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Signed Permutation Statistics

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We derive multivariate generating functions that count signed permutations by various statistics, using the hyperoactahedral generalization of methods of Garsia and Gessel. We also derive the distributions over inverse descent classes of signed permutations for two of these statistics individually (the major index and inversion number). These results show that, in contrast to the case for (unsigned) permutations, these two statistics are not generally equidistributed. We also discuss applications to statistics on the wreath product $C_k \wr S_n$ of a cyclic group with the symmetric group.

1. Introduction

There is an abundance of literature on permutations statistics (see [7] for some references) and their joint distributions. Three statistics which have frequently been studied are the *number of descents* $d(\pi)$, the *major index* $\text{maj}(\pi)$, and the *number of inversions* (or *Coxeter group length*) $\text{inv}(\pi)$ of a permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & & \pi_n \end{pmatrix}$$

defined by

$$d(\pi) = \#\{i: 1 \leqslant i \leqslant n-1,\ \pi_i > \pi_{i+1}\},$$

$$\text{maj}(\pi) = \sum_{\substack{\pi_i > \pi_{i+1} \\ 1 \leqslant i \leqslant n-1}} i,$$

$$\text{inv}(\pi) = \#\{(i, j): 1 \leqslant i < j \leqslant n,\ \pi_i > \pi_j\}$$

In [7], Garsia and Gessel use the theory of P-partititons (see [11, Section 4.5]) to derive a multivariate generating function for the joint distribution of

$$(d(\pi), \text{inv}(\pi), \text{maj}(\pi))$$

as π ranges over the the *symmetric group* S_n of all permutations on n elements. They then develop some of the theory of multipartite P-partitions, and use this to derive a generating function for the joint distribution of

$$(d(\pi), d(\pi^{-1}), \text{maj}(\pi), \text{maj}(\pi^{-1}))$$

(see also [4, 8] for some alternative derivations of this distribution).

The goal of this paper is to prove hyperoctahedral generalizations of these and other results. In Section 2, we define the analogues of $\text{d}(\pi)$, $\text{inv}(\pi)$, $\text{maj}(\pi)$, and a new statistic $n(\pi)$, where π is an element of the *hyperoctahedral group* B_n of signed permutations. We then review the definitions and facts needed from the theory of P-partitions and multipartite P-partitions for *signed posets P*.

In Sections 3 and 4, we use these to derive the two main results: the joint distributions of

$$(d(\pi), \text{inv}(\pi), \text{maj}(\pi), n(\pi))$$

and

$$(d(\pi), d(\pi^{-1}), \text{maj}(\pi), \text{maj}(\pi^{-1}), n(\pi))$$

as π ranges over B_n.

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Section 5 deals with the distribution of $(n(\pi), \operatorname{maj}(\pi))$ and of $(n(\pi), \operatorname{inv}(\pi))$ over *inverse descent classes* in B_n (to be defined later). Here the situation contrasts with that of S_n, where $\operatorname{maj}(\pi)$ and $\operatorname{inv}(\pi)$ have identical distributions as π ranges over a fixed inverse descent class (see [6]). Our results do not indicate any simple relation between their analogous distributions in B_n. However, there *is* a simple relation between their distributions over all of B_n, and in Section 6 we give a bijective proof of this utilizing a well-known bijection of Foata [5].

By setting the variable a which counts the new statistic $n(\pi)$ equal to zero, one immediately recovers the known analogues of all of these results for the symmetric group S_n. In Section 7, we show that by setting $a = k-1$, we obtain new results on the distribution of certain statistics on the wreath product $C_k \wr S_n$ of a cyclic group with the symmetric group S_n.

2. Preliminaries

Let B_n denote the *hyperoctahedral group* of all permutations and sign changes of the co-ordinates in $\mathbf{R}^n$. Letting e_i denote the ith standard basis vector, we will use the two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & & \pi_n \end{pmatrix}$$

where $\pi_i \in \{\pm 1, \ldots, \pm n\}$ to mean that $\pi(e_i) = \operatorname{sgn}(\pi_i) e_{|\pi_i|}$. It will be convenient for us to think of B_n as a *Coxeter group* with *root system* Φ_n, *positive roots* Φ_n^+, and *simple roots* Π_n given by

$$\begin{aligned} \Phi_n &= \{\pm e_i, \pm e_i \pm e_j : 1 \leq i \neq j \leq n\}, \\ \Phi_n^+ &= \{+e_k, +e_i + e_j, +e_i - e_j : 1 \leq k \leq n, i < j \leq n\}, \\ \Pi_n &= \{+e_n, +e_i - e_{i+1} : 1 \leq i < n\}, \end{aligned}$$

where we label the simple roots $\alpha_1, \ldots, \alpha_n$ as follows: $\alpha_i = +e_i - e_{i+1}$ if $i < n$, and $\alpha_n = +e_n$ (see [2] for terminology and background about Coexeter groups and root systems). By analogy with the symmetric group, the *descent set* of π is defined to be

$$D(\pi) = \{i : \pi(\alpha_i) \in -\Phi_n^+\},$$

the *number of descents* is

$$d(\pi) = \# D(\pi)$$

and the *major index* is

$$\operatorname{maj}(\pi) = \sum_{i \in D(\pi)} i.$$

Also by analogy with S_n, the *number of inversions* (or *Coxeter group length*) is defined by

$$\operatorname{inv}(\pi) = \#\{\alpha \in \Phi_n^+ : \pi^{-1}(\alpha) \in -\Phi_n^+\}.$$

One further statistic that we will use is the *number of negative signs*, defined by

$$n(\pi) = \#\{i : \pi_i < 0\}.$$

For example, if

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & +2 & +5 & -1 & -4 \end{pmatrix}$$

then $D(\pi)=\{1,4,5\}$, so $d(\pi)=3$, $\text{maj}(\pi)=1+4+5=10$, $n(\pi)=3$, and one can check that $\text{inv}(\pi)=14$. Note that a signed permutation $\pi\in B_n$ has $n(\pi)=0$ if and only if π is actually just a permutation in S_n.

We now recall some definitions and results from [9, 10]. A (*natural*) *signed poset* P is a subset $P\subseteq\Phi_n^+$ satisfying the following closure property: if $\alpha,\beta\in P$ and $a\alpha+b\beta\in\Phi_n$ for some $a,b>0$, then $a\alpha+b\beta\in P$. Given a subset $A\subseteq\Phi_n^+$, the *signed poset generated by* A is the smallest set P containing A which has the above closure property. Given P, a signed poset, say a vector $f\in\mathbf{Z}^n$, is a *P-partition* if $\langle\alpha,f\rangle\geq 0$ for all $\alpha\in P$ (where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbf{R}^n$). Denote by $\mathscr{A}(P)$ the set of all P-partitions, and define the *Jordan–Hölder set* of P by

$$\mathscr{L}(P)=\{\pi\in B_n: P\subseteq\pi\Phi_n^+\}.$$

The main result about P-partitions is then as follows:

THEOREM 2.1 [10, *Theorem* 3.3].

$$\mathscr{A}(P)=\coprod_{\pi\in\mathscr{L}(P)}\mathscr{A}(\pi\Phi_n^+).$$

where $\coprod$ *denotes the disjoint union of sets.*

By chasing through the definitions, one obtains the following more concrete description of the sets $\mathscr{A}(\pi\Phi_n^+)$:

PROPOSITION 2.2 [10, Proposition 3.4]. *$\mathscr{A}(\pi\Phi_n^+)$ is the set all vectors $(f_1,\ldots,f_n)\in\mathbf{Z}^n$ such that* $\text{sgn}(f_i)=\text{sgn}(\pi_i)$ *and*

$$|f_{\pi_1}|\sim_1|f_{\pi_2}|\sim_2\cdots|f_{\pi_n}|\sim_n 0,$$

where $\sim_i$ is the relation $>$ if $i\in D(\pi)$ and $\sim_i$ is $\geq$ otherwise.

As an example, let $n=2$ and $P=\{+e_1,+e_1-e_2\}$. Then

$$\mathscr{L}(P)=\left\{\begin{pmatrix}1&2\\+1&+2\end{pmatrix},\begin{pmatrix}1&2\\+1&-2\end{pmatrix},\begin{pmatrix}1&2\\+2&-1\end{pmatrix}\right\},$$

and

$$\begin{aligned}\mathscr{A}(P)&=\{(f_1,f_2)\in\mathbf{Z}^2: f_1\geq 0, f_1\geq f_2\}\qquad\text{by definition}\\&=\{f_1\geq f_2\geq 0\}\\&\quad\amalg\{f_1\geq|f_2|>0, f_2<0\}\\&\quad\amalg\{|f_2|>f_1\geq 0, f_2<0\}\qquad\text{by the last two results.}\end{aligned}$$

To deal with bipartite P-partitions, we introduce the lexicographic order on $\mathbf{R}^2$:

$$\begin{pmatrix}f_1\\g_1\end{pmatrix}\geq_L\begin{pmatrix}f_2\\g_2\end{pmatrix}\qquad\text{if we have (1) } f_1\geq f_2 \text{ and (2) } f_1=f_2 \text{ implies } g_1\geq g_2.$$

One then defines the set of bipartite P-partitions by

$$\mathscr{A}_2(P)=\left\{\begin{pmatrix}f\\g\end{pmatrix}\in(\mathbf{Z}^n)^2:\begin{pmatrix}\langle\alpha,f\rangle\\\langle\alpha,g\rangle\end{pmatrix}\geq_L\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

The main result required about bipartite P-partitions is the following:

THEOREM 2.3 [9, *Proposition* 3.4.2, *Lemma* 3.4.3].

$$\mathscr{A}_2(P)=\coprod_{\substack{(\pi_1,\pi_2)\in(B_n)^2\\\pi_2\pi_1\in\mathscr{L}(P)}}\left\{\begin{pmatrix}f\\g\end{pmatrix}\in(\mathbf{Z}^n)^2: g\in\mathscr{A}(\pi_2\Phi_n^+),\ \pi_2^{-1}(f)\in\mathscr{A}(\pi_1\Phi_n^+)\right\}.$$

We will also use the following notations: $[n]$ denotes the set $\{1,2,\ldots,n\}$. If $F(x)$ is a formal power series in x, then $F(x)|_{x\to q}$ denotes the result of replacing x by q. We further define the following:

$$[n]_p = 1(1+p)(1+p+p^2)\cdots(1+p+p^2+\cdots+p^{n-1}),$$
$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$
$$(a;q_1,q_2)_{r,s} = \prod_{\substack{0\le i\le r-1\\0\le j\le s-1}} (1-aq_1^iq_2^j),$$
$$[n]!_p = [1]_p[2]_p\cdots[n]_p,$$
$$[\hat{n}]!_{a,p} = (-ap;p)_n[n]!_p,$$
$$\binom{n}{k}_p = \frac{[n]!_p}{[k]!_p[n-k]!_p},$$
$$\binom{\hat{n}}{k}_{a,p} = \frac{[\hat{n}]!_{a,p}}{[k]!_p[\widehat{n-k}]!_{a,p}},$$
$$\binom{n}{k_1\cdots k_r}_p = \frac{[n]!_p}{[k_1]!_p[k_2]!_p\cdots[k_r]!_p},$$
$$\binom{\hat{n}}{\hat{k}_0k_1\cdots k_r}_{a,p} = \frac{[\hat{n}]!_{a,p}}{[\hat{k}_0]!_{a,p}[k_1]!_p\cdots[k_r]!_p}.$$

3. The Four-variate Distribution of $(d(\pi), \mathrm{inv}(\pi), \mathrm{maj}(\pi), n(\pi))$

Following [7], we will count vectors $f\in\mathbf{Z}^n$ according to some statistics in two ways: one direct, the other a 'Mahonian' count that involves P-partitions. Equating these two expressions will then yield the result.

For $f\in\mathbf{Z}^n$, define

$$\max(f) = \max\{|f_i|\}_{i=1,\ldots,n}, \qquad |f| = \sum_{i=1}^n |f_i|, \qquad n(f) = \#\{i: f_i<0\}.$$

This gives three of the statistics. By Theorem 2.1, there is a unique $\pi\in B_n$ such that $f\in\mathscr{A}(\pi\Phi_n^+)$, which we will call $\pi(f)$. We then define a fourth statistic by $\mathrm{inv}(f) = \mathrm{inv}(\pi(f))$ (one can give a more direct, but clumsier definition of $\mathrm{inv}(f)$). Note that by Proposition 2.2, one could equivalently define $n(f)=n(\pi(f))$.

We now set about calculating

$$\sum_{f\in\mathbf{Z}^n} t^{\max(f)}q^{|f|}p^{\mathrm{inv}(f)}a^{n(f)}$$

in two ways.

For the direct approach, we begin by noting that

$$\sum_{f\in\mathbf{Z}^n} t^{\max(f)}q^{|f|}p^{\mathrm{inv}(f)}a^{n(f)} = (1-t)\sum_{k\ge0} t^k \sum_{\substack{f\in\mathbf{Z}^n\\\max(f)\le k}} q^{|f|}p^{\mathrm{inv}(f)}a^{n(f)}. \tag{1}$$

If we let $\omega(f) = x_{|f_1|}\cdots x_{|f_n|}$, then $q^{|f|} = \omega(f)|_{x_i\to q^i}$. Therefore one has

$$\sum_{\substack{f\in\mathbf{Z}^n\\\max(f)\le k}} q^{|f|}p^{\mathrm{inv}(f)}a^{n(f)} = \sum_{\substack{f\in\mathbf{Z}^n\\\max(f)\le k}} \omega(f)p^{\mathrm{inv}(f)}a^{n(f)}\Big|_{x_i\to q^i}$$
$$= \sum_{\mu_0+\cdots+\mu_k=n} x_0^{\mu_0}\cdots x_k^{\mu_k} \sum_{f\in\mathbf{Z}^n,\ \#\{i:\,|f_i|=j\}=\mu_j} p^{\mathrm{inv}(f)}a^{n(f)}\Big|_{x_i\to q^i}.$$

LEMMA 3.1.

$$\sum_{f\in\mathbf{Z}^n,\ \#\{i:\,|f_i|=j\}=\mu_j} p^{\mathrm{inv}(f)}a^{n(f)} = \binom{\hat{n}}{\hat{\mu}_0\mu_1\cdots\mu_k}_{a,p} = \frac{[\hat{n}]!_{a,p}}{[\hat{\mu}_0]!_{a,p}[\mu_1]!_p\cdots[\mu_k]!_p}.$$

PROOF. Let $S=\{\mu_k,\mu_k+\mu_{k-1},\ldots,\mu_k+\cdots+\mu_1\}\subseteq[n]$. It is not hard to see from Proposition 2.2 that the map $f\mapsto\pi(f)$ gives a bijection between $\{f\in\mathbf{Z}^n,\ \#\{i:|f_i|=j\}=\mu_j\}$ and $\{\pi\in B_n: D(\pi)\subseteq S\}$, the inverse of which may be described as follows: if we let $(s_1,\ldots,s_n)$ be the sequence which has μ_k occurrences of k followed by μ_{k-1} occurrences of $k-1$, etc., then the inverse map sends π to $(f_1,\ldots,f_n)$, where $f_i=\mathrm{sgn}(\pi_i)s_{|\pi_i|}$. Thus we have

$$\sum_{f\in\mathbf{Z}^n,\ \#\{i:\,|f_i|=j\}=\mu_j} p^{\mathrm{inv}(f)}a^{n(f)} = \sum_{D(\pi)\subseteq S} p^{\mathrm{inv}(\pi)}a^{n(\pi)}. \tag{2}$$

Let W_S denote the *parabolic subgroup* of B_n (see [2]) which is isomorphic to $S_{\mu_k}\times S_{\mu_{k-1}}\times B_{\mu_0}$ and has S_{μ_k} acting on the first μ_k co-ordinates, $S_{\mu_{k-1}}$ acting on the next μ_{k-1} co-ordinates, etc. It is known [2], Chapter 4, Section 1, Example 3] that any $\sigma\in B_n$ can be written uniquely as $\sigma=\pi\tau$, where $D(\pi)\subseteq S$ and $\tau\in W_S$ and, furthermore $\mathrm{inv}(\sigma)=\mathrm{inv}(\pi)+\mathrm{inv}(\tau)$. This decomposition also has the easily verified property that $n(\sigma)=n(\pi)+n(\tau)$. Hence we have,

$$\sum_{D(\pi)\subseteq S} p^{\mathrm{inv}(\pi)}a^{n(\pi)} \sum_{\tau\in W_S} p^{\mathrm{inv}(\tau)}a^{n(\tau)} = \sum_{\sigma\in B_n} p^{\mathrm{inv}(\sigma)}a^{n(\sigma)}$$

and therefore

$$\sum_{D(\pi)\subseteq S} p^{\mathrm{inv}(\pi)}a^{n(\pi)} = \frac{\sum_{\sigma\in B_n} p^{\mathrm{inv}(\sigma)}a^{n(\sigma)}}{\sum_{\tau\in B_{\mu_0}} p^{\mathrm{inv}(\tau)}a^{n(\tau)} \sum_{\tau\in S_{\mu_1}} p^{\mathrm{inv}(\tau)}\cdots\sum_{\tau\in S_{\mu_k}} p^{\mathrm{inv}(\tau)}}.$$

The result will then follow once we verify that

$$\sum_{\tau\in S_n} p^{\mathrm{inv}(\tau)} = [n]!_p$$

and

$$\sum_{\tau\in B_n} p^{\mathrm{inv}(\tau)}a^{n(\tau)} = [\hat{n}]!_{a,p}.$$

The first of these is well known (see, e.g., [11, Corollary 1.3.10]). The second fact will be proven in Section 6, but we remark that it follows from the first by considering S_n as a parabolic subgroup of B_n. One need only check that

$$\sum_{\substack{\pi\in B_n\\ D(\pi)\subseteq\{n\}}} p^{\mathrm{inv}(\pi)}a^{n(\pi)} = (-ap;p)_n.$$

Verification of this is left to the reader. □

Continuing the derivation, from the lemma we have that

$$\begin{aligned}\sum_{\substack{f\in\mathbf{Z}^n\\ \max(f)\leq k}} q^{|f|}p^{\mathrm{inv}(f)}a^{n(f)} &= \sum_{\mu_0+\cdots+\mu_k=n} \frac{[\hat{n}]!_{a,p}}{[\hat{\mu}_0]!_{a,p}[\mu_1]!_p\cdots[\mu_k]!_p} x_0^{\mu_0}\cdots x_k^{\mu_k}\Big|_{x_i\to q^i}\\ &= [\hat{n}]!_{a,p}\sum_{\mu_0+\cdots+\mu_k=n}\frac{(q^0)^{\mu_0}}{[\hat{\mu}_0]!_{a,p}}\frac{(q^1)^{\mu_1}}{[\mu_1]!_p}\cdots\frac{(q^k)^{\mu_k}}{[\mu_k]!_p}\\ &= [\hat{n}]!_{a,p}\cdot\text{coefficient of } u^n \text{ in } \hat{e}[u]_{a,p}e[qu]_p\cdots e[q^ku]_p,\end{aligned}$$

where

$$e[u]_p = \sum_{n\geq 0}\frac{u^n}{[n]!_p}$$

and

$$\hat{e}[u]_{a,p} = \sum_{n\geq 0} \frac{u^n}{[\hat{n}]!_{a,p}}.$$

Combining this with equation (1) gives an expression for our direct count:

$$\frac{1}{[\hat{n}]!_{a,p}(1-t)} \sum_{f\in \mathbf{Z}^n} t^{\max(f)} q^{|f|} p^{\mathrm{inv}(f)} a^{n(f)}$$
$$= \text{coefficient of } u^n \text{ in } \sum_{k\geq 0} t^k \hat{e}[u]_{a,p} e[qu]_p \cdots e[q^k u]_p. \quad (3)$$

We now embark on the Mahonian approach. We will apply Theorem 2.1 to the case in which $P = \varnothing$, the *empty* signed poset, and hence $\mathscr{L}(P) = B_n$. This gives

$$\mathbf{Z}^n = \coprod_{\pi\in B_n} \mathscr{A}(\pi\Phi_n^+)$$

and hence that

$$\sum_{f\in\mathbf{Z}^n} t^{\max(f)} q^{|f|} p^{\mathrm{inv}(f)} a^{n(f)} = \sum_{\pi\in B_n} \sum_{f\in\mathscr{A}(\pi\Phi_n^+)} t^{\max(f)} q^{|f|} p^{\mathrm{inv}(f)} a^{n(f)}$$
$$= \sum_{\pi\in B_n} p^{\mathrm{inv}(\pi)} a^{n(\pi)} \sum_{f\in\mathscr{A}(\pi\Phi_n^+)} t^{\max(f)} q^{|f|}.$$

For a fixed π, by Proposition 2.2 we have $f \in \mathscr{A}(\pi\Phi_n^+)$ iff $\mathrm{sgn}(f_i) = \mathrm{sgn}(\pi_i)$ and

$$|f_{|\pi_1|}| \geq \cdots \geq |f_{|\pi_n|}| \geq 0,$$

with the ith occurrence of $\geq$ replaced by $>$ whenever $i \in D(\pi)$. Thus we may bijectively encode any such f as a partition $\lambda = (\lambda_1 \geq \cdots \lambda_n \geq 0)$ with at most n parts by setting

$$\lambda_i = |f_{|\pi_i|}| - \#\{j \in D(\pi): j \geq i\}.$$

The key properties of this encoding are that $\max(f) = \max(\lambda) + d(\pi)$ and $|f| = |\lambda| + \mathrm{maj}(\pi)$. We conclude that

$$\sum_{f\in\mathscr{A}(\pi\Phi_n^+)} t^{\max(f)} q^{|f|} = t^{d(\pi)} q^{\mathrm{maj}(\pi)} \sum_{\lambda \text{ of length}\leq n} t^{\max(\lambda)} q^{|\lambda|}$$
$$= \frac{t^{d(\pi)} q^{\mathrm{maj}(\pi)}}{(tq;q)_n},$$

where the last equality is a standard argument in partitions. This gives us that

$$\sum_{f\in\mathbf{Z}^n} t^{\max(f)} q^{|f|} p^{\mathrm{inv}(f)} a^{n(f)} = \sum_{\pi\in B_n} \frac{t^{d(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{inv}(\pi)} a^{n(\pi)}}{(tq;q)_n}.$$

Combining this with equation (3) yields the final answer:

$$\sum_{n\geq 0} \frac{u^n}{(t;q)_{n+1}[\hat{n}]!_{a,p}} \sum_{\pi\in B_n} t^{d(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{inv}(\pi)} a^{n(\pi)} = \sum_{k\geq 0} t^k \hat{e}[u]_{a,p} e[qu]_p \cdots e[q^k u]_p.$$

4. The Five-variate Distribution of $(d(\pi), d(\pi^{-1}), \mathrm{maj}(\pi), \mathrm{maj}(\pi^{-1}), n(\pi))$

Again we follow [7] and count in two ways the expression

$$\sum_{\binom{f}{g}\in\mathscr{A}_2(\Phi_n^+)} t_1^{\max(f)} t_2^{\max(g)} q_1^{|f|} q_2^{|f|} a^{n(g)}.$$

Using the direct approach, first rewrite the above sum as

$$(1-t_1)(1-t_2)\sum_{k_1\geq 0,\,k_2\geq 0} t_1^{k_1}t_2^{k_2} \sum_{\substack{\binom{f}{g}\in\mathscr{A}_2(\Phi_n^+)\\ \max(f)\leq k_1,\,\max(g)\leq k_2}} q_1^{|f|}q_2^{|f|}a^{n(g)}.$$

Next note that by definition, $\binom{f}{g}\in\mathscr{A}_2(\Phi_n^+)$ iff

$$\binom{f_1}{g_1}\geq_L\cdots\geq_L\binom{f_n}{g_n}\geq_L\binom{0}{0}$$

and hence is completely determined by the multiset

$$\left\{\binom{f_i}{g_i}\right\}_{i=1,\ldots,n}.$$

We conclude that

$$\sum_{\substack{\binom{f}{g}\in\mathscr{A}_2(\Phi_n^+)\\ \max(f)\leq k_1,\max(g)\leq k_2}} q_1^{|f|}q_2^{|f|}a^{n(g)}$$

is equal to the coefficient of u^n in

$$\prod_{\substack{0\leq i\leq k_1\\ 0\leq j\leq k_2}}\frac{1}{1-uq_1^{|i|}q_2^{|j|}}\prod_{\substack{1\leq i\leq k_1\\ 1\leq -j\leq k_2}}\frac{1}{1-auq_1^{|i|}q_2^{|j|}}$$

and finally that

$$\sum_{\binom{f}{g}\in\mathscr{A}_2(\Phi_n^+)} t_1^{\max(f)}t_2^{\max(g)}q_1^{|f|}q_2^{|g|}a^{n(g)}$$

is equal to the coefficient of u^n in

$$(1-t_1)(1-t_2)\sum_{k_1\geq 0,\,k_2\geq 0}\frac{t_1^{k_1}t_2^{k_2}}{(u;q_1,q_2)_{k_1+1,\,k_2+1}(auq_1q_2;q_1,q_2)_{k_1,k_2}}.$$

For the Mahonian count, we apply Theorem 2.3. Since $\mathscr{L}(\Phi_n^+)=\{1\}$, we have

$$\mathscr{A}_2(\Phi_n^+)=\coprod_{\substack{\pi_1,\pi_2\in B_n\\ \pi_2\pi_1=1}}\left\{\binom{f}{g}: g\in\mathscr{A}(\pi_2\Phi_n^+),\ \pi_2^{-1}(f)\in\mathscr{A}(\pi_1\Phi_n^+)\right\}$$

and therefore

$$\begin{aligned}
&\sum_{\binom{f}{g}\in\mathscr{A}_2(\Phi_n^+)} t_1^{\max(f)}t_2^{\max(g)}q_1^{|f|}q_2^{|g|}a^{n(g)}\\
&=\sum_{\substack{\pi_1,\pi_2\in B_n\\ \pi_2\pi_1=1}}\ \sum_{\substack{g\in\mathscr{A}(\pi_2\Phi_n^+)\\ \pi_2^{-1}(f)\in\mathscr{A}(\pi_1\Phi_n^+)}} t_1^{\max(f)}t_2^{\max(g)}q_1^{|f|}q_2^{|g|}a^{n(g)}\\
&=\sum_{\pi\in B_n}\ \sum_{\pi(f)\in\mathscr{A}(\pi\Phi_n^+)} t_1^{\max(f)}q_1^{|f|}\sum_{g\in\mathscr{A}(\pi^{-1}\Phi_n^+)} t_2^{\max(g)}q_2^{|g|}a^{n(g)}\\
&=\sum_{\pi\in B_n}\frac{t_1^{d(\pi)}q_1^{\mathrm{maj}(\pi)}}{(t_1q_1;q_1)_n}\frac{t_2^{d(\pi^{-1})}q_2^{\mathrm{maj}(\pi^{-1})}q^{n(\pi)}}{(t_2q_1;q_2)_n},
\end{aligned}$$

where the last equality follows from reasoning similar to the Mahonian count in Section 3. Equating this expression with the direct count yields the final answer:

$$\sum_{n\geq 0}\frac{u^n}{(t_1;q)_{n+1}(t_2;q_2)_{n+1}}\sum_{\pi\in B_n} t_1^{d(\pi)}t_2^{d(\pi^{-1})}q_1^{\mathrm{maj}(\pi)}q_2^{\mathrm{maj}(\pi^{-1})}a^{n(\pi)}$$

$$=\sum_{k_1\geq 0,\,k_2\geq 0}\frac{t_1^{k_1}t_2^{k_2}}{(u;q_1,q_2)_{k_1+1,k_2+1}(auq_1q_2;q_1,q_2)_{k_1,k_2}}.$$

5. Distributions over Inverse Descent Classes

Given $S \subseteq [n]$, the set

$$\mathcal{D}_S = \{\pi \in B_n \colon D(\pi^{-1}) = S\}$$

is called an *inverse descent class*. For S_n, it is known that maj and inv are equidistributed over inverse descent classes (see [6] for a bijective proof), and that this distribution may be expressed neatly as a determinant. In this section we derive similar determinantal expressions for the distribution of maj and inv over $\mathcal{D}_s$ which show that they are *not* in general equidistributed.

For the remainder of this section, we adopt the following convention: if $S = \{s_1, \ldots, s_r\}$, then $s_0 = 0$.

Proposition 5.1. *Let*

$$\alpha^{\mathrm{inv}}(S) = \sum_{D(\pi^{-1}) \subseteq S} p^{\mathrm{inv}(\pi)} a^{n(\pi)}.$$

Then

$$\alpha^{\mathrm{inv}}(S) = \binom{\hat{n}}{s_1 - s_0 \quad s_2 - s_1 \cdots s_r - s_{r-1} \quad \widehat{n - s_r}}_{a,p}$$

$$= \binom{\widehat{n - s_0}}{s_1 - s_0}_{a,p} \binom{\widehat{n - s_1}}{s_2 - s_1}_{a,p} \cdots \binom{\widehat{n - s_{r-1}}}{s_r - s_r - 1}_{a,p}.$$

Proof. The third expression above is easily seen to be equal to the second, so we only need to show that the first two are equal. In the proof of Lemma 3.1, we showed that the second is the same as

$$\sum_{D(\pi) \subseteq S} p^{\mathrm{inv}(\pi)} a^{n(\pi)}.$$

But $\mathrm{inv}(\pi) = \mathrm{inv}(\pi^{-1})$ and $n(\pi) = n(\pi^{-1})$, so the latter is equal to

$$\sum_{D(\pi^{-1}) \subseteq S} p^{\mathrm{inv}(\pi)} a^{n(\pi)}.$$ □

Corollary 5.2. *Let*

$$\beta^{\mathrm{inv}}(S) = \sum_{\pi \in D_S} p^{\mathrm{inv}(\pi)} a^{n(\pi)}.$$

Then

$$\beta^{\mathrm{inv}}(S) = [\hat{n}]!_{a,p} \det(a(i, j)) = \det(b(i, j)),$$

where the $(r+1) \times (r+1)$ *matrices* $(a(i, j))$, $(b(i, j))$ *are defined by*

$$a(i, j) = \begin{cases} \dfrac{1}{[s_i - s_{j-1}]!_p} & \text{if } i \leq r, \\[2ex] \dfrac{1}{[\widehat{n - s_{j-1}}]!_{a,p}} & \text{if } i = r + 1, \end{cases}$$

$$b(i, j) = \begin{cases} \dbinom{\widehat{n - s_{j-1}}}{s_i - s_{j-1}}_{a,p} & \text{if } i \leq r, \\[2ex] 1 & \text{if } i = r + 1. \end{cases}$$

PROOF. Since

$$\alpha^{\text{inv}}(S) = \sum_{T \subseteq S} \beta^{\text{inv}}(T),$$

by the principle of inclusion–exclusion, we have

$$\beta^{\text{inv}}(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha^{\text{inv}}(T).$$

Using the two expressions for $\alpha^{\text{inv}}(S)$ given by Proposition 5.1, it is a standard argument [11, Proposition 2.2.6] to deduce from this the two determinantal expressions for $\beta^{\text{inv}}(S)$. □

For the statistic maj, we use the theory of P-partitions. Let P_n be the signed poset generated by

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} = \{+e_i - e_{i+1}\}_{i=1}^{n-1},$$

and let P_S be the signed poset generated by $\{\alpha_i \in \Pi_n : i \notin S\}$. Again following [7], we will count $\sum_{f \in \mathscr{A}(P_S)} q^{|f|} a^{n(f)}$ in two ways. By Theorem 2.1,

$$\sum_{f \in \mathscr{A}(P_S)} q^{|f|} a^{n(f)} = \sum_{\pi \in \mathscr{L}(P_S)} \sum_{f \in \mathscr{A}(\pi \Phi_n^+)} q^{|f|} a^{n(f)} = \sum_{\pi \in \mathscr{L}(P_S)} \frac{q^{\text{maj}(\pi)} a^{n(\pi)}}{(q;q)_n},$$

where the second equality follows by the same argument as in Section 3. Now notice that

$$\begin{aligned} \pi \in \mathscr{L}(P_S) &\Leftrightarrow P_S \subseteq \pi \Phi_n^+ \\ &\Leftrightarrow \pi^{-1}(P_S) \subseteq \Phi_n^+ \\ &\Leftrightarrow D(\pi^{-1}) \subseteq S \end{aligned}$$

and so we have our Mahonian count:

$$\sum_{f \in \mathscr{A}(P_S)} q^{|f|} a^{n(f)} = \sum_{\pi \in \mathscr{L}(P_S)} \sum_{f \in \mathscr{A}(\pi \Phi_n^+)} q^{|f|} a^{n(f)} = \sum_{D(\pi^{-1}) \subseteq S} \frac{q^{\text{maj}(\pi)} a^{n(\pi)}}{(q;q)_n}.$$

For the direct count, notice that $f \in \mathbf{Z}^n$ is a P_S-partition iff its first s_1 co-ordinates are in decreasing order, its next $s_2 - s_1$ co-ordinates are in decreasing order, etc., and its last $n - s_r$ co-ordinates are *non-negative* and in decreasing order. This may be rephrased as saying that its first s_1 co-ordinates form a P_{s_1}-partition, its next $s_2 - s_1$ co-ordinates form a $P_{s_2 - s_1}$-partition, etc., and its last $n - s_r$ co-ordinates form a $\Phi_{n-s_r}^+$-partition. Therefore,

$$\sum_{f \in \mathscr{A}(P_S)} q^{|f|} a^{n(f)} = \sum_{f \in \mathscr{A}(P_{s_1 - s_0})} q^{|f|} a^{n(f)} \cdots \sum_{f \in \mathscr{A}(P_{s_r - s_{r-1}})} q^{|f|} a^{n(f)} \sum_{f \in \mathscr{A}(\Phi_{n-s_r}^+)} q^{|f|} a^{n(f)}.$$

If we let

$$M_n(a, q) = \sum_{\pi \in \mathscr{L}(P_n)} q^{\text{maj}(\pi)} a^{n(\pi)},$$

the usual argument allows us to rewrite this as

$$\sum_{f \in \mathscr{A}(P_S)} q^{|f|} a^{n(f)} = \frac{M_{s_1 - s_0}(a, q)}{(q;q)_{s_1 - s_0}} \cdots \frac{M_{s_r - s_{r-1}}(a, q)}{(q;q)_{s_r - s_{r-1}}} \frac{1}{(q;q)_{n-s_r}},$$

since $\mathscr{L}(\Phi^+_{n-s_r}) = \{1\}$. Equating this with the Mahonian count gives the following:

THEOREM 5.3. *Let*

$$\alpha^{\mathrm{maj}}(S) = \sum_{D(\pi^{-1}) \subseteq S} q^{\mathrm{maj}(\pi)}.$$

Then

$$\begin{aligned}\alpha^{\mathrm{maj}}(S) &= [n]!_q \frac{M_{s_1-s_0}(a,q)}{[s_1-s_0]!_q} \cdots \frac{M_{s_r-s_{r-1}}(a,q)}{[s_r-s_{r-1}]!_q} \cdot \frac{1}{[n-s_r]!_q} \\ &= M_{s_1-s_0}(a,q)\binom{n-s_0}{s_1-s_0}_q \cdots M_{s_r-s_{r-1}}(a,q)\binom{n-s_{r-1}}{s_r-s_{r-1}}_q \\ &= M_{s_1-s_0}(a,q) \cdots M_{s_r-s_{r-1}}(a,q)\binom{n}{s_1-s_0 \cdots s_r-s_{r-1}\, n-s_r}.\end{aligned}$$

COROLLARY 5.4. *Let* $\beta^{\mathrm{maj}}(S) = \sum_{\pi \in \mathscr{D}_S} q^{\mathrm{maj}(\pi)}$. *Then*

$$\beta^{\mathrm{inv}}(S) = [n]!_q \det(a'(i,j)) = \det(b'(i,j)),$$

where the $(r+1) \times (r+1)$ *matrices* $(a'(i,j))$, $(b'(i,j))$ *are defined by*

$$a'(i,j) = \begin{cases} \dfrac{M_{s_i-s_{j-1}}(a,q)}{[s_i-s_{j-1}]!q} & \text{if } i \leq r, \\ \dfrac{1}{[n-s_{j-1}]!_q} & \text{if } i = r+1, \end{cases}$$

$$b'(i,j) = \begin{cases} M_{s_i-s_{j-1}}(a,q)\dbinom{n-s_{j-1}}{s_i-s_{j-1}}_p & \text{if } i \leq r, \\ 1 & \text{if } i = r+1. \end{cases}$$

PROOF. See the proof of Corollary 5.2. □

In the light of these last two results, it is useful to have more explicit expressions for the polynomials $M_n(a, q)$. The author is grateful to D. Foata for pointing out that the next proposition implies that $M_n(a, q)$ is the same as the *Rogers–Szegö polynomial* $H_n(aq)$, about which a great deal is known (see [1, p. 49, Problems 3–9]).

PROPOSITION 5.5.

$$\sum_{n \geq 0} \frac{M_n(a,q)x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty (axq;q)_\infty}.$$

PROOF. We know from our previous calculations that

$$M_n(a,q)/(q;q)_n = \sum_{f \in \mathscr{A}(P_n)} q^{|f|} a^{n(f)},$$

and hence that the left-hand side in the proposition is

$$\sum_{n \geq 0} \sum_{f \in \mathscr{A}(P_n)} q^{|f|} a^{n(f)} x^n.$$

Recall also that $f = (f_1, \ldots, f_n) \in \mathscr{A}(P_n)$ iff the f_is are in decreasing order. Hence this last expression is equal to

$$\sum_{n \geq 0} \sum_{\substack{(f_1,\ldots,f_n) \in \mathbf{Z}^n \\ f_1 \geq \cdots \geq f_n}} q^{|f_1| + \cdots + |f_n|} a^{\#\{i : f_i < 0\}} x^n \prod_{i \geq 0} (1 - xq^{|i|})^{-1} \prod_{i < 0} (1 - axq^{|i|})^{-1}.$$ □

6. A Bijection Relating $\operatorname{maj}(\pi)$ and $\operatorname{inv}(\pi)$

By taking $S=[n]$ in Proposition 5.1 and Theorem 5.3, we have

$$\sum_{\pi\in B_n} p^{\operatorname{inv}(\pi)}a^{n(\pi)}=[\hat{n}]!_{a,p}=(-ap;p)_n[n]!_p \tag{4}$$

and

$$\sum_{\pi\in B_n} p^{\operatorname{maj}(\pi)}a^{n(\pi)}=M_1(a,p)^n[n]!_p=(1+ap)^n[n]!_p \tag{5}$$

and hence we have the following relation between the distributions of l and maj over all of B_n:

$$(-ap;p)_n\sum_{\pi\in B_n} p^{\operatorname{maj}(\pi)}a^{n(\pi)}=(1+ap)^n\sum_{\pi\in B_n} p^{\operatorname{inv}(\pi)}a^{n(\pi)}. \tag{6}$$

The objective of this section is to sketch a bijective proof of (6). Our approach will be to prove (4) and (5) using bijections that relate maj and l in B_n to maj and l in S_n, respectively, and then combine these with Foata's bijection ϕ [5] which relates maj to l in S_n. It would be interesting to find a more direct bijection, as this might also shed some light on the relation between $\alpha^{\operatorname{inv}}(S)$ and $\alpha^{\operatorname{maj}}(S)$ or $\beta^{\operatorname{inv}}(S)$ and $\beta^{\operatorname{maj}}(S)$ in general.

To prove (4) bijectively, on the right-hand side we interpret $(-ap;p)_n$ as counting subsets T of $[n]$ jointly according to their cardinality and the statistic $\operatorname{maj}(T)=\sum_{i\in T} i$, and we interpret $[n]!_p$ as $\sum_{\pi\in S_n} p^{\operatorname{inv}(\pi)}$. Hence (4) will follow if we can establish a bijection $\delta\colon B_n\to 2^{[n]}\times S_n$ such that if $\delta(\pi)=(T,\sigma)$ then $\operatorname{inv}(\pi)=\operatorname{maj}(T)+\operatorname{inv}(\sigma)$. Given $\pi\in B_n$, since S_n is a parabolic subgroup of B_n, we can write π uniquely in the form $\pi=\tau\sigma$, where $\sigma\in S_n$ and $D(\tau)\subseteq\{n\}$. In fact, $D(\tau)\subseteq\{n\}$ is equivalent to the following condition: for some k, we have

$$\tau=\begin{pmatrix}1 & 2 & \cdots & n-k & n-k+1 & n-k+2 & \cdots & n\\ j_1 & j_2 & & j_{n+k} & -i_1 & -i_2 & \cdots & -i_k\end{pmatrix},$$

where $0<j_1<\cdots<j_{n-k}$ and $i_1>\cdots i_k>0$. So if we let $T=\{n+1-i_1,\ldots,n+1-i_k\}$, then τ is completely determined by T and, furthermore, one can check that $\operatorname{inv}(\tau)=\operatorname{maj}(T)$, $n(\tau)=\#T$. Hence the map $\delta(\pi)=(T,\sigma)$ has the properties we want. To give an example, if

$$\pi=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ -2 & +6 & +5 & +1 & -4 & +3\end{pmatrix}$$

then

$$\pi=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ +1 & +3 & +5 & +6 & -4 & -2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ +6 & +4 & +3 & +1 & +5 & +2\end{pmatrix}=\tau\sigma,$$

so $\delta(\pi)=(\{3,5\},\sigma)$.

To prove (5) bijectively, on the right-hand side we interpret $(1+ap)^n$ as counting subsets $T\subseteq[n]$ according to their cardinality, and $[n]!_p$ as $\sum_{\pi\in S_n} p^{\operatorname{maj}(\pi)}$ (see [5]). Hence (5) will follow if we can establish a bijection $\varepsilon\colon B_n\to 2^{[n]}\times S_n$ such that if $\varepsilon(\pi)=(T,\sigma)$ then $\operatorname{maj}(\pi)=\#T+\operatorname{maj}(\sigma)$ and $n(\pi)=\#T$. Given

$$\pi=\begin{pmatrix}1 & \cdots & n\\ \pi_1 & \cdots & \pi_n\end{pmatrix},$$

we let $T=\{i\colon -i \text{ appears among the } \pi_j\text{'s}\}$. We then define σ by repeatedly applying the following operator κ: $B_n \to B_n$ to π until we reach an unsigned permutation σ:

(1) Let i be the smallest element in T, so that $\pi_j = -i$ for some j.
(2) Replace $\pi_j = -i$ by $\pi_j = +i$.
(3) Let Λ be the linear order

$$+1 < +2 < \cdots < +n < -n < \cdots < -2 < -1$$

(it is easy to check that $i \in D(\pi)$ iff $\pi_i > \pi_{i+1}$ in this order, where $\pi_{n+1} = n+1$ by convention). If $+i = \min_\Lambda\{+i, \pi_{j-1}, \pi_{j+1}\}$ then stop. Else either π_{j-1} or π_{j+1} is $\min_\Lambda\{+i, \pi_{j-1}, \pi_{j+1}\}$. Assume without loss of generality that this minimum is π_{j+1} (the other case is symmetric). Let l be the smallest positive integer such that

$$\pi_{j+1} < \pi_{j+2} < \cdots \pi_{j+l} < +i$$

using the Λ order. Replace the consecutive sequence $+i, \pi_{j+1}, \pi_{j+2}, \ldots, \pi_{j+l}$ in π with the sequence $\pi_{j+1}, \pi_{j+2}, \ldots, \pi_{j+l}, +i$.

We must apply κ exactly $\#T$ times to reach a permutation $\sigma \in S_n$, and we then define $\varepsilon(\pi) = (T, \sigma)$. For example, if

$$\pi_1 \cdots \pi_n = +8 - 2 + 1 + 3 - 4 + 7 - 5 + 6$$

then $T = \{2, 4, 5\}$, and the results of three successive applications of κ are shown below:

$$\begin{aligned} &+8 + 1 + 2 + 3 - 4 + 7 - 5 + 6, \\ &+8 + 1 + 2 + 4 + 3 + 7 - 5 + 6, \\ &+8 + 1 + 2 + 4 + 3 + 7 + 5 + 6. \end{aligned}$$

It is not hard to check that, given T, we can reverse each application of κ, so that ε is a bijection. Also, after each application of κ, the major index and the statistic n are both lowered by 1, so that $\mathrm{maj}(\pi) = \#T + \mathrm{maj}(\sigma)$ and $n(\pi) = \#T$ as desired.

We can now offer a bijection for (6). Let $\phi: S_n \to S_n$ be Foata's bijection [5], having the property that $\mathrm{inv}(\phi(\sigma)) = \mathrm{maj}(\sigma)$. We then define our bijection $\Psi: 2^{[n]} \times B_n \to 2^{[n]} \times B_n$ to be the composite

$$(T, \pi) \overset{id\times\varepsilon}{\mapsto} (T, U, \sigma) \overset{id\times id\times\phi}{\mapsto} (T, U, \rho) \overset{(\text{swap } T,\, U)\times id}{\mapsto} (U, T, \rho) \overset{id\times\delta^{-1}}{\mapsto} (U, \zeta).$$

Note that

$$\begin{aligned} \mathrm{maj}(T) + \mathrm{maj}(\pi) &= \mathrm{maj}(T) + \mathrm{maj}(\sigma) + \#U \\ &= \mathrm{maj}(T) + \mathrm{inv}(\rho) + \#U \\ &= \mathrm{inv}(\zeta) + \#U \end{aligned}$$

and

$$\begin{aligned} \#T + n(\pi) &= \#T + \#U \\ &= \#U + n(\zeta). \end{aligned}$$

Hence the existence of the bijection Ψ proves (6).

7. Application: $C_k \wr S_n$ Statistics

For $k \geq 2$, we let $C_k \wr S_n$ denote the wreath product of a cyclic group C_k of order k with the symmetric group S_n. A typical element π in $C_k \wr S_n$ sends the ith standard basis vector e_i to $\zeta^{l_i} e_{m_i}$, where $\zeta = e^{2\pi i/k}$ and $0 \leq l_i < k$. We will denote this element π by

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \zeta^{l_1} m_1 & \zeta^{l_2} m_2 & & \zeta^{l_n} m_n \end{pmatrix}.$$

Note that $C_2 \wr S_n$ is the same as B_n. It turns out that some of the natural statistics on $C_k \wr B_n$ are related in an almost trivial fashion to statistics on B_n, and hence we can count these statistics by simply specializing a in the results of Sections 3 and 4.

For this purpose, we define a surjective set map $\phi: C_k \wr S_n \to B_n$ by sending π to $\hat{\pi}$, where

$$\hat{\pi}(e_i) = \begin{cases} +e_{m_i} & \text{if } l_i = 0, \\ -e_{m_i} & \text{if } l_i > 0. \end{cases}$$

It is worth noting that although ϕ is *not* a group homomorphism (unless $k = 2$), we do have that $\phi(\pi^{-1}) = \phi(\pi)^{-1}$.

As a group, $C_k \wr S_n$ is generated by the set

$$S = \{s_1, \ldots, s_n\} = \left\{ \begin{pmatrix} 12 \\ 21 \end{pmatrix}, \begin{pmatrix} 23 \\ 32 \end{pmatrix}, \ldots, \begin{pmatrix} n-1 & n \\ n & n-1 \end{pmatrix}, \begin{pmatrix} n \\ \zeta n \end{pmatrix} \right\}.$$

We define the *length* or the *number of inversions* of π by

$$\operatorname{inv}(\pi) = \min\{r: \pi = s_{i_1} s_{i_2} \cdots s_{i_r} \text{ for some } s_{i_j}\}$$

and we define the *descent set* of π in $C_k \wr S_n$ by

$$D(\pi) = \{i: \operatorname{inv}(\pi s_i^{-1}) = \operatorname{inv}(\pi) - 1\},$$

from which we derive the *descent number* $d(\pi) = D(\pi)$ and the *major index* $\operatorname{maj}(\pi) = \sum_{i \in D(\pi)} i$. The next proposition summarizes the relation between these statistics in $C_k \wr S_n$ and their counterparts in B_n.

PROPOSITION 7.1. (*1*) $\operatorname{inv}(\pi) = \operatorname{inv}(\phi(\pi)) + \sum_{i: l_i \geq 1} (l_i - 1)$.
(*2*) $D(\pi) = D(\phi(\pi))$.
(*3*) $D(\pi^{-1}) = D(\phi(\pi)^{-1})$.
(*4*) *For any* $\hat{\pi} \in B_n$, *we have*

$$\sum_{\substack{\pi \in C_k \wr S_n \\ \phi(\pi) = \hat{\pi}}} p^{\operatorname{inv}(\pi)} = p^{\operatorname{inv}(\hat{\pi})}([k-1]_p)^{n(\hat{\pi})}.$$

PROOF. The proofs of items (1)–(3) are routine verifications, which we leave to the reader. To prove (4), note that choosing a π which maps to a given $\hat{\pi}$ amounts to choosing l_i between 1 and $k-1$ for each i such that $m_i \in N(\hat{\pi})$. By part (1), we then have

$$\operatorname{inv}(\pi) = \operatorname{inv}(\phi(\pi)) + \sum_{m_i \in N(\hat{\pi})} (l_i - 1).$$

Therefore

$$\sum_{\substack{\pi \in C_k \wr S_n \\ \phi(\pi) = \hat{\pi}}} p^{\operatorname{inv}(\pi)} = p^{\operatorname{inv}(\hat{\pi})} \prod_{m_i \in N(\hat{\pi})} \sum_{l_i = 1}^{k-1} p^{l_i - 1} = p^{\operatorname{inv}(\hat{\pi})}([k-1]_p)^{n(\hat{\pi})}. \qquad \square$$

If we now specialize the four-variate distribution from Section 3 to $a = [k-1]_p$ and the five-variate distribution of Section 4 to $a = k - 1$, we reap the following two corollaries:

COROLLARY 7.2.

$$\sum_{n \geq 0} \frac{u^n}{(t;q)_{n+1}[n]!_p(-p[k-1]_p;p)_n} \sum_{\pi \in C_k \wr S_n} t^{d(\pi)} q^{\operatorname{maj}(\pi)} p^{\operatorname{inv}(\pi)}$$
$$= e[u]_{[k-1]_p, p} \sum_{m \geq 0} t^m e[qu]_p e[q^2 u]_p \cdots e[q^m u]_p.$$

COROLLARY 7.3.

$$\sum_{n\geq 0}\frac{u^n}{(t_1;q_1)_{n+1}(t_2;q_2)_{n+1}}\sum_{\pi\in C_k\wr S_n} t_1^{d(\pi)}t_2^{d(\pi^{-1})}q_1^{\mathrm{maj}(\pi)}q_2^{\mathrm{maj}(\pi^{-1})}$$

$$=\sum_{\substack{k_1\geq 0\\ k_2\geq 0}} t_1^{k_1}t_2^{k_2}\frac{1}{(u;q_1,q_2)_{k_1+1,k_2+1}((k-1)uq_1q_2;q_1,q_2)_{k_1,k_2}}.$$

REMARK. Setting $p=q=1$ and replacing u by $ku(1-t)$ in the first corollary above yields

$$\sum_{n\geq 0}\frac{u^n}{n!}\sum_{\pi\in C_k\wr S_n} t^{d(\pi)}=\frac{e^{u(1-t)}}{1-te^{ku(1-t)}},$$

a generalization of the well-known generating function for the *Eulerian polynomials*. In [12], Steingrímsson defines another descent statistic $\tilde{d}(\pi)$ for π in $C_k\wr S_n$ which is easily seen to be equidistributed with $d(\pi)$. He also defines the *exceedance*

$$e(\pi)=\#\{i: m_i>i \text{ or } (m_i=i \text{ and } l_i>0)\}$$

and shows that it is equidistributed with $\tilde{d}(\pi)$ (and hence also with $d(\pi)$). We take this as evidence that although there does not seem to be a canonical Mahonian distribution on $C_k\wr S_n$, there *does* seem to be a canonical Eulerian distribution, given by the exponential generating function above.

We should also remark that simultaneously with the preparation of this paper, the distribution of the descent statistic d over B_n (e.g. the case $p=q=a=1$ in the main result of Section 3) was computed by Stembridge [13] and the joint distribution (d, n) of descents and negative signs (the case $p=q=1$ in the aforementioned result) was computed by Brenti [3].

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