remark 2.2. For a staircase partition Δ_b, we have $\mathcal{B}(\Delta_b) = \Delta_b$, and it is easy to check that these staircase partitions are the *only* partitions to be fixed under $\mathcal{B}$.

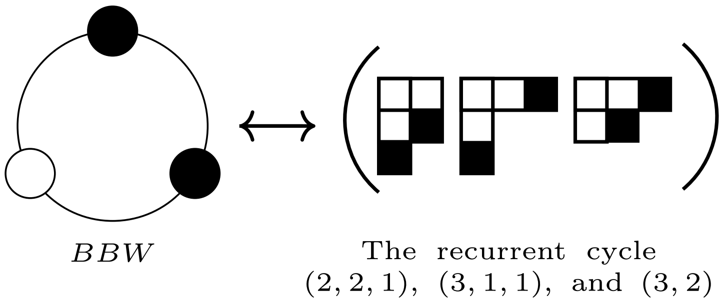


2.1.1 Past Results

One of the first key results involving Bulgarian Solitaire was Brandt's ([Bra82]) discovery and characterization of recurrent cycles for non-triangular n. He showed that, after defining the unique a and b such that $n = \binom{b+1}{2} - a$, for $0 \leqslant a \leqslant b-1$, then the number $C_a(b)$ of recurrent cycles for n equals

$$C_a(b) = \frac{1}{n} \sum_{d|(b,a)} \phi(d) \binom{b/d}{a/d}$$

where $\phi(d)$ is Euler's phi function. Brandt's discovery of recurrent cycles flowed naturally into the existence of a bijection between elements of Bulgarian Solitaire recurrent cycles and black/white beaded necklaces. An visual of the bijection is:



Specifically, for any $n = 1+2+\cdots+(k-1)+r$, where $0 \leqslant r \leqslant k-1$, elements of a Bulgarian Solitaire recurrent cycle can be written in the form $\lambda = (\lambda_1 + \delta_1, \lambda_2 + \delta_2 + \cdots + \lambda_l + \delta_l)$, where each δ_i is either 1 or 0 and

$$\sum_{i=1}^{l} \delta_i = r$$

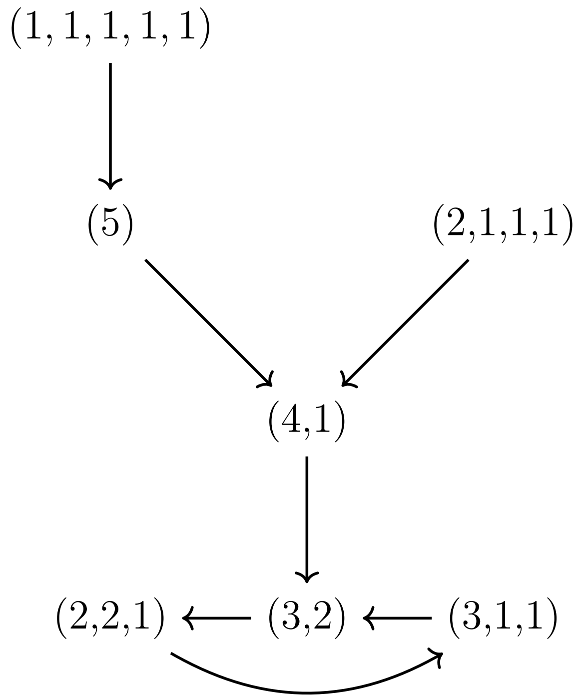
The bijection assigns black beads to each δ_i that is 1, and white beads to the δ_i that are 0.

A useful tool for understanding Bulgarian Solitaire behavior is the *game graph*.

Definition 2.3. A *game graph* for Bulgarian solitaire is a digraph whose nodes are partitions of n, and has edges from $\lambda \to \mathcal{B}(\lambda)$.

An example Bulgarian Solitaire game graph for $n = 5$ is below.

n=5



Each *level* of the game graph is a move, and we say two partitions are in the same level if they are the same number of moves away from the recurrent cycle. In the example above, the zero level is the recurrent cycle $\{(2, 2, 1), (3, 2), (3, 1, 1)\}$, the first level is $\{(4, 1)\}$, the second level is $\{(5), (2, 1, 1, 1)\}$, and the third level is $\{(1, 1, 1, 1, 1)\}$.

In the same paper that he identified recurrent cycles, Brandt also conjectured that the number of levels in the game graph for triangular $n = 1 + 2 + \cdots + k$ was at most $k^2 - k + 1$. Igusa in [Igu85] introduced a method of looking at *gaps*, or the differences between sizes of adjacent piles. Nguyen and I produced a similar concept of working with these gaps in [HN23], which can be applied to the forward game as follows

Definition 2.4. For a given partition of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, define the *gaps of* λ to be $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ where

$$\mu_i = \lambda_i - \lambda_{i+1}$$

and we assume $\lambda_{l+1} = 0$

Igusa used this concept to prove that not only did Brandt's conjecture of reaching a stable configuration in at most $k(k - 1)$ moves hold, but for $n < 1 + 2 + \cdots + k$, the game graph of n has strictly less than $k^2 - k + 1$ levels. Later on, this result was improved on by Griggs and Ho in [GH98], to show that for $n < 1 + 2 + \cdots + k$ and $k \geqslant 4$, a cycle is reached in at most $k^2 - 2k - 1$ moves.

Another area of interest in Bulgarian Solitaire is the question of which partitions you can only reach by starting with them. In a biblical analogy, these partitions were named *Garden of Eden partitions*.

Definition 2.5. In Bulgarian Solitaire, a *Garden of Eden partition* is a partition $\lambda \vdash n$, such that there is no partition λ' satisfying $\lambda = \mathcal{B}(\lambda')$.

The following result from Hopkins and Jones in [HJ06, Corollary 1] determines exactly every Garden of Eden partition for Bulgarian Solitaire.

Proposition 2.6. *A partition $\lambda \vdash n$, such that $\lambda = (\lambda_1, \dots, \lambda_l)$ is a Garden of Eden partition if and only if $\lambda_1 < l - 1$*

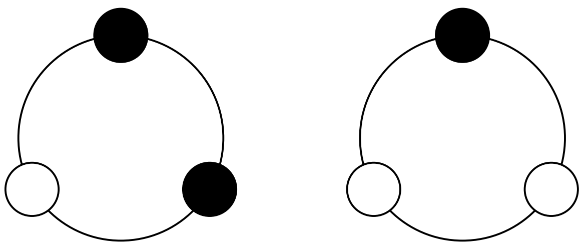
Eriksson and Jonsson proved in [EJ17] that in the triangular case, $n = \binom{k+1}{2}$, as k grows, the level sizes have the generating function

$$H_W(x) = \frac{(1-x)^2}{1-3x+x^2}$$

which converges to the evenly indexed Fibonacci numbers. They also introduced the concept of the *quasi-infinite game tree*, which Pham in her thesis ([Pha22]) last year generalized to the idea of the *quasi-infinite forest*. Nguyen and I ([HN23]) improve on her discussion of the quasi-infinite forest in our paper. Using the quasi-infinite forest, Pham proved that

Theorem 2.7. *For primitive necklaces P with $|P| \geqslant 3$, there is a power series H_P in $\mathbb{Z}[[x]]$ such that the sequence of generating functions $(D_{P^k})_{k=0}^{\infty}$ converges to H_P. Moreover, H_P is a rational function having denominator polynomial of degree at most $|P|$.*

Along the lines of this Theorem, Pham introduced the idea of dual necklaces P and P', where P' is formed by reversing the order of P and swapping black and white beads. For example, if $P = BBW$, then $P' = BWW$.



$P = BBW$ $P' = BWW$

She conjectured that $H_P(x)$ and $H_{P'}(x)$ would have the same denominator. Recently, Nguyen and I used our μ method of gaps with Reverse Bulgarian Solitaire to prove Pham's conjecture in the specific cases of

$$H_{B(WB)^k}(x) = H_{W(BW)^k}(x)$$

and that

$$H_{B(W)^k}(x) \text{ and } H_{W(B)^k}(x)$$

share the same denominator.

Pham also made the following very surprising conjecture ([Pha22, Conjecture 3.1.4]).

Conjecture 2.8. *For any primitive necklace P with $|P| \geqslant 3$, there is an integer c_P such that for $k \geqslant 2$,*

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1}|\mathcal{O}_P|$$

for some constant c_P that depends only on P. Moreover, if P and P' are obtained from each other by reversing their order swapping black and white beads, then $c_P = c_{P'}$.

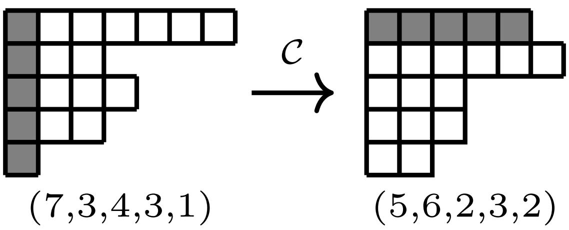
The existence (or nonexistence) of this c_P is what initially started our research into Bulgarian Solitaire, and one area we decided to investigate was if c_P (assuming Pham's conjecture holds) is unique to Bulgarian Solitaire, or if it appears in the variants as well. The first place we checked was the first variant released: Carolina Solitaire.

2.2 Carolina Solitaire

Now we move into the first of the variants that we look at in this paper. Carolina Solitaire was introduced to Griggs and Ho by Andrey Andreev in 1997, which they then publicized in [GH98]. In this game, the play $\mathcal{C}$ is the same as Bulgarian Solitaire, except that the piles are now labeled in order, and after taking the l cards, you place them in front of the old piles. As mentioned above, since the piles are now ordered, instead of working with integer partitions, we use integer compositions. On compositions, this means that for a given composition $c \models n$ of length l,

$$\mathcal{C}(c) = (l, c_1 - 1, c_2 - 1, \ldots, c_l - 1)$$

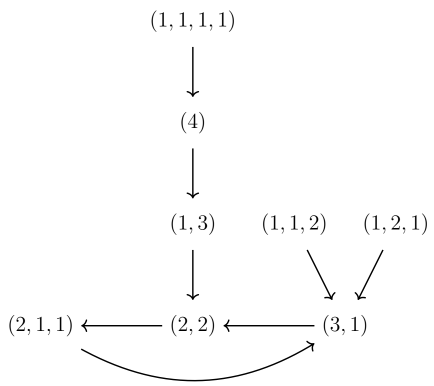
On Young Diagrams, $\mathcal{C}$ is taking the first column and placing it in as the first row.



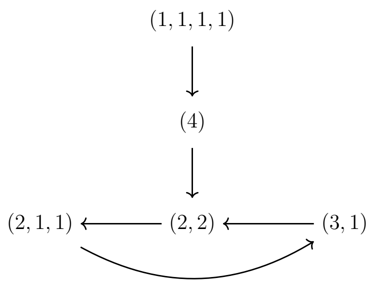
Note that no reordering is necessary in this case, since we are working with compositions now. Griggs and Ho showed that there are many analogies between Bulgarian and Carolina Solitaire. One of the first that they found is that

Theorem 2.9. *A composition, $c \models n$, is in a recurrent cycle under $\mathcal{C}$ if and only if the corresponding partition, $\lambda \vdash n$, is in a recurrent cycle under $\mathcal{B}$.*

This means that even though the Carolina Solitaire game graph may be larger than Bulgarian Solitaire, understanding the behavior of Bulgarian Solitaire recurrent cycles exactly determines the same behavior in Carolina Solitaire cycles.



The Carolina Solitaire Game Graph
for $n = 4$



The Bulgarian Solitaire Game Graph
for $n = 4$

They also proved the following.

Theorem 2.10. *For a triangular number, $n = 1 + 2 + \cdots + k$, playing the Carolina Solitaire move will reach a fixed composition $(k, k-1, \ldots, 1)$ in at most $k^2 - 1$ moves.*

The strong similarities for both recurrent cycles and level sizes between Carolina Solitaire and Bulgarian Solitaire raises the question of if other results that have come up in Bulgarian

Solitaire might apply similarly to Carolina Solitaire. Specifically, we wanted to find out if a analogous c_p might exist for Carolina Solitaire. However, using the data from Table 1, we found that in the case of BBW, we would need $16c_p = 16318$, and since 16 does not divide 16318, we conclude

Proposition 2.11. *Carolina Solitaire orbits do not exhibit the same geometric growth as Bulgarian Solitaire orbits: In Carolina Solitaire, there do not exist constants c_P for each primitive necklace P with $|P| \geqslant 3$, such that*

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1}|\mathcal{O}_P|$$

Given how much more quickly the number of compositions grows than the number of partitions, we were heavily limited by computing power, and were unable to produce sufficient data to investigate Carolina Solitaire orbits' sizes further.

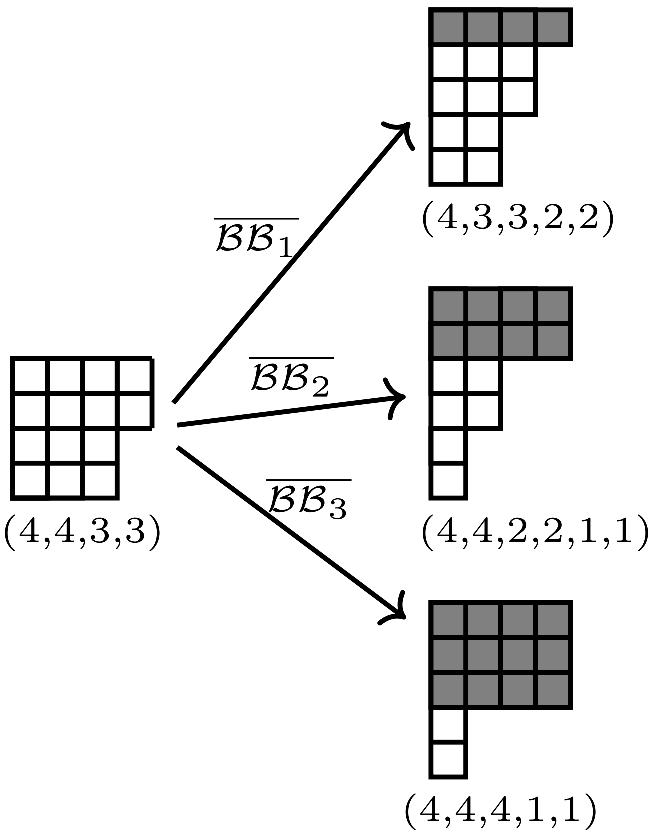
3 New Non-Deterministic Variants

3.1 Block Bulgarian Solitaire

Now we introduce the first of two new non-deterministic variants that can be thought of as extensions of Bulgarian Solitaire. In Block Bulgarian Solitaire, the player instead of taking just a single card from each pile, now has the ability to take either 1 card from each pile or 2 cards from each pile, up to k cards from each pile, where k is the size of the smallest pile. If they took i cards from each pile, then they make i new piles each of size l, where l was the number of piles before the move. In partition notation, the Block Bulgarian Solitaire Move is defined as follows: For a partition $\lambda \vdash n$, with smallest nonzero pile of size k,

$$\overline{\mathcal{BB}_i}(\lambda) = (\lambda_1 - i, \lambda_2 - i, \ldots, \lambda_l - i, l, l, \ldots, l)$$

where $0 < i \leqslant k$ (and we place the i piles of size l where they belong in weakly descending order). An example Block Bulgarian Solitaire move on Young Diagrams is below.



Note that playing $\overline{\mathcal{BB}_1}(\lambda)$ is the Bulgarian Solitaire move $\mathcal{B}(\lambda)$, so every Bulgarian Solitaire game tree is contained within the Block Bulgarian Solitaire game graph. This non-deterministic variant adds a great deal of complexity to the well-understood Bulgarian Solitaire, but also provides an interesting similarity.

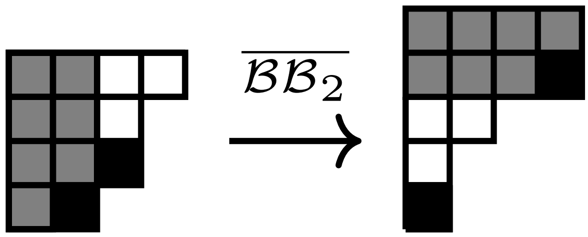
Theorem 3.1. *Block Bulgarian Solitaire has a single sink containing exactly the elements of Bulgarian Solitaire recurrent cycles.*

Recall that when we use the term "Block Bulgarian Solitaire sink", we mean the sink of the strongly connected components digraph for a Block Bulgarian Solitaire game graph.

To prove Theorem 3.1, we first prove the following lemma.

Lemma 3.2. *When possible, in a Bulgarian Solitaire recurrent cycle, playing the Block Bulgarian Solitaire 2-move ($\overline{\mathcal{BB}_2}$) will switch you into a new Bulgarian Solitaire recurrent cycle that has the same number of black and white beads, just in a permutation of their order.*

Before proving this lemma, it might help to show a simple motivating example.



(4,3,3,2) (4,4,2,1,1)

In the example, we can see that our initial partition is in the recurrent cycle corresponding to $BBWWW$, while after playing $\overline{\mathcal{BB}_2}$, the new partition is in the recurrent cycle corresponding to $BWBWW$.

Proof. The number of black and white beads cannot change since we are still partitioning the same n. Thus we only need to show that for some λ in a Bulgarian Solitaire recurrent cycle, such that the smallest part has size 2, then $\overline{\mathcal{BB}_2}(\lambda)$ is in a different Bulgarian Solitaire recurrent cycle.

First, note that in order to have $\overline{\mathcal{BB}_2}(\lambda)$ be a legal move, λ must be $(\lambda_1, \lambda_2, \dots, 2)$, so the corresponding necklace must end in $\dots BW$. Playing $\overline{\mathcal{BB}_2}(\lambda)$ will then create 2 new piles of length l, where l is the length of λ. Furthermore, since λ is in a Bulgarian Solitaire recurrent cycle, the largest pile in λ, λ_1 must have size either l or $l+1$ (depending on if λ starts with W or B). In either case, $\lambda_1 - 2 < l$, so when we play $\overline{\mathcal{BB}_2}(\lambda)$, the 2 piles of length l will be the largest piles and will be added to the front of $\lambda' = \overline{\mathcal{BB}_2}(\lambda)$. Then λ' will start with $(l, l, \lambda_1 - 2, \dots)$, which corresponds to $WB\dots$ as a necklace. Thus λ' is still a necklace, so it is a Bulgarian Solitaire recurrent cycle. But because we permuted the B and W at the end when we played $\overline{\mathcal{BB}_2}(\lambda)$, the necklace for λ' is not a rotation of the necklace for λ. Thus we must be in a different Bulgarian Solitaire recurrent cycle. □

Now we are ready to prove Theorem 3.1 directly.

Proof. First, note that from any given starting partition, playing $\overline{\mathcal{BB}_1}$ is always an option, and playing it enough times will eventually lead us into a Bulgarian Solitaire recurrent cycle.

Once we are in a Bulgarian Solitaire recurrent cycle, by [Bra82] the smallest nonzero pile can either have size 1 or size 2. Then the only possible moves from a Bulgarian Solitaire recurrent cycle are $\overline{\mathcal{BB}_1}$ or $\overline{\mathcal{BB}_2}$.

Playing $\overline{\mathcal{BB}_1}$ is simply the Bulgarian Solitaire move, which keeps us in the same recurrent cycle.

By Lemma 3.2, playing $\overline{\mathcal{BB}_2}$ will switch us into a new Bulgarian Solitaire recurrent cycle.

Since $\overline{\mathcal{BB}_1}$ and $\overline{\mathcal{BB}_2}$ both keep us in a Bulgarian Solitaire recurrent cycle, it follows that the sink of Block Bulgarian Solitaire can only contain Bulgarian Solitaire recurrent cycles.

Because every Bulgarian Solitaire recurrent cycle for a given n is just a different ordering of the black and white beads in the necklace, playing $\overline{\mathcal{BB}_2}$ enough times will eventually yield every possible permutation of black and white beads, and thus every possible recurrent cycle.

Thus every Bulgarian Solitaire recurrent cycle is in the same connected component of the Block Bulgarian Solitaire game graph.

Therefore the sink of Block Bulgarian Solitaire contains exactly the Bulgarian Solitaire recurrent cycles. □

Remark 3.3. In the situation where $\overline{\mathcal{BB}_2}(\lambda)$ is not possible for any λ in the Bulgarian Solitaire recurrent cycle, then there is no consecutive BW in the recurrent cycle, which only happens if we are in the case where $\lambda = B^k$ or $\lambda = W^k$, which are both ways to represent a staircase partition. In this case, the sink is clearly the Bulgarian Solitaire recurrent cycle since λ is fixed.

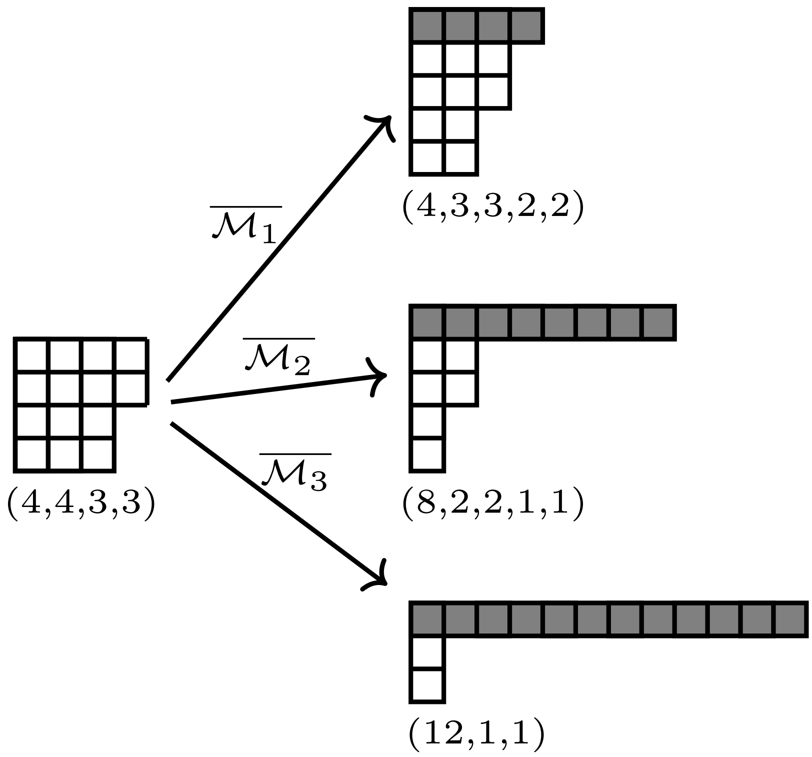
This theorem shows that while Block Bulgarian Solitaire is a more complex variant of Bulgarian Solitaire, understanding the behavior of Bulgarian Solitaire will still be valuable for Block Bulgarian Solitaire.

3.2 Minnesota Solitaire

The second non-deterministic extension of Bulgarian Solitaire is Minnesota Solitaire. In Minnesota Solitaire, the player again can choose up to k cards from each pile. The player then forms a single new pile of length il. In partition notation, the Minnesota Solitaire move is defined as follows: For a partition $\lambda \vdash n$, with smallest pile of size k,

$$\overline{\mathcal{M}_i}(\lambda) = (\lambda_1 - i, \lambda_2 - i, \ldots, \lambda_l - i, il)$$

where $0 < i \leqslant k$ (and we place the pile of size il where it belongs in weakly descending order). Some example Minnesota Solitaire moves on Young Diagrams are below.



Like Block Bulgarian Solitaire, playing $\overline{\mathcal{M}_1}(\lambda)$ is simply playing $\mathcal{B}(\lambda)$, so Minnesota Solitaire can also be thought of as an extension of Bulgarian Solitaire. With this variant, we were unable to prove anything definitive, but the data in Table 4 supports the following conjecture.

Conjecture 3.4. *Minnesota Solitaire has a single sink containing every element of the Bulgarian Solitaire recurrent cycles for n.*

Table 4 shows that the sink(s) contains more than just the elements of the Bulgarian Solitaire recurrent cycle.

Table 5 suggests another conjecture, this time regarding the number of elements in the sink for certain n.

Conjecture 3.5. *For a nearly triangular $n = T_k - 1$, the Minnesota Solitaire sink contains exactly $2(k-1)$ elements.*

4 New Deterministic Variants

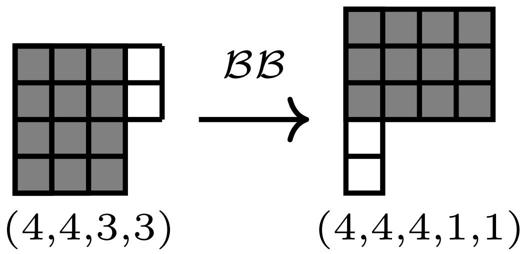
The non-deterministic variants, while interesting on their own, provide a level of complexity that makes them more difficult to work with than the deterministic games. With this in mind, we propose a few modifications that convert Block Bulgarian Solitaire and Minnesota Solitaire into deterministic variants that are more manageable. While Bulgarian Solitaire can be thought of as Minimal Block Bulgarian and Minimal Minnesota Solitaire, we choose instead to take the maximal number of cards.

4.1 Maximal Block Bulgarian Solitaire

First, we create a deterministic variant of Block Bulgarian Solitaire by requiring the player to take the maximum number of cards possible (k). In this case, we have the move $\mathcal{BB}$, where for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l-1}, k)$, we have

$$\mathcal{BB}(\lambda) = (l, l, \ldots, l, \lambda_1 - k, \lambda_2 - k, \ldots, \lambda_{l-1} - k)$$

where there are k copies of l in the new partition (reordered in nonincreasing order as necessary).



Since we now have a deterministic game, a reasonable area to look at is how Block Bulgarian Solitaire compares with regular Bulgarian Solitaire with respect to recurrent cycles of size 1, that is, fixed partitions. We see a few in the diagrams, mostly squares, but we can generalize this using the idea of *square staircase partitions.* A square staircase is just a visual representation of the product of a square times a triangular number, and since it isn't hard to see that anything other than a square staircase is not fixed under $\mathcal{BB}$, we have the following analogue to Gardner's initial Bulgarian Solitaire conjecture.

Theorem 4.1. *A partition $\lambda \vdash n$ is fixed under $\mathcal{BB}$ if and only if it is a square staircase.*

We prove Theorem 4.1 directly.

Proof. Suppose we have a square staircase partition $\Lambda_{(a,b)} \vdash n$, such that $\Lambda_{(a,b)} = (ab, ab, \ldots, ab, a(b-1), a(b-1), \ldots, a, a, \ldots, a)$ (as defined in Definition 1.3) Then $\Lambda_{(a,b)}$ has length ab and smallest pile size a. Then we have

$$\mathcal{BB}(\Lambda_{(a,b)}) = (ab - a, ab - a, \ldots, ab - a, a(b-1) - a, \ldots, 0, 0, \ldots, 0, ab, ab, \ldots, ab)$$

Reordering and simplifying, we get

$$\mathcal{BB}(\Lambda_{(a,b)}) = (ab, ab, \ldots, ab, a(b-1), a(b-1), \ldots, a, a, \ldots, a)$$

Thus square staircase partitions are fixed under $\mathcal{BB}$.

Now consider some partition $\lambda \vdash n$, such that $\lambda = (\lambda_1, \dots, \lambda_l)$, with $\lambda_l = k$ and r total piles of size k. Then

$$\mathcal{BB}(\lambda) = (\lambda_1 - k, \lambda_2 - k, \dots, 0, kr, \dots, kr)$$

Since $k \neq 0$, λ can be fixed under $\mathcal{BB}$ only if $\lambda_1 = \lambda_2 = \cdots = \lambda_r = kr$, $\lambda_{r+1} = \cdots = \lambda_{2r} = kr - k = \lambda_1 - k$, and so on. But this is exactly the definition of a square staircase. Thus square staircases are the fixed partitions under $\mathcal{BB}$. □

Here is an alternative way of interpreting Theorem 4.1, using Definition 1.3 from Section 1.1.3.

Corollary 4.2. *The number of fixed partitions of n under $\mathcal{BB}$ is equal to the number of ways n can be written as $a^2\binom{b+1}{2}$, for integers a and b.*

An easy way to construct numbers that have multiple fixed partitions is by using triangular numbers that are also perfect squares.

Remark 4.3. One can create a number that has any number of partitions that are fixed under $\mathcal{BB}$ simply by taking the product of enough square triangular numbers.

However these are not the only numbers that have more than 1 fixed partition. Taking n to be 100800, we have solutions $(a_1, b_1) = (2, 224)$ and $(a_2, b_2) = (60, 7)$. But neither $\binom{8}{2} = 28$ nor $\binom{225}{2} = 25200$ are perfect squares.

Remark 4.4. We can see even in the triangular case that the number of recurrent cycles in Maximal Block Bulgarian Solitaire is not the same as the number in regular Bulgarian Solitaire or Carolina Solitaire.

Table 6 contains more data on Maximal Block Bulgarian Solitaire recurrent cycles.

4.1.1 Reverse Maximal Block Bulgarian Solitaire

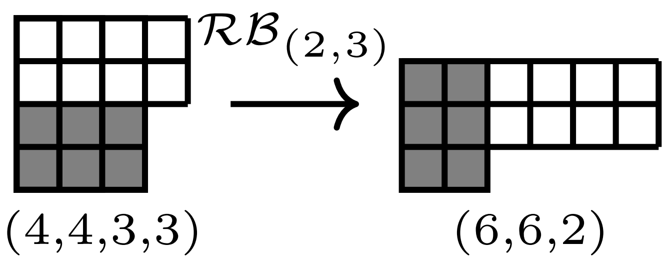
There are times where the reverse is easier to work with, especially when trying to understand level sizes. In this case, we define the *Reverse Maximal Block Bulgarian Solitaire move, $\mathcal{RB}$* as follows

Definition 4.5. Starting with an initial partition $\lambda \vdash n$ of length l, $\mathcal{RB}(\lambda)_{(r,k)}$ is defined by taking r piles of length k such that $k > l - r$, and by adding r cards to k piles, including piles of 0 cards as necessary.

In partition notation, we have for a partition $\lambda \vdash n$ of length l, and $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+j} = k$, then for $r \leqslant j$

$$\mathcal{RB}_{(r,k)} = (\lambda_1 + r, \lambda_2 + r, \dots, \lambda_i + r, \lambda_{i+r+1} + r, \dots, \lambda_l + r, r, r, \dots, r)$$

where there are $k + r - l$ piles of length r.



Using Reverse Block Bulgarian Solitaire, along with the fact that squares are trivially square staircases, we then are able to show that

Proposition 4.6. *The last level set before reaching a $k \times k$ square has exactly $k-1$ elements.*

Reverse Block Bulgarian Solitaire gives us all the tools we need to prove Proposition 4.6.

Proof. Given $n = k^2 > 1$ ($n = 1$ has a single element (1), and thus the level set before (1) has 0 elements), and starting from the partition $\lambda = (k, k, \ldots, k)$, we have k piles of size k, and the length of λ is also k. By Definition 4.5, $\mathcal{RB}(\lambda)$ is defined if $k > l - r = k - r$. Then $\mathcal{RB}(\lambda)$ is defined for $1 \leqslant r \leqslant k$, which is k total moves. However, playing $\mathcal{RB}_k(\lambda)$ is simply taking k piles of length k, and adding k cards to each remaining pile, which are all 0 before adding the cards back in. Thus $\mathcal{RB}_k(\lambda) = \lambda$ (which is necessary since a $k \times k$ square is fixed under $\mathcal{BB}$). Furthermore, playing each $r < k$ will yield a distinct partition, as increasing r will increase the number of piles that are 0 before adding the cards back in. Hence the number of elements in the level set immediately preceding a $k \times k$ square will be exactly $k - 1$. □

Reverse games also lend themselves to identifying Garden of Eden partitions. In a similar idea to Definition 2.5, in Maximal Block Bulgarian Solitaire, a Garden of Eden partition is a partition $\lambda \vdash n$ such that there does not exist another partition $\lambda' \vdash n$ such that $\lambda = \mathcal{BB}(\lambda')$.

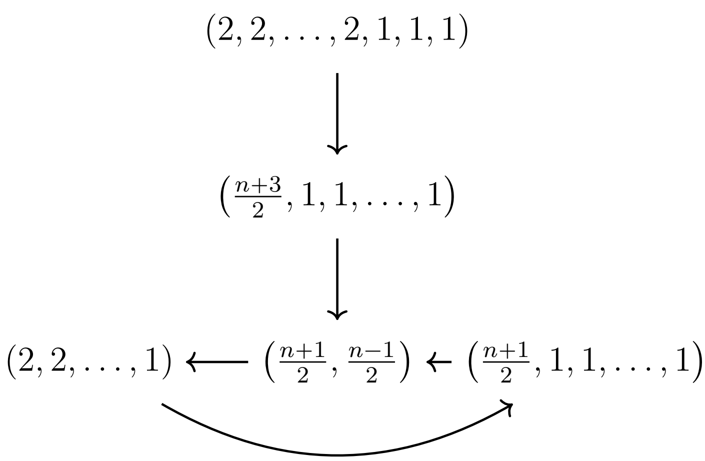
Another way to think about Garden of Eden partitions is that they are the sinks of the reverse game. In the proof of Proposition 4.7, we use this concept to show that our game graph is complete.

Right away, from Definition 4.5, we can recognize that the Garden of Eden partitions are those where $k + r \leqslant l$ for any legal choice of k and r. One such type of partition that satisfies the requirements to be a Garden of Eden partition is the staircase partition. Since staircase partitions are also square staircases, this means that in Maximal Block Bulgarian Solitaire, the game graph for a staircase partition is just the staircase.

In Section 5.6, a pattern that appears is that we see in several graphs a connected component that follows the pattern of a direct path of length 2 into a three-cycle. As it turns out, this connected component appears for any odd $n \geqslant 5$.

Proposition 4.7. *For any odd $n \geqslant 5$, the Maximal Block Bulgarian Solitaire game graph has a connected component with level sizes [3,1,1].*

The game graph in Proposition 4.7 looks like this



The Maximal Block Bulgarian Solitaire
Game Graph for odd $n \geqslant 5$

We prove Proposition 4.7 directly, using the Reverse Block Bulgarian Solitaire move and Garden of Eden partitions.

Proof. Suppose for a given odd n, that we make our starting partition be $\lambda = (2, 2, \ldots, 2, 1, 1, 1)$, where there are $\frac{n-3}{2}$ copies of 2. Then $\mathcal{BB}(\lambda) = \lambda' = (\frac{n+3}{2}, 1, 1, \ldots, 1)$, where there are $\frac{n-3}{2}$ copies of 1. Playing again, we have $\mathcal{BB}(\lambda') = \lambda'' = (\frac{n+1}{2}, \frac{n-1}{2})$. From here, we get $\mathcal{BB}(\lambda'') = \lambda^{(3)} = (2, 2, \ldots, 2, 1)$, with $\frac{n-1}{2}$ copies of 2. Next $\mathcal{BB}(\lambda^{(3)}) = \lambda^{(4)} = (\frac{n+1}{2}, 1, 1, \ldots, 1)$, with $\frac{n-1}{2}$ copies of 1. Finally, playing $\mathcal{BB}(\lambda^{(4)}) = \lambda''$, so we have found the recurrent cycle.

If we try to play $\mathcal{RB}(\lambda)$, we can either play the piles of 2 (which we could play up to $\frac{n-3}{2}$ of them, or we could play the piles of 1, of which there are 3. The length of λ is $l = \frac{n-3}{2} + 3 = \frac{n+3}{2}$. Then in order to play from here, we need $2 > \frac{n+3}{2} - \frac{n-3}{2} = 3$, or $1 > \frac{n+3}{2} - 3 = \frac{n-3}{2}$. The first case is clearly impossible, and in the second case, we need $1 > \frac{n-3}{2}$, so $n \leqslant 3$. Thus for $n \geqslant 5$, λ is a Garden of Eden partition.

If we try to play $\mathcal{RB}(\lambda')$, our options are to either play 1 pile of length $\frac{n+3}{2}$, or we can try to play $\frac{n-3}{2}$ piles of length 1. If we play the pile of length $\frac{n+3}{2}$, we get $\mathcal{RB}(\lambda') = \lambda$ (since that's just the reverse of the move we did to get to λ'). If we instead wish to play the piles of length 1, we need $1 > \frac{n-3}{2} + 1 - \frac{n-3}{2}$=1. Thus the only Reverse Maximal Block Bulgarian Solitaire move from λ' returns us to λ.

If we wish to play from λ'', we can either play the pile of $\frac{n+1}{2}$ or the pile of $\frac{n-1}{2}$, both of which are playable. But we already know Forward Maximal Block Bulgarian has two partitions (λ' and $\lambda^{(4)}$) that play into λ''. Thus we already know both possible Reverse Maximal Block Bulgarian Solitaire plays will stay in the current game graph.

If we wish to play $\lambda^{(3)}$, we check if we can play either $\frac{n-1}{2}$ piles of 2, which is playable, or we can play 1 pile of 1, which is clearly not playable. Thus, the only Reverse Maximal Block Bulgarian Solitaire play from $\lambda^{(3)}$ is back to λ''.

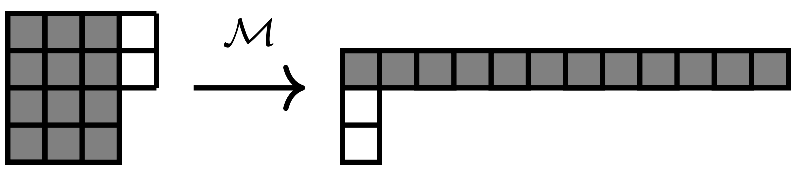
Finally, looking at $\lambda^{(4)}$, we can try to play the pile of $\frac{n+1}{2}$ or the $\frac{n-1}{2}$ piles of 1. The first pile is playable, but if we try to play the piles of 1, we find $1 > \frac{n-1}{2} + 1 - \frac{n-1}{2} = 1$, so the only playable move is the first pile. Then this must play into $\lambda^{(3)}$ to fit with the Forward Maximal Block Bulgarian Solitaire.

Thus we have defined the entire orbit explicitly for any odd $n \geqslant 5$. □

4.2 Maximal Minnesota Solitaire

The final game we introduce here is the deterministic variant of Minnesota Solitaire, called Maximal Minnesotan Solitaire. In this game, the Maximal Minnesota Solitaire move, $\mathcal{M}$ is defined as playing $\overline{\mathcal{M}_k}$, so for a given partition $\lambda \vdash n$ with smallest pile of size k,

$$\mathcal{M}(\lambda) = (\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_l - k, kl)$$



(4,4,3,3) (12,1,1)

It turns out that once again, a special set of partitions is fixed under $\mathcal{M}$. In this case,

Proposition 4.8. *The fixed partitions under $\mathcal{M}$ are the stretched staircase partitions.*

Proof. First, consider some stretched staircase partition $\Gamma_{(a,b)} \vdash n$. Then by Definition 1.4, we know that the smallest nonzero pile in $\Gamma_{(a,b)}$ has size a and $\Gamma_{(a,b)}$ has length b. Then playing $\mathcal{M}(\Gamma_{(a,b)})$ will return

$$\mathcal{M}(\Gamma_{(a,b)}) = (ab - a, a(b-1) - a, \ldots, a(b-(b-2)) - a, a - a, ab)$$

Then by reordering in nonincreasing order and simplifying, we find

$$\mathcal{M}(\Gamma_{(a,b)}) = (ab, a(b-1), a(b-2), \ldots, a)$$

Thus all stretched staircases are fixed under $\mathcal{M}$.

If we have a partition $\lambda \vdash n$ that is not a stretched staircase, then if our smallest nonzero pile has size k, for some $\lambda_i \in \lambda$ of length l, it must be the case that $\lambda_i - k \neq \lambda_{i+1}$ (including $\lambda_{l+1} = 0$). Then when we play $\mathcal{M}(\lambda)$, we get

$$\mathcal{M}(\lambda) = (\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_i - k, \ldots, 0, kl)$$

Since $k \neq 0$, the only way λ can be fixed under $\mathcal{M}$ is if for $j = 1, 2, \ldots, l$, we have $\lambda_j - k = \lambda_{j+1}$. But this only happens when λ is a stretched staircase! Thus the partitions that are fixed under $\mathcal{M}$ are the stretched staircase partitions. □

Unlike the number of square staircase partitions, the number of stretched staircase partitions of n is a known quantity. Wiseman first posted this claim in [Wis19] without proof. We prove it here as a lemma.

Lemma 4.9. *The number of stretched staircase partitions of n is equal to the number of triangular numbers that divide n.*

Proof. This lemma follows directly from Definition 1.4. Since a stretched staircase partition is of the form $(ab, a(b-1), \ldots, a)$, which summed together gives $n = a\binom{b+1}{2}$, we have a stretched staircase partition if and only if $\binom{b+1}{2}$ divides n. □

Theorem 4.10 combines Proposition 4.8 and Lemma 4.9.

Theorem 4.10. *The number of fixed partitions of n under $\mathcal{M}$ is equal to the number of triangular numbers that divide n.*

This sequence is much easier to understand than the number of square staircase partitions.

Remark 4.11. Once again the number of recurrent cycles in Maximal Minnesota Solitaire is not the same as the number in regular Bulgarian Solitaire or Maximal Block Bulgarian Solitaire.

Table 9 contains more data on Maximal Minnesota Solitaire recurrent cycles.

4.2.1 Reverse Maximal Minnesota Solitaire

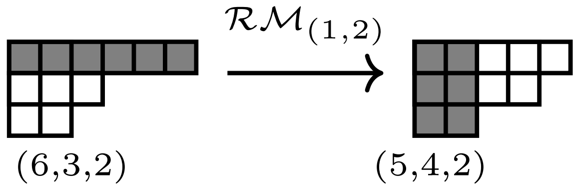
We can define the Reverse Maximal Minnesota Solitaire move as follows:

Definition 4.12. For a starting partition $\lambda \vdash n$ of length l, we define the Reverse Maximal Minnesota Solitaire move $\mathcal{RM}_{(i,k)}$ by taking the ith pile, and distributing those cards evenly among the remaining piles (including piles of size 0 as necessary).

On partitions, this move is

$$\mathcal{RM}_{(i,k)} = (\lambda_1 + k, \lambda_2 + k, \ldots, \lambda_{i-1} + k, \lambda_{i+1} + k, \ldots, \lambda_l + k, k, k, \ldots, k)$$

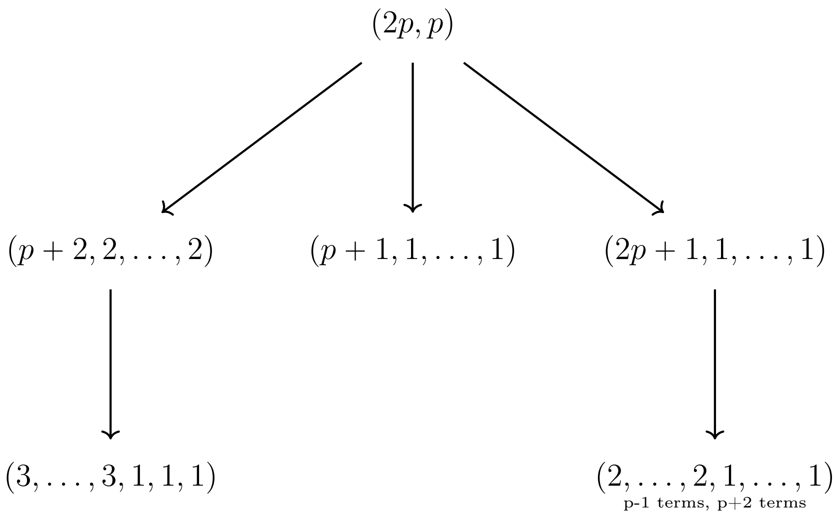
where k divides λ_i, and the number of piles of size k is $\frac{\lambda_i}{k}$. Note that $\mathcal{RM}_{(i,k)}$ is a legal move if and only if $\frac{\lambda_i}{k} > l$.



Again looking for Garden of Eden partitions in Maximal Minnesota Solitaire, we can see from the definition that a partition $\lambda \vdash n$ is a Garden of Eden partition if and only if $\lambda_1 \leqslant l$. Then once again, the staircase partitions (where $\lambda_1 = l$) are both fixed partitions under $\mathcal{M}$ and Garden of Eden Partitions. Using Reverse Maximal Minnesota Solitaire, we can then prove the following about the orbit containing the stretched staircase $(2p, p)$.

Proposition 4.13. *For a prime number, $p > 2$, the Reverse Maximal Minnesota Solitaire game graph for the $[2p, p]$ starting has level sizes of $1, 3, 2$.*

The Reverse Maximal Minnesota Solitaire game graph in this case looks like this



Proof. First, note that $\lambda = (2p, p)$ is a stretched staircase, so by Theorem 4.10, it must be fixed under $\mathcal{M}$. Then playing Reverse Maximal Minnesota Solitaire, we can either play the pile of size $2p$ or the pile of size p. Since $p > 1$, both piles are playable. If we choose to play the pile of size p, then, since p is prime, we can only give one card to p piles, making $\lambda' = (2p + 1, 1, \ldots, 1)$ (where there are $p - 1$ piles of size 1). From λ', we can only play the pile of size $2p + 1$, and since $2p + 1$ is not divisible by p, we again can only give one card to each pile, making $\lambda'' = (2, 2, \ldots, 2, 1, 1, \ldots, 1)$ (where there are $p - 1$ piles of size 2, and $3p - 2(p - 1) = p + 2$ piles of size 1). Since $2 < p - 1 + p + 2$ for any prime $p > 2$, λ'' must be a Garden of Eden Partition.

Now we go back and start by playing the pile of size $2p$. Note that giving p cards to 2 piles just gives λ again. The other options for plays are to give 1 card to $2p$ piles, giving $\lambda^{(3)} = (p+1, 1, 1, \ldots, 1)$, where there are $2p-1$ piles of size 1. Then, since $p+1 < 2p-1$ for any prime $p > 2$, $\lambda^{(3)}$ is a Garden of Eden partition.

Finally, if we play $2p$, and give 2 cards to p piles, then we make $\lambda^{(4)} = (p+2, 2, \ldots, 2)$ ($p-1$ piles of size 2). Then the pile of size $p+2$ is playable, and since $p-2$ does not divide $p+2$, we can only distribute 1 card to $p+2$ piles. This gives $\lambda^{(5)} = (3, 3, \ldots, 3, 1, 1, 1)$ ($p-1$ piles of size 3). Then, since $3 < p+1$ for any prime $p > 2$, $\lambda^{(5)}$ is a Garden of Eden partition.

Hence we have identified the entire orbit containing the stretched staircase $(2p, p)$. □

Another pattern involves stretching the partition (1) by the squares of primes. Table 11 shows the motivation behind Proposition 4.14.

Proposition 4.14. *Stretching the partition* $[1]$ *by squares of primes* p^2*, the level sizes of the game graph are* $1, 2, 1, 1, ...$*, where the number of 1s after the 2 is equal to* p*.*

Proof. The partition $\lambda = (p^2)$ is a stretched staircase, so by Proposition 4.8, we know it is fixed under $\mathcal{M}$. Then, since p is prime, playing Reverse Maximal Minnesota Solitaire has two possible moves, namely $(1, 1)$ and $(1, p)$.

If we play $\mathcal{RM}_{(1,1)}(\lambda)$, then we will have p^2 piles each of size 1, that is

$$\mathcal{RM}_{(1,1)}(\lambda) = (1, 1, \ldots, 1)$$

Since $p^2 > 1$, we have found a Garden of Eden Partition.

If we instead play $\mathcal{RM}_{(1,p)}(\lambda)$, then we will have p piles of size p, so

$$\mathcal{RM}_{(1,p)}(\lambda) = (p, p, \ldots, p) = \lambda^{(0)}$$

Since p is prime, it follows that $(p-1) \nmid p$, so the only possible move on $\lambda^{(0)}$ is to play $\mathcal{RM}_{(1,1)}(\lambda^{(0)})$. This gives

$$\mathcal{RM}_{(1,1)}(\lambda^{(0)}) = (p+1, p+1, \ldots, p+1, 1) = \lambda^{(1)}$$

where there are $p-1$ piles of size $p+1$. Since there are p piles in total, we can only play the piles of size strictly greater than $p-1$ (meaning piles of size at least p). Since the only factor of $p+1$ that is at least p is itself, the only playable move is $\mathcal{RM}_{(1,1)}(\lambda^{(1)})$. This gives

$$\mathcal{RM}_{(1,1)}(\lambda^{(1)}) = (p+2, p+2, \ldots, p+2, 2, 1, 1) = \lambda^{(2)}$$

where there are $p-2$ piles of size $p+2$ (and $p+1$ total piles).

Similarly, in the general case, since there are $p+a-1$ piles in total, we can only play the piles with size at least $p+a-1$, which are exactly those of size $p+a$. Since the only factor of $p+a$ that is at least $p+a-1$ is itself, the only playable move is $\mathcal{RM}_{(1,1)}(\lambda^{(a)})$. After $p-1$ plays, we will reach the following partition

$$\lambda^{(p-1)} = (2p-1, p-1, p-2, p-2, \ldots, 2, 2, 1, 1)$$

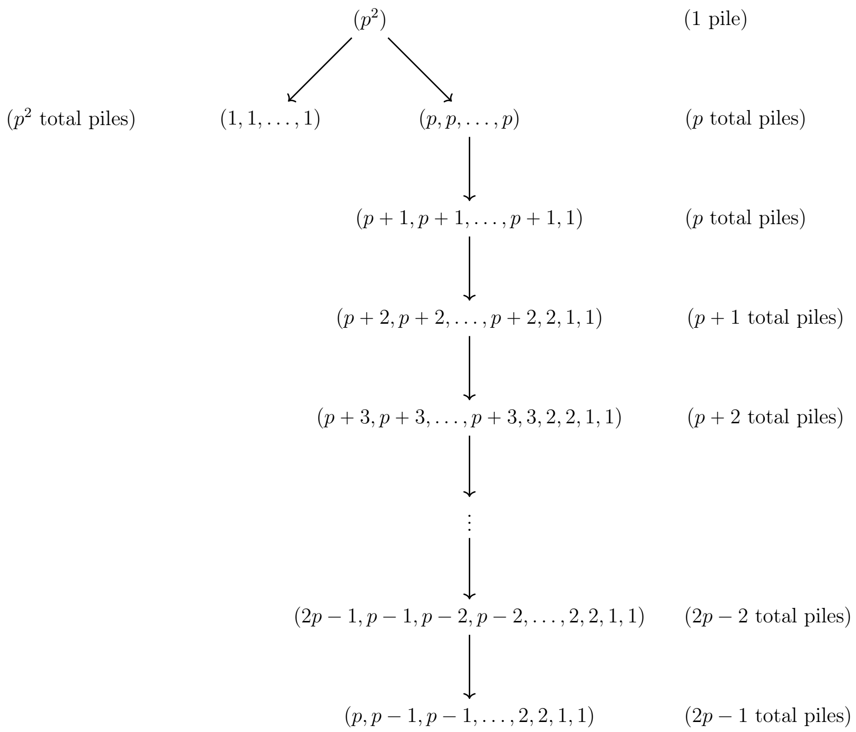
where there are $2p-2$ total piles. Then the only playable pile is the one of size $2p-1$, and since $(2p-2) \nmid (2p-1)$ for $p > 1$, the only possible play is $\mathcal{RM}_{(1,1)}(\lambda^{(p-1)})$. This gives

$$\mathcal{RM}_{(1,1)}(\lambda^{(p-1)}) = (p, p-1, p-1, \ldots, 2, 2, 1, 1) = \lambda^{(p)}$$

which has $2p-1$ total piles. Then, since $p < 2p-1$ for $p > 1$, we have reached a Garden of Eden Partition.

Then the zero level is λ, the first level is $\lambda^{(0)}$ and $(1, 1, \ldots, 1)$, and each $\lambda^{(n)}$ for $1 \leqslant n \leqslant p$ is in its own level of size 1. Then the level sizes are $1, 2, 1, 1, \ldots, 1$, where there are exactly p levels of size 1 after the first level. □

The game graph in this case looks like this



5 Additional Data

5.1 Orbit Sizes for Necklaces in Carolina Solitaire

Here we provide data on $|\mathcal{O}_N|$ for necklaces $N = P^k$ in order to determine empirically if the c_p Pham conjectured for Bulgarian Solitaire exists in Carolina Solitaire.

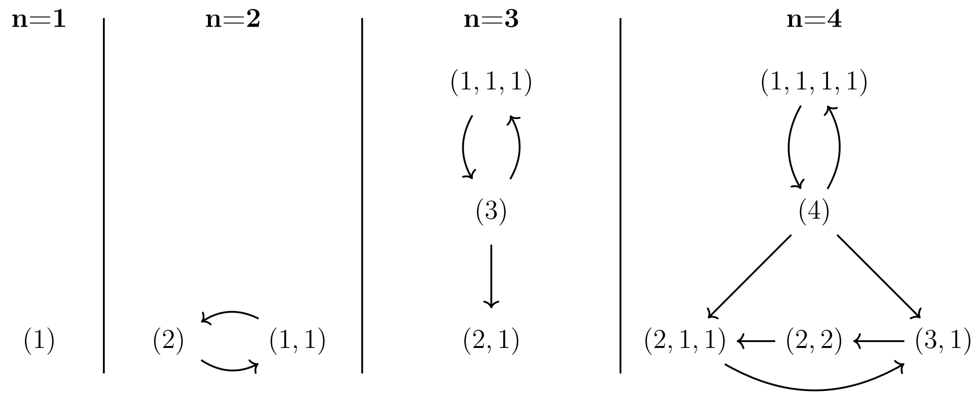
5.1.1 BBW

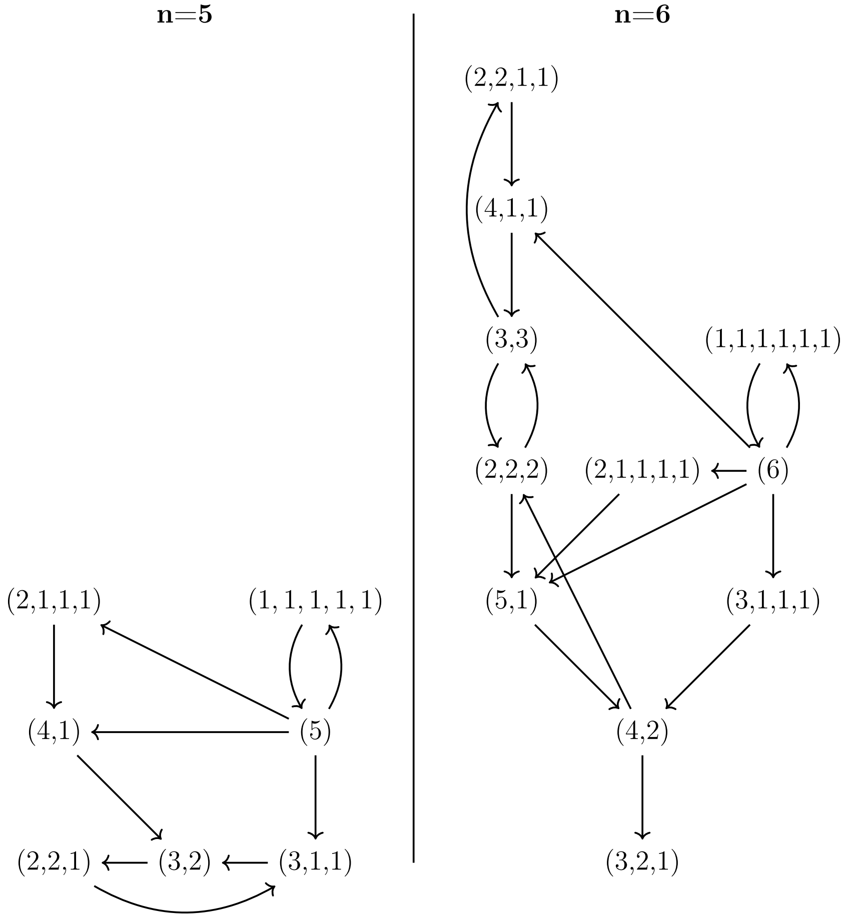
N	$\|\mathcal{O}_N\|$
BBW	16
$(BBW)^2$	16318

Table 1: Orbit sizes for powers of BBW

5.2 Small Game Graphs for Block Bulgarian Solitaire

Here we provide Block Bulgarian Solitaire game graphs for small n. Note that in each of these, the sink is exactly the Bulgarian Solitaire recurrent cycle for n, which supports Theorem 3.1.





5.3 Sinks for Block Bulgarian Solitaire

Here we provide the data that we found from the game graphs in a more organized fashion. This data led to Theorem 3.1.

n	Block Bulgarian Solitaire Sink	Corresponding Necklaces
1	(1)	$B = W^2$
2	(2),(1,1)	BW, WB
3	(2,1)	$B^2 = W^3$
4	(2,2), (3,1), (2,1,1)	WBW, BWW, WWB
5	(2,2,1), (3,2), (3,1,1)	WBB, BBW, BWB
6	(3,2,1)	$B^3 = W^4$
7	(3,2,1,1), (3,2,2), (4,2,1), (3,3,1)	$WWWB$, $WWBW$, $BWWW$, $WBWW$
8	(4,2,1,1), (4,2,2), (4,3,1), (3,3,1,1), (3,3,2), (3,2,2,1)	$BWWB$, $BWBW$, $BBWW$, $WBWB$, $WBBW$, $WWBB$
9	(3,3,2,1), (4,3,2), (4,3,1,1), (4,2,2,1)	$WBBB$, $BBBW$, $BWBB$
10	(4,3,2,1)	$B^4 = W^5$

Table 2: Block Bulgarian Solitaire Sinks

5.4 Sink Sizes for Block Bulgarian Solitaire

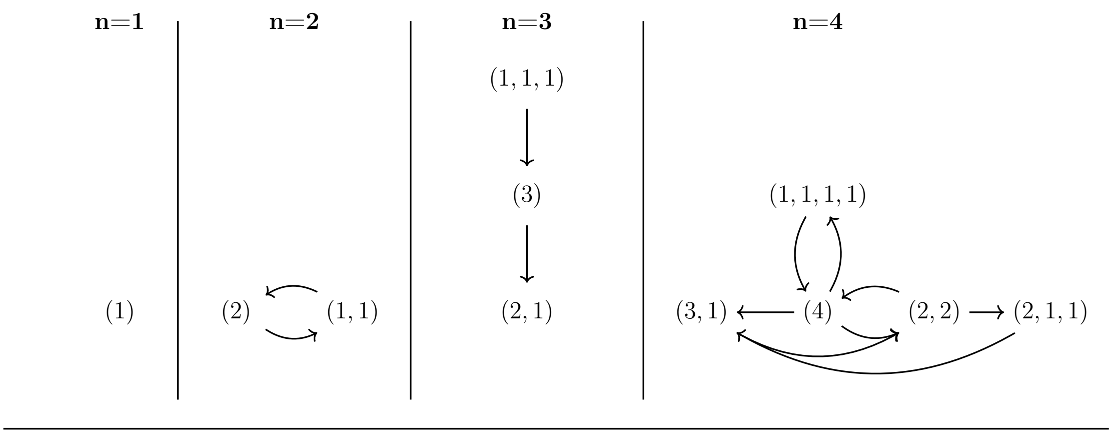
Here we provide data on the size of the Block Bulgarian Solitaire sink for $n \leqslant 30$. Note that thanks to Theorem 3.1, we know the size to be the size of each Bulgarian Solitaire recurrent cycle times the number of recurrent cycles (as found by Brandt in [Bra82]).

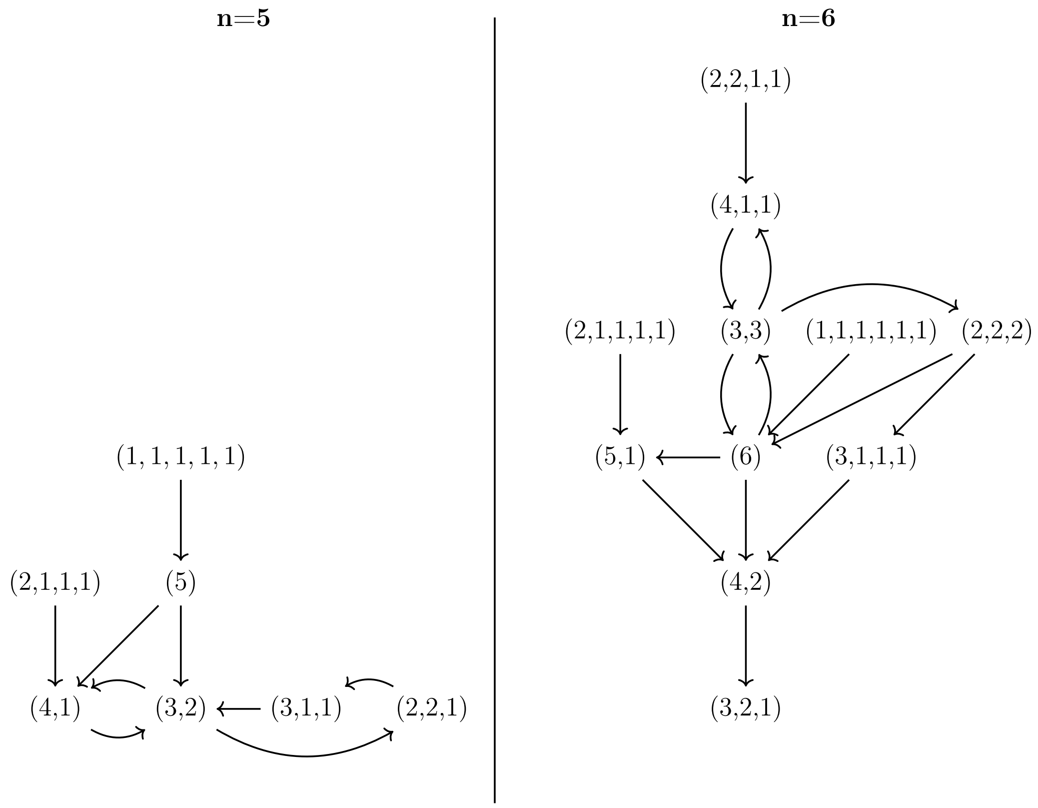
n	Sink Size	n	Sink Size
1	1	16	6
2	2	17	15
3	1	18	20
4	3	19	15
5	3	20	6
6	1	21	1
7	4	22	7
8	6	23	21
9	4	24	35
10	1	25	35
11	5	26	21
12	10	27	7
13	10	28	1
14	5	29	8
15	1	30	28

Table 3: Sizes of the Block Bulgarian Solitaire Sink for $n \leqslant 30$

5.5 Small Game Graphs for Minnesota Solitaire

Here we provide Minnesota Solitaire game graphs for small n. Note that the sinks in $n = 4$ and $n = 5$ contain more elements than the Bulgarian Solitaire recurrent cycle, so we cannot make the same claim as we did for Theorem 3.1. However, each n has a single sink containing the Bulgarian Solitaire recurrent cycle, which leads to Conjecture 4.8.





5.6 Sinks for Minnesota Solitaire

Here we provide sinks for Minnesota Solitaire, which shows that unlike Block Bulgarian Solitaire, the sinks do not contain only the Bulgarian Solitaire recurrent cycles, but up to $n = 10$, the data agrees with Conjecture 3.4.

n	Minnesota Solitaire Sink	Corresponding Necklaces
1	(1)	$B = W^2$
2	(2),(1,1)	BW, WB
3	(2,1)	$B^2 = W^3$
4	(2,2), (3,1), (2,1,1), (4)	WBW, BWW, WWB, n/a
5	(2,2,1), (3,2), (3,1,1), (4,1)	WBB, BBW, BWB, n/a
6	(3,2,1)	$B^3 = W^4$
7	(3,2,1,1), (3,2,2), (4,2,1), (3,3,1), (4,3), (6,1), (5,2)	$WWWB$, $WWBW$, $BWWW$, $WBWW$ n/a, n/a, n/a
8	(4,2,1,1), (4,2,2), (4,3,1), (3,3,1,1) (3,3,2), (3,2,2,1), (7,1), (8), (4,4), (6,2), (5,2,1), (6,1,1), (5,3)	$BWWB$, $BWBW$, $BBWW$, $WBBW$ $WBBW$, $WWBB$, n/a, n/a n/a, n/a, n/a, n/a, n/a
9	(3,3,2,1), (4,3,2), (4,3,1,1), (4,2,2,1) (5,3,1), (6,2,1)	$WBBB$, $BBBW$, $BWBB$ n/a, n/a
10	(4,3,2,1)	$B^4 = W^5$

Table 4: Minnesota Solitaire Sinks

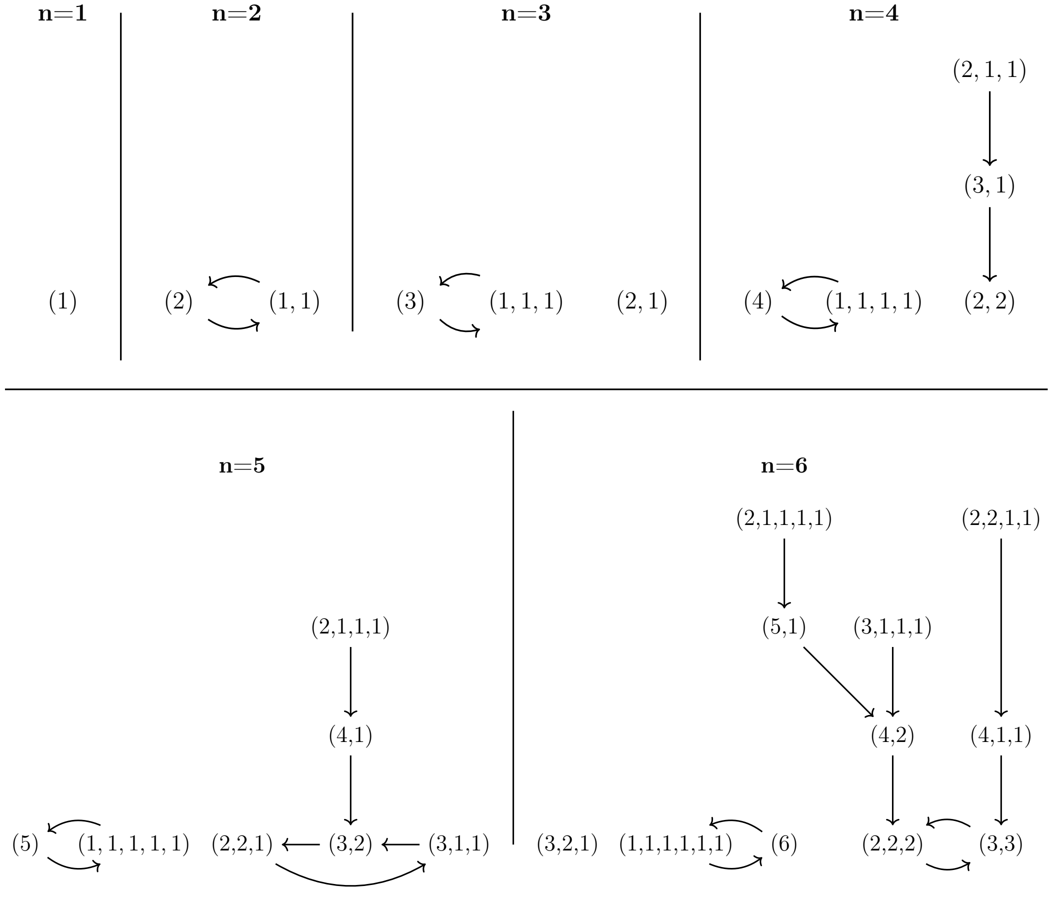
5.7 Sink Sizes for Minnesota Solitaire

Here we provide data on the size of the Minnesota Solitaire sink for $n \leqslant 30$. This data led to Conjecture 3.5.

n	Sink Size	n	Sink Size
1	1	16	88
2	2	17	106
3	1	18	145
4	4	19	171
5	4	20	10
6	1	21	1
7	7	22	316
8	13	23	383
9	6	24	471
10	1	25	527
11	26	26	671
12	40	27	12
13	42	28	1
14	8	29	1133
15	1	30	1379

Table 5: Sizes of the Minnesota Solitaire Sink for $n \leqslant 30$

5.8 Small Maximal Block Bulgarian Solitaire Game Graphs



5.9 Small Recurrent Cycles for Maximal Block Bulgarian Solitaire

Here we provide data on recurrent cycles for Maximal Block Bulgarian Solitaire. Highlighted in green are the 3-cycles that occur in every odd $n > 3$, and led to Proposition 4.7. The recurrent cycles of size 1 are highlighted in blue. They are all square staircase partitions, which led to Theorem 4.1. We can also compare this table to Table 9, and see that Maximal Minnesota Solitaire usually has fewer recurrent cycles than Maximal Block Bulgarian Solitaire, however that is not true in every case (such as $n = 15$).

n	Recurrent Cycles	Cycles
1	$\{(1)\}$	1
2	$\{(2),(1,1)\}$	1
3	$\{(3),(1,1,1)\}$, $\{(2,1)\}$	2
4	$\{(4),(1,1,1,1)\}$, $\{(2,2)\}$	2
5	$\{(5),(1,1,1,1,1)\}$, $\{(3,2),(3,1,1),(2,2,1)\}$	2
6	$\{(6),(1,1,1,1,1,1)\}$, $\{(2,2,2),(3,3)\}$, $\{(3,2,1)\}$	3
7	$\{(7),(1,1,1,1,1,1,1)\}$, $\{(3,3,1),(3,2,2)\}$, $\{(4,1,1,1),(2,2,2,1),(4,3)\}$	3
8	$\{(8),(1,1,1,1,1,1,1,1)\}$, $\{(4,2,2),(3,3,1,1),(3,3,2)\}$, $\{(4,4),(2,2,2,2)\}$	3
9	$\{(9),(1,1,1,1,1,1,1,1,1)\}$, $\{(5,1,1,1,1),(5,4),(2,2,2,2,1)\}$, $\{(3,3,3)\}$, $\{(4,2,2,1),(3,3,2,1),(4,3,1,1),(4,3,2)\}$	4
10	$\{(10),(1,1,1,1,1,1,1,1,1,1)\}$, $\{(4,3,2,1)\}$, $\{(5,5),(2,2,2,2,2)\}$, $\{(4,4,2),(4,2,2,2),(4,3,3),(3,3,2,2),(4,4,1,1),(3,3,3,1)\}$	4
11	$\{(11),(1,1,1,1,1,1,1,1,1,1,1)\}$, $\{(3,3,3,2),(5,3,3),(4,4,1,1,1)\}$, $\{(5,2,2,2),(3,3,3,1,1),(4,4,3)\}$, $\{(4,4,2,1),(4,3,2,2),(4,3,3,1)\}$, $\{(6,1,1,1,1,1),(2,2,2,2,2,1),(6,5)\}$	5
12	$\{(12),(1,1,1,1,1,1,1,1,1,1,1,1)\}$, $\{(3,3,3,3),(4,4,4)\}$, $\{(4,4,2,2)\}$, $\{(2,2,2,2,2,2),(6,6)\}$, $\{(3,3,3,2,1),(5,4,3),(5,2,2,2,1),(5,4,1,1,1)\}$	5
13	$\{(13),(1,1,1,1,1,1,1,1,1,1,1,1,1)\}$, $\{(4,3,3,3),(4,4,4,1)\}$, $\{(5,5,3),(5,2,2,2,2),(3,3,3,3,1),(5,5,1,1,1),(3,3,3,2,2),(5,4,4)\}$, $\{(5,3,3,1,1),(4,4,2,2,1),(5,4,2,2),(4,4,3,2)\}$, $\{(4,4,3,1,1),(5,3,3,2)\}$, $\{(2,2,2,2,2,2,1),(7,6),(7,1,1,1,1,1,1)\}$	6
14	$\{(14),(1,1,1,1,1,1,1,1,1,1,1,1,1,1)\}$, $\{(2,2,2,2,2,2,2),(7,7)\}$, $\{(5,3,3,3),(4,4,4,2),(4,4,3,3),(5,5,2,2),(4,4,4,1,1),(4,4,2,2,2)\}$, $\{(6,4,4),(3,3,3,3,2),(5,5,1,1,1,1)\}$, $\{(6,2,2,2,2),(5,5,4),(3,3,3,3,1,1)\}$, $\{(5,3,3,2,1),(5,4,2,2,1),(5,4,3,2),(5,4,3,1,1),(4,4,3,2,1)\}$	6
15	$\{(15)(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)\}$, $\{(6,3,3,3),(4,4,4,3),(4,4,4,1,1,1)\}$, $\{(5,4,3,2,1)\}$, $\{(5,5,5),(3,3,3,3,3)\}$, $\{(4,4,3,3,1),(5,5,2,2,1),(4,4,4,2,1),(5,4,4,1,1),(5,3,3,3,1),(5,4,2,2,2),(5,5,3,1,1),(5,4,3,3),(5,5,3,2),(4,4,3,2,2),(5,3,3,2,2),(5,4,4,2)\}$, $\{(8,1,1,1,1,1,1,1),(2,2,2,2,2,2,2,1),(8,7)\}$, $\{(6,5,4),(6,2,2,2,2,1),(6,5,1,1,1,1),(3,3,3,3,2,1)\}$	7
16	$\{(16),(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)\}$, $\{(4,4,4,4)\}$, $\{(2,2,2,2,2,2,2,2),(8,8)\}$, $\{(4,4,4,2,2),(5,5,3,3),(5,5,2,2,2)\}$, $\{(3,3,3,3,3,1),(3,3,3,3,2,2),(6,2,2,2,2,2),(6,6,1,1,1,1),(6,6,4),(6,5,5)\}$, $\{(5,5,3,2,1),(5,4,3,2,2),(5,4,4,2,1),(5,4,3,3,1)\}$	6

Table 6: Maximal Block Bulgarian Solitaire Recurrent Cycles

5.10 Number of Square Staircase Partitions for Select n

Here we provide some data on the number of square staircase partitions for small n, as well as some larger n of interest.

n	$\#\Lambda_(a,b)$	n	$\#\Lambda_(a,b)$	n	$\#\Lambda_(a,b)$
1	1	11	0	36	2
2	0	12	1	144	2
3	1	13	0	44100	3
4	1	14	0	100800	2
5	0	15	1	11524800	3
6	1	16	1		
7	0	17	0		
8	0	18	0		
9	1	19	0		
10	1	20	0		

Table 7: The number of square staircase partitions for select n

5.11 Maximal Block Bulgarian Solitaire Level Sizes for Squares

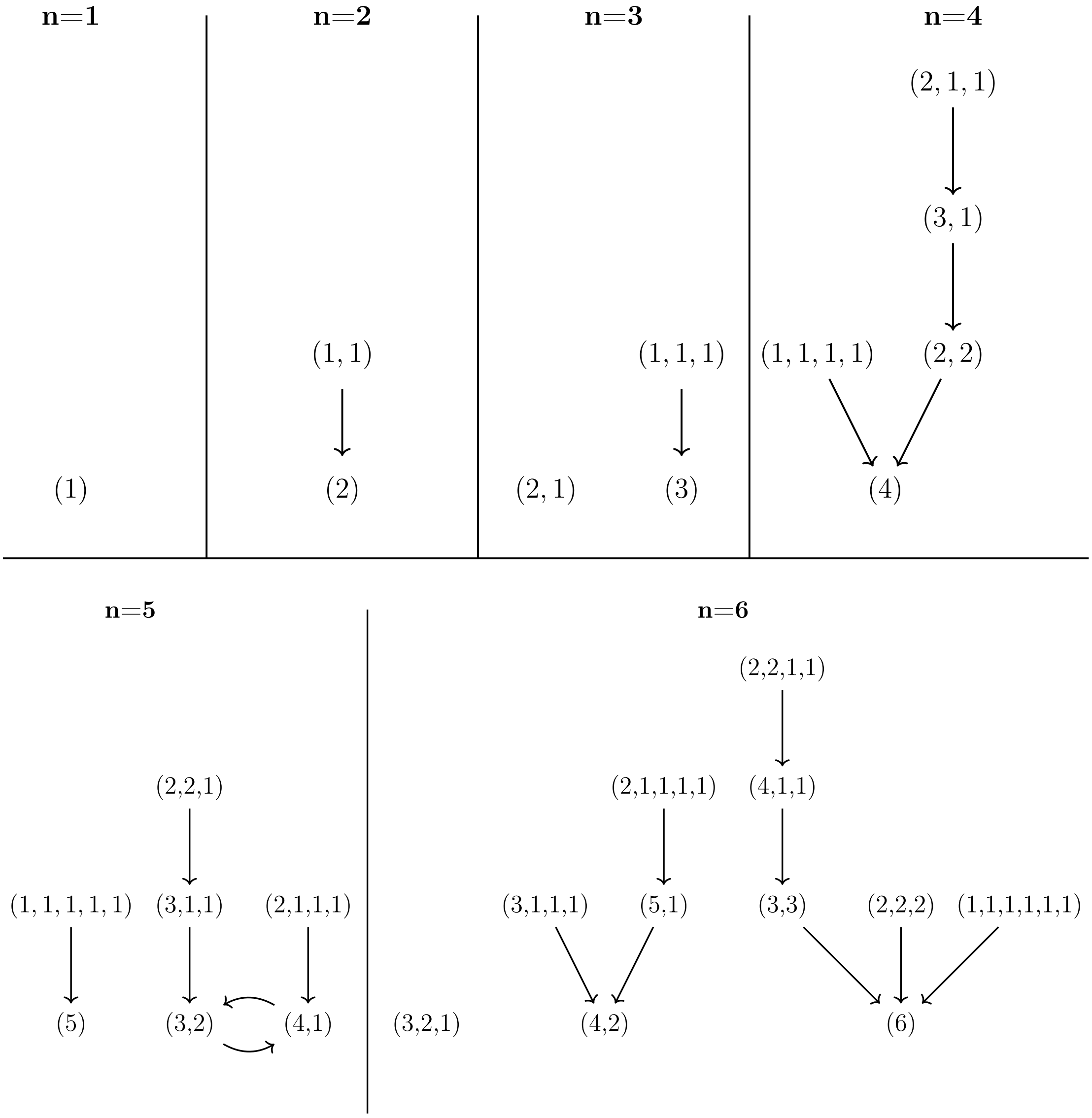
Here we provide data on level sets for $k \times k$ squares in Maximal Block Bulgarian Solitaire. The data strongly supported Proposition 4.6, but we were unable to find any other definite patterns. There are a few possible sequences in OEIS that may coincide with the second level set or the number of levels, but due to computing limitations, we were unable to collect enough data to determine anything concrete.

n	k	Level Sizes	Levels	Orbit Size
1	1	(1)	1	1
4	2	(1,1,1)	3	3
9	3	(1,2,4,4,3,1)	6	15
16	4	(1,3,8,11,12,11,6,1)	8	53
25	5	(1,4,14,28,40,43,32,15,5,1)	10	183
36	6	(1,5,21,51,90,123,122,80,36,14,5,1)	12	549
49	7	(1,6,30,90,192,312,380,335,215,100,38,20,11,3)	14	1733

Table 8: Maximal BBS Levels and Orbit Sizes for Squares

5.12 Small Maximal Minnesota Solitaire Game Graphs

Here we provide some Minnesota Solitaire game graphs for small n. Notice that in the $n = 4$ case, we have level sizes of $[1, 2, 1, 1]$, which fits with Conjecture 4.14.



5.13 Small Recurrent Cycles for Maximal Minnesota Solitaire

Here we provide data on recurrent cycles for Maximal Minnesota Solitaire. We highlight the recurrent cycles of size one in blue. Note that they are all stretched staircase partitions, which supports Proposition 4.8. Fitting with Corollary 4.2 and Theorem 4.10, Maximal Minnesota Solitaire has at least as many recurrent cycles of size 1 as Maximal Block Bulgarian Solitaire.

n	Recurrent Cycles	Number of Cycles
1	$\{(1)\}$	1
2	$\{(2)\}$	1
3	$\{(3)\}$, $\{(2,1)\}$	2
4	$\{(4)\}$	1
5	$\{(5)\}$, $\{(3,2),(4,1)\}$	2
6	$\{(6)\}$, $\{(4,2)\}$, $\{(3,2,1)\}$	3
7	$\{(7)\}$, $\{(6,1),(5,2),(4,3)\}$	2
8	$\{(8)\}$	1
9	$\{(9)\}$, $\{(6,3)\}$, $\{(8,1),(7,2),(5,4)\}$, $\{(5,3,1),(6,2,1),(4,3,2)\}$	4
10	$\{(10)\}$, $\{(4,3,2,1)\}$, $\{(6,4),(8,2)\}$, $\{(5,3,2),(6,3,1)\}$	4
11	$\{(11)\}$, $\{(7,4),(8,3),(10,1),(9,2),(6,5)\}$	2
12	$\{(12)\}$, $\{(8,4)\}$, $\{(6,4,2)\}$, $\{(6,5,1),(8,3,1),(5,4,3),(9,2,1),(7,3,2)\}$	4
13	$\{(13)\}$, $\{(10,3),(12,1),(7,6),(9,4),(11,2),(8,5)\}$	2
14	$\{(14)\}$, $\{(12,2),(8,6),(10,4)\}$, $\{(6,5,3),(9,3,2),(7,6,1)\}$, $\{(5,4,3,2),(6,4,3,1),(7,4,2,1),(8,3,2,1)\}$	4
15	$\{(15)\}$, $\{(10,5)\}$, $\{(5,4,3,2,1)\}$, $\{(14,1),(11,4),(13,2),(8,7)\}$, $\{(9,6),(12,3)\}$, $\{(12,2,1),(11,3,1),(7,6,2),(9,4,2),(8,6,1),(7,5,3),(6,5,4),(10,3,2)\}$, $\{(8,4,3),(9,5,1)\}$, $\{(7,4,3,1),(8,4,2,1),(6,4,3,2)\}$	8
16	$\{(16)\}$, $\{(7,6,3),(9,5,2),(8,5,3),(9,6,1),(9,4,3)\}$	2

Table 9: Maximal Minnesota Solitaire Recurrent Cycles

5.14 Number of Stretched Staircase Partitions for Select n

Here we provide the data that led to Theorem 4.10.

n	$\#\Gamma_{(a,b)}$	n	$\#\Gamma_{(a,b)}$	n	$\#\Gamma_{(a,b)}$
1	1	11	1	21	3
2	1	12	3	22	1
3	2	13	1	23	1
4	1	14	1	24	3
5	1	15	3	25	1
6	3	16	1	26	1
7	1	17	1	27	2
8	1	18	3	28	2
9	2	19	1	29	1
10	2	20	2	30	5

Table 10: Number of fixed partitions under $\mathcal{M}$ for n up to 30

5.15 Data for Proposition 4.14

Here we provide data in support of Proposition 4.14.

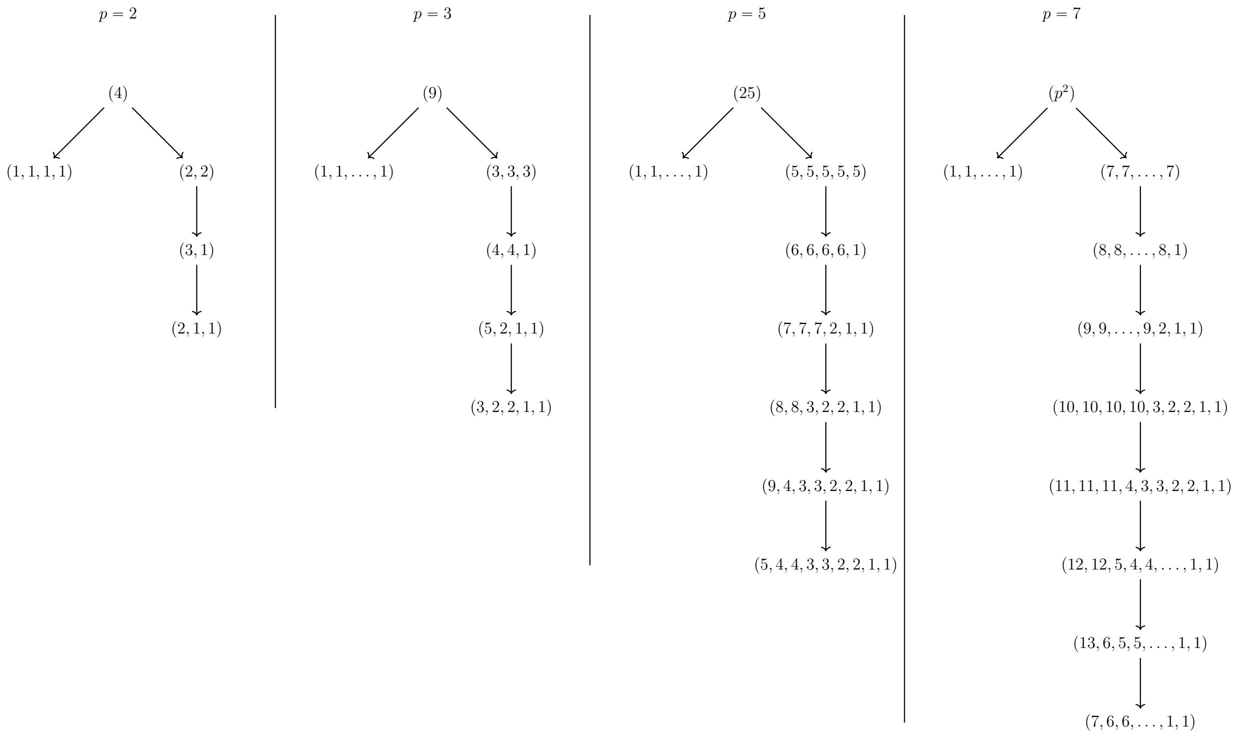
5.15.1 Level Sizes

n	Level Sizes
4	[1,2,1,1]
9	[1,2,1,1,1]
25	[1,2,1,1,1,1,1]
49	[1,2,1,1,1,1,1,1,1]

Table 11: Level sizes for stretching the partition (1) by squares of primes

5.15.2 Game Graphs for Proposition 4.14

Here we show the game graphs corresponding to the previous section.



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