Critical groups of McKay-Cartan matrices

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Acknowledgements

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Plan

- Motivation: chip firing with avalanche-finite matrices
- McKay-Cartan matrices
- Theorem 1: size and structure of critical groups
- Theorem 2: critical groups for the reflection representation of $\mathfrak{S}_n$.

Chip firing

A matrix $C = (c_{ij})$ in $\mathbb{Z}^{\ell \times \ell}$ with $c_{ij} \leq 0$ for all $i \neq j$ is called a *Z-matrix*.

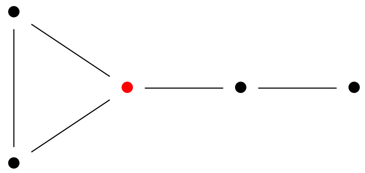
Given a Z-matrix C, we call the elements $v = (v_1, ..., v_\ell)^t \in \mathbb{N}^\ell$ *chip configurations*, and we define a dynamical system on the set of such configurations as follows:

- A configuration v is *stable* if $v_i < c_{ii}$ for $i = 1, ..., \ell$.
- If v is unstable, then choose some i so that $v_i \geq c_{ii}$ and form a new configuration $v' = (v'_1, ..., v'_\ell)^t$ where $v'_j = v_j - c_{ij}$ for $j = 1, ..., \ell$. The result v' is called the *C-toppling* of v at position i.

A Z-matrix is called an *avalanche-finite matrix* if any chip configuration can be brought to a stable one by a sequence of these topplings.

Example: chip firing on graphs

If C is the (reduced) Laplacian matrix of a graph Γ, then toppling at position i corresponds to sending chips along the edges incident to vertex i. One of the vertices (corresponding the the removed row and column in the Laplacian matrix) is a "black hole".

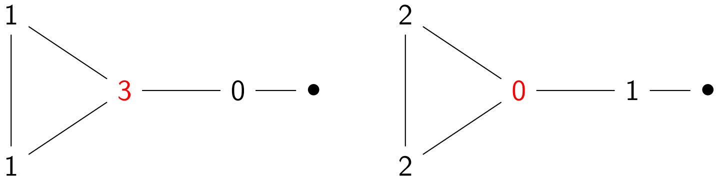


Reduced Laplacian is

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ \color{red}{-1} & \color{red}{-1} & \color{red}{3} & \color{red}{-1} \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Example: chip firing on graphs

Give each node some number of chips, this corresponds to picking the configuration v. In this case $v = (1, 1, 3, 0)$.



Subtracting the 3rd row of the Laplacian:

$$v = (1, 1, 3, 0) - (-1, -1, 3, -1) = (2, 2, 0, 1) = v'$$

Critical groups

If C is an avalanche-finite matrix the *critical group* of C is

$$K(C) := \text{coker } (\mathbb{Z}^\ell \xrightarrow{C} \mathbb{Z}^\ell) = \mathbb{Z}^\ell/\text{im } (C)$$

This gives a finite group, since avalanche-finite matrices are invertible.

- It turns out that *critical configurations* (those which are stable and recurrent) form a set of coset representatives in $K(C)$.
- Another special set of configurations, the *superstable configurations* also form a set of coset representatives.

Smith normal form basics

Let R be a ring and $A \in R^{n \times n}$ be a matrix. A matrix S is called the *Smith normal form* of A if:

- There exist invertible matrices $P, Q \in R^{n \times n}$ such that $S = PAQ$.
- S is a diagonal matrix $S = \mathrm{diag}(s_1, ..., s_n)$ with $s_i | s_{i+1}$ for $i = 1, ..., n-1$.

Proposition

Let $A \in R^{n \times n}$ be a matrix and suppose A has Smith normal form $S = diag(s_1, ..., s_n)$. Then coker $(A : R^n \to R^n) \cong \bigoplus_{i=1}^{n} R/(s_i)$

Proposition

Let $A \in R^{n \times n}$ be a matrix. If R is a PID then A has a Smith normal form.

Notation

For the rest of the talk:

- G is a finite group
- $1_G = \chi_0, \chi_1, ..., \chi_\ell$ are its irreducible complex characters
- γ is a faithful (not-necessarily-irreducible) n-dimensional representation of G with character χ_γ

McKay-Cartan matrices

Let M be the $(\ell+1) \times (\ell+1)$ integer matrix with entries m_{ij} defined by

$$\chi_\gamma \cdot \chi_i = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

The *extended McKay-Cartan matrix* $\widetilde{C}$ is

$$\widetilde{C} := nI - M$$

and the *McKay-Cartan matrix* C is the $\ell \times \ell$ submatrix formed by removing the row and column corresponding to χ_0 from $\widetilde{C}$. The *critical group* is $K(\gamma) = K(C)$.

Theorem (G. Benkart, C. Klivans, and V. Reiner)

The McKay-Cartan matrix associated to a faithful representation γ is an avalanche-finite matrix.

Example: McKay-Cartan matrix for $\mathfrak{S}_4$

Let $G = \mathfrak{S}_4$ and γ be the reflection representation. This corresponds to the partition $(3, 1)$.

	e	(12)	(123)	(1234)	(12)(34)
χ_0	1	1	1	1	1
$\chi_\gamma = \chi_1$	3	1	0	-1	-1
χ_2	2	0	-1	0	2
χ_3	3	-1	0	1	-1
χ_4	1	-1	1	-1	1

Example: McKay-Cartan matrix for $\mathfrak{S}_4$

Let $G = \mathfrak{S}_4$ and γ be the reflection representation.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \widetilde{C} = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

Example: McKay-Cartan matrix for $\mathfrak{S}_4$

$$C = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \xrightarrow{SNF} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Thus $K(\gamma) = \text{coker}\,(C) \cong \mathbb{Z}/4\mathbb{Z}$.

Theorem 1

Let $e = c_0, c_1, ..., c_\ell$ be a set of conjugacy class representatives for G, then:

i.

$$\prod_{i=1}^{\ell}(n - \chi_\gamma(c_i)) = |K(\gamma)| \cdot |G|$$

ii. *If χ_γ is real-valued, and $\chi_\gamma(c)$ is an integer character value achieved by m different conjugacy classes, then $K(\gamma)$ contains a subgroup isomorphic to $(\mathbb{Z}/(n - \chi_\gamma(c))\mathbb{Z})^{m-1}$.*

Example: checking Theorem 1

We will use the reflection representation of $\mathfrak{S}_4$ again. Recall that $K(\gamma) \cong \mathbb{Z}/4\mathbb{Z}$.

	e	(12)	(123)	(1234)	(12)(34)
$\chi_\gamma = \chi_1$	3	1	0	-1	-1

i. In this case we have

$$\prod_{i=1}^{\ell}(n - \chi_\gamma(c_i)) = 2 \cdot 3 \cdot 4 \cdot 4 = 96$$

$$|K(\gamma)| \cdot |G| = 4 \cdot 4! = 96$$

ii. χ_γ is real-valued and has the repeated character value -1 with multiplicity 2. Thus $K(\gamma)$ should have a subgroup isomorphic to $(\mathbb{Z}/4\mathbb{Z})^1$.

Critical groups for reflection representations of $\mathfrak{S}_n$.

In the previous example, Theorem 1 was enough to uniquely determine $K(\gamma)$. However, this does not happen in general. It turns out we can use the following proposition:

Proposition (A. Miller and V. Reiner)

Suppose an $(\ell+1)\times(\ell+1)$ *integer matrix* A *has a Smith normal form over* $\mathbb{Z}[t]$ *for* $tI - A$*, then this Smith form must be*

$$\begin{pmatrix} s_{\ell+1}(t) & & \\ & \ddots & \\ & & s_1(t) \end{pmatrix}$$

where

$$s_i(t) = \prod_{\substack{\lambda \\ \mu(\lambda)\geq i}} (t-\lambda)$$

and where $\mu(\lambda)$ *denote the dimension of the* λ*-eigenspace for* A*.*

Why is this proposition useful?

- It turns out that $tI - \widetilde{C}$ has a Smith form over $\mathbb{Z}[t]$ when γ is the reflection representation of $\mathfrak{S}_n$.
- Setting $t = 0$ then gives us the critical group $K(\gamma)$.

Two facts

1. For all $\lambda \vdash n$:

$$\chi_{(n-1,1)} \cdot \chi_\lambda = C(\lambda)\chi_\lambda + \sum \chi_\mu$$

   Where the sum is over those partitions μ which can be obtained from λ by removing and then adding a box. And $C(\lambda)$ is one less than the number of corners of λ (Ballantine and Orellana)

2. The map $UD - tI$ in Young's lattice has a Smith normal form over $\mathbb{Z}[t]$ (Cai and Stanley).

Putting it together

Theorem 2

Let γ be the reflection representation of $\mathfrak{S}_n$ and let $\widetilde{C}$ be the associated extended McKay-Cartan matrix. Let $p(k)$ denote the number of partitions of the integer k. Then

$$K(\gamma) \cong \bigoplus_{i=2}^{p(n)-p(n-1)} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{\substack{1 \leq k \leq n \\ p(k)-p(k-1) \geq i}} k$$

Example: checking Theorem 2

k	1	2	3	4	5	6
$p(k)$	1	2	3	5	7	11

For $n = 4$, we have

$$q_2 = \left(\prod_{\substack{1 \leq k \leq 4 \\ p(k) - p(k-1) \geq 2}} k \right) = 4$$

Thus $K(\gamma) \cong \mathbb{Z}/4\mathbb{Z}$.

Example: using Theorem 2

k	1	2	3	4	5	6
$p(k)$	1	2	3	5	7	11

For a more interesting example, let's try the case $n = 6$:

$$q_2 = \left(\prod_{\substack{1 \leq k \leq 6 \\ p(k) - p(k-1) \geq 2}} k \right) = 4 \cdot 5 \cdot 6 = 120$$

$$q_3 = \left(\prod_{\substack{1 \leq k \leq 6 \\ p(k) - p(k-1) \geq 3}} k \right) = 6$$

$$q_4 = 6$$

Thus $K(\gamma) \cong \mathbb{Z}/120\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

Thanks for coming

Are there any questions?

References

C. Ballantine and R. Orellana, A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_{\lambda}$. *Sm. Lothar. Combin.* 54A (2005/07).

G. Benkart, C. Klivans, and V. Reiner, Chip firing on Dynkin diagrams and McKay quivers. arXiv:1601.06849 [math.CO], 2016.

N. Biggs, Algebraic graph theory, 2nd edition. Cambridge University Press, Cambridge, 1993.

T.W.X. Cai and R. Stanley, The Smith normal form of a matrix associated with Young's lattice. *Proc. Amer. Math. Soc.* **143** (2015), no.11.

D. Lorenzini, Smith normal form and Laplacians. *J. of Combinatorial Theory*, Series B 98 (2008), no. 6, 1271-1300.