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# Signed Posets*

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We define a new object, called a *signed poset*, that bears the same relation to the *hyperoctahedral group* $B_n$ (i.e., signed permutations on $n$ letters), as do posets to the *symmetric group* $S_n$. We then prove hyperoctahedral analogues of the following results: (1) the generating function results from the theory of $P$-partitions; (2) the fundamental theorem of finite distributive lattices (or Birkhoff's theorem) relating a poset to its distributive lattice of order ideals; (3) the edgewise-lexicographic shelling of upper-semimodular lattices; (4) MacMahon's calculation of the distribution of the major index for permutations. © 1993 Academic Press, Inc.

CONTENTS



## 1. INTRODUCTION: POSETS VIEWED IN TERMS OF A ROOT SYSTEM

In this section, we review some basic notions about posets and rephrase them in terms of the *root system* associated to the symmetric group $S_n$. Although this rephrasing may seem unnecessary, it will motivate our subsequent definitions of hyperoctahedral analogues.

DEFINITION. Let $e_i$ denote the $i$th standard basis vector in $\mathbf{R}^n$. The *root system* for $S_n$ is the set of vectors

$$\Phi = \{e_i - e_j : 1 \leqslant i \neq j \leqslant n\} \subseteq \mathbf{R}^n,$$

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with *positive roots*

$$\Phi^+ = \{e_i - e_j : 1 \leqslant i < j \leqslant n\},$$

and *simple roots*

$$\Pi = \{e_i - e_{i+1} : 1 \leqslant i < n\}.$$

We will have need of only a few of the properties of root systems, but the interested reader may find out more about them in [Bo]. The few properties of $\Phi$, $\Phi^+$, $\Pi$ that we need (and that are easy to check) are:

1. $\Phi = \Phi^+ \amalg -\Phi^+$, where $-\Phi^+ = \{-\alpha : \alpha \in \Phi^+\}$ and "$\amalg$" denotes disjoint union.

2. If for $S \subseteq \Phi$ we denote by

$$\bar{S}^{\mathrm{PLC}} = \left\{\alpha \in \Phi : \alpha = \sum_{\beta \in S} c_\beta \beta \text{ for some } c_\beta \geqslant 0\right\}$$

the set of positive linear combination of elements of $S$ that lie in $\Phi$, then we have that $\Phi^+ = \bar{\Pi}^{\mathrm{PLC}}$.

3. If we consider $S_n$ as acting on $\mathbf{R}^n$ by permuting coordinates, this action preserves $\Phi$.

Given a poset $P$ with $n$ elements, without loss of generality we may assume $P$ is a partial order $<_P$ on the set $\{1, 2, ..., n\}$ (this is someties called a *labelled poset*). We will identify such a partial order with a subset $P \subseteq \Phi$ in the following way: $e_i - e_j \in P$ if and only if $<_P j$, i.e.,

$$P = \{e_i - e_j : 1 \leqslant i \neq j \leqslant n,\ i <_P j\}.$$

We then have the following

PROPOSITION 1.1. *The above identification gives a one-to-one correspondence between partial orders $<_P$ on $\{1, 2, ..., n\}$ and subsets $P \subseteq \Phi$ satisfying*

1. $\alpha \in P \Rightarrow -\alpha \notin P$,
2. $\bar{P}^{\mathrm{PLC}} = P$.

*Proof.* Property 1 corresponds to *antisymmetry* of $<_P$:

$$i <_P j \Rightarrow j \not<_P i$$

$$e_i - e_j \in P \Rightarrow e_j - e_i \notin P.$$

Property 2 corresponds to transitivity of $<_P$:

$$i <_P j,\ j <_P k \Rightarrow i <_P k$$

$$e_i - e_j,\ e_j - e_k \in P \Rightarrow (e_i - e_j) + (e_j - e_k) = e_i - e_k \in P.$$

Irreflexivity of $<_P$ corresponds to the fact that $\underline{0} \notin P$. ∎

The notion of an order ideal of a poset fits nicely with the above correspondence.

DEFINITION. Let $P$ be a partial order on $\{1, 2, ..., n\}$. Then $I \subseteq \{1, 2, ..., n\}$ is an *ideal* of $P$ if $j \in I$ and $i <_P j$ implies $i \in I$. We can rephrase this in terms of the above correspondence if we identify subsets $I \subseteq \{1, 2, ..., n\}$ with their *characteristic vector* (i.e., the vector $(\varepsilon_1, ..., \varepsilon_n) \in \{1, 0\}^n$ with $\varepsilon_i = 1$ if $i \in I$ and 0 else). We have that $I$ is an ideal of $P$ exactly when $\langle \alpha, I \rangle \geqslant 0$ for all $\alpha \in P$ (where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbf{R}^n$).

One can also define the poset $J(P)$ of ideals of $P$ ordered under inclusion. This ordering corresponds to the componentwise partial order on the characteristic vectors, with $1 > 0$ in each component.

The basic result about $J(P)$ is known as the Fundamental Theorem Of Finite Distributive Lattices, or Birkhoff's theorem (see [St1, Theorem 3.4.1] for a proof):

THEOREM 1.2. *$J(P)$ is a finite distributive lattice, i.e., it satisfies the distributive law*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \in J(P).$$

*Furthermore, every finite distributive lattice $L$ is isomorphic to $J(P)$ for some poset $P$, which is uniquely defined up to poset isomorphism.*

Note that the notion of poset isomorphism is quite natural to define under our correspondence: two posets $P_1$, $P_2$ are isomorphic if there exists some permutation $\pi \in S_n$ such that $\pi P_1 = P_2$.

Another notion which can be rephrased quite naturally is that of a $P$-partition, a common generalization of partitions, partitions into distinct parts, compositions, plane partitions, and column-strict tableaux (for more about $P$-partitions, see [St2]).

DEFINITION. Given a partial order $P$ on $\{1, 2, ..., n\}$, a *$P$-partition* is a function $f\colon \{1, 2, ..., n\} \to \mathbf{N}$ satisfying

$$f(i) \geqslant f(j) \quad \text{if} \quad i <_P j$$

$$f(i) > f(j) \quad \text{if} \quad i <_P j \text{ and } i > j.$$

Rephrasing this in terms of our correspondence, a $P$-partition is a vector $f \in \mathbf{N}^n$ satisfying

$$\langle \alpha, f \rangle \geqslant 0 \quad \text{for all} \quad \alpha \in P$$
$$\langle \alpha, f \rangle > 0 \quad \text{for all} \quad \alpha \in P \cap -\Phi^+.$$

In subsequent sections, we define hyperoctahedral analogues of all of these concepts and prove analogues of some of the basic results known about them.

## 2. Signed Posets

In this section we introduce and give examples of signed posets, our hyperoctahedral analogues of posets.

Definition. Let $B_n$ denote the *hyperoctahedral group* or the group of *signed permutations*, i.e., all permutations and sign changes of the coordinates in $\mathbf{R}^n$. The *root system* for $B_n$ is the set of vectors

$$\Phi = \{\pm e_i : 1 \leqslant i \leqslant n\} \cup \{\pm e_i \pm e_j : 1 \leqslant i < j \leqslant n\},$$

with *positive roots*

$$\Phi^+ = \{+e_i : 1 \leqslant i \leqslant n\} \cup \{+e_i + e_j, +e_i - e_j : 1 \leqslant i < j \leqslant n\},$$

and *simple roots*

$$\Pi = \{e_i - e_{i+1} : 1 \leqslant i < n\} \cup \{+e_n\}.$$

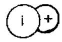
One can easily check that Properties 1–3 that held for $\Phi$ and $S_n$ in Section 1 also hold here for $\Phi$ and $B_n$.

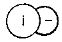
Motivated by Proposition 1.1, we define a *signed poset $P$* (*on $n$ elements*) to be a subset $P \subseteq \Phi$ satisfying

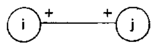
1. $\alpha \in P \Rightarrow -\alpha \notin P$
2. $\bar{P}^{\mathrm{PLC}} = P$.

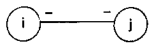
We will say that two signed posets $P_1$, $P_2$ are *isomorphic* (written $P_1 \cong P_2$) if there exists a signed permutation $w \in B_n$ such that $wP_1 = P_2$.

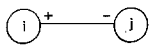
It is convenient to have a pictorial representation of a signed poset $P$, so we define its *signed digraph* $D(P)$ as follows. $D(P)$ is a graph on vertex set $\{1, 2, ..., n\}$ with certain labelled edges and labelled loops attached to the vertices:

If $+e_i \in P$, attach a loop like this: 

If $-e_i \in P$, attach a loop like this: 

If $+e_i + e_j \in P$, attach an edge like this: 

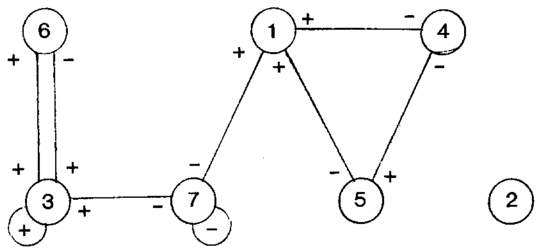
If $-e_i - e_j \in P$, attach an edge like this: 

If $+e_i - e_j \in P$, attach an edge like this: 

EXAMPLE. The set $P = \{+e_1 - e_2, +e_1 + e_2\}$ is *not* a signed poset, since

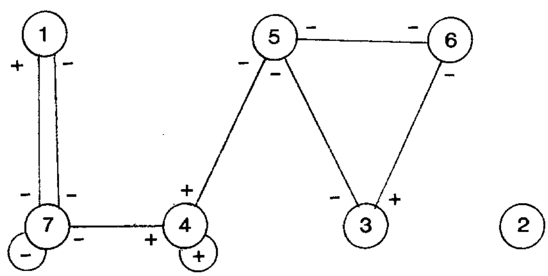
$$+e_1 = \tfrac{1}{2}(+e_1 - e_2) + \tfrac{1}{2}(+e_1 + e_2) \in \bar{P}^{\mathrm{PLC}}$$

but $+e_1 \notin P$, violating axiom 2 (the "transitivity" axiom). Neither is $P = \{+e_1 + e_2, -e_1 - e_2\}$, since it violates axiom 1 (the "antisymmetry" axiom). Figure 1 shows two examples of signed posets $P_1$, $P_2$ and their signed digraphs, along with the isomorphism $P_1 \cong P_2$.



$P_1 = \{+e_3, -e_7, +e_1-e_4, +e_1-e_5, +e_1-e_7$

$+e_3+e_6, +e_3-e_6, +e_3-e_7, -e_4+e_5\}$



$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -5 & +2 & -7 & +6 & +3 & +1 & -4 \end{pmatrix} P_1$

FIG. 1. An example of two isomorphic signed posets.

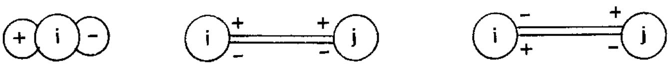



FIG. 2. Configurations not allowed.

*Remark.* In [Re], signed posets are called $B_n$-parsets because they are a special case of the notion of a *parset* defined for arbitrary root systems. The terminology "signed poset" seems more in line with other hyper-octahedral terminology such as "signed permutation" and "signed sets." We also hope that the terminology "signed digraph" is consistent with the theory of *signed graphs* developed by Zaslavsky [Za].

The axioms of signed posets dictate that certain configurations of edges and loops in $D(P)$ cannot occur, and certain configurations imply the existence of more edges and loops. These rules are summarized in Figs. 2 and 3.

*Remark.* Since $\bar{P}^{\mathrm{PLC}} = P$, one might try to simplify $D(P)$ by eliminating those edges and loops which are implied by others. It is in fact true that for any signed poset $P$, there is a unique minimal subset $H \subseteq P$ such that $\bar{H}^{\mathrm{PLC}} = P$ (this is a reflection of the easy-to-check fact that $\bar{\cdot}^{\mathrm{PLC}}$ is a *convex closure*; see [EJ, Theorem 2.1]). Thus if we were to include only the loops and edges that correspond to elements of $H$, we would obtain the analogue

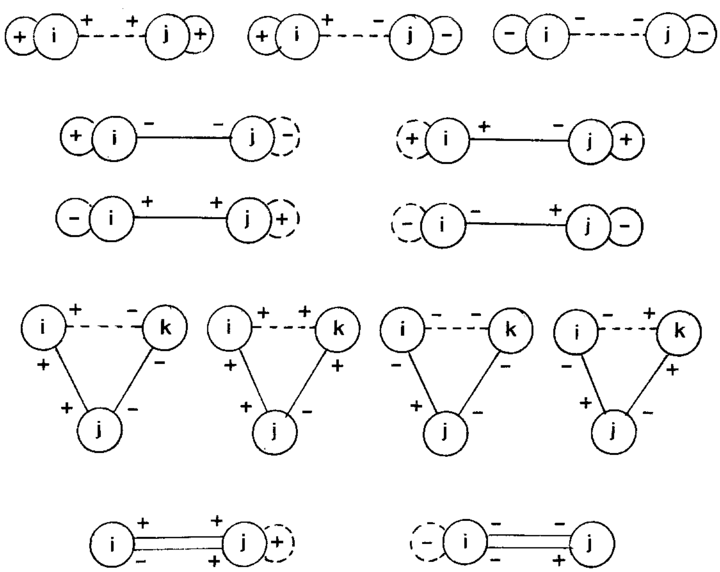


FIG. 3. Configurations implying more edges (implied edges shown dotted).

of the *Hasse diagram* of $P$. However, we feel that these Hasse diagrams are harder to understand than $D(P)$ (because "transitivity" is more complicated for signed posets than it was for posets).

One further notion for which we need an analogue is that of a subposet.

DEFINITION. Let $P$ be a signed poset (on $n$ elements) and $T \subseteq \{1, 2, ..., n\}$. The *induced signed subposet of $P$ on $T$* is the signed poset $P_T$ on $\# T$ elements consisting of only those roots in $P$ whose nonzero coordinates lie in $T$ (strictly speaking, we should also re-index the coordinates in $T$ to be $\{1, 2, ..., \# T\}$).

EXAMPLE. For the signed poset

$$P_1 = \{+e_3, -e_7, +e_1-e_7, +e_1-e_4, +e_1-e_5, \\ +e_3+e_6, +e_3-e_6, +e_3-e_7, -e_4+e_5\},$$

from Fig. 1, and $T = \{2, 3, 4, 6\}$, we have that (without re-indexing)

$$P_{1T} = \{+e_3, +e_3+e_6, +e_3-e_6\}.$$

## 3. $P$-Partitions, Generating Functions, and $J(P)$

In this section, we define $P$-partitions for signed posets $P$ and investigate some of their various generating functions and counting functions. We also introduce the poset $J(P)$ of ideals of $P$. Our notation and exposition are intended to parallel those of [St2]. In fact, by embedding posets as signed posets in a certain fashion, we can arrange that all of our definitions and results when particularized to these embedded posets yield a result analogous to that of [St2]. This embedding is done as follows:

DEFINITION. Given a poset $P$ on $\{1, 2, ..., n\}$, we define its *positive embedding* $P^+$ by

$$P^+ = \{+e_i - e_j : 1 \leqslant i \neq j \leqslant n,\ i <_P j\} \\ \cup \{+e_i + e_j : 1 \leqslant i < j \leqslant n\} \cup \{+e_i : 1 \leqslant i \leqslant n\}.$$

One can see readily that $P^+$ is a signed poset.

Motivated by our rephrasing of the definition of $P$-partitions, ideals, and $J(P)$ given in Section 1, we make the following definitions.

DEFINITION. For $P$ a signed poset (on $n$ elements), a *P-partition* is a vector $f \in \mathbf{Z}^n$ satisfying

$$\langle \alpha, f \rangle \geqslant 0 \quad \text{for all} \quad \alpha \in P$$
$$\langle \alpha, f \rangle > 0 \quad \text{for all} \quad \alpha \in P \cap -\Phi^+.$$

We denote by $\mathscr{A}(P)$ the set of all $P$-partitions. For $m \in \mathbf{N}$, define

$$\mathscr{A}(P; m) = \{(f_1, ..., f_n) \in \mathscr{A}(P) : |f_i| \leqslant m \ \forall i\}.$$

Define $F(P, x)$ to be the formal power series in variables $x_1, x_{-1}, ..., x_n, x_{-n}$ given by

$$F(P, x) = \sum_{f \in \mathscr{A}(P)} x^f,$$

where $x^f = \prod_{i=1}^n x_{\operatorname{sgn}(f_i) \cdot i}^{|f_i|}$ (e.g., $x^{(-5,1,3,-2)} = x_{-1}^5 x_2^1 x_3^3 x_{-4}^2$). Let $U_m(P, x)$ be the formal power series in one variable $x$ defined by

$$U_m(P, x) = \sum_{f \in \mathscr{A}(P; m)} x^{|f_1| + \cdots + |f_n|}$$

and

$$U(P, x) = \lim_{m \to \infty} U_m(P, x) = \sum_{f \in \mathscr{A}(P)} x^{|f_1| + \cdots + |f_n|} = F(P, x)\,|_{x_{\pm i} \to x}.$$

Define a poset

$$J(P) = \{f \in \{+1, -1, 0\}^n : \langle \alpha, f \rangle \geqslant 0 \ \forall \alpha \in P\}$$

with partial order inherited from $\{+1, -1, 0\}^n$ by setting $+1 > 0$, $-1 > 0$ and extending componentwise. We will call an element $I \in J(P)$ an *ideal* of $P$.

EXAMPLE. Let $P = \{+e_2 - e_1, +e_2\}$. Then

$$\mathscr{A}(P) = \{(f_1, f_2) \in \mathbf{Z}^2 : f_2 > f_1, f_2 \geqslant 0\}$$

and

$$F(P, x) = \sum_{f_2 > f_1, f_2 \geqslant 0} x^f = \sum_{f_2 > f_1 \geqslant 0} x^f + \sum_{f_2 \geqslant 0 > f_1} x^f$$
$$= \frac{x_2}{(1 - x_2)(1 - x_1 x_2)} + \frac{x_{-1}}{(1 - x_{-1})(1 - x_2)}$$

and thus

$$U(P, x) = \frac{x}{(1-x)^2} + \frac{x}{(1-x)(1-x^2)} = \frac{x^2 + 2x}{(1-x)(1-x^2)}.$$

$D(P)$ and $J(P)$ are shown in Fig. 4.

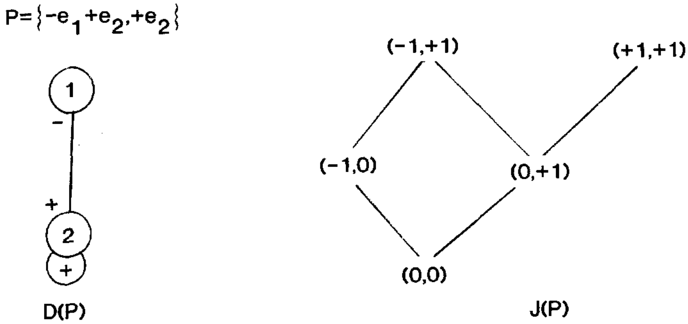



FIG. 4. An example of $D(P)$ and $J(P)$.

Our first step toward nice expressions for our generating functions is to relate $P$-partitions to chains in $J(P)$. The next definition gives us a way of relating a vector in $\mathbf{Z}^n$ to a chain in $\{+1, -1, 0\}^n$.

DEFINITION. Given $f \in \mathbf{Z}^n$, let $\{|f_i|\}_{i=1,\dots,n} = \{n_1, \dots, n_k\}$ with $n_1 > \cdots > n_k$. Define a chain $c(f)$ of vectors $I_1 < \cdots < I_k$ in $\{+1, -1, 0\}^n$ by

$$I_i(j) = \begin{cases} \operatorname{sgn}(f_j) & \text{if } |f_j| \geqslant n_i \\ 0 & \text{else} \end{cases}$$

for $j = 1, \dots, n$. For example, if $f = (-5, +1, +3, -2, -3)$ then $\{|f_i|\}_{i=1,2,3,4,5} = \{5, 3, 2, 1\}$ and $c(f)$ is

$$(-1, 0, 0, 0, 0) \leqslant (-1, 0, +1, 0, -1) \leqslant (-1, 0, +1, -1, -1) \leqslant (-1, +1, +1, -1, -1).$$

One way to visualize this is as follows. Given $f$, make a histogram that has the coordinates $f_i$ along the bottom, and a column of $\pm 1$'s (depending on $\operatorname{sgn}(f_i)$) of height $|f_i|$ above each $f_i$, filling in zeroes elsewhere. Then $I_1, \dots, I_k$ are the set of (distinct) rows read from top to bottom. For instance, in the example above, we have

$$\begin{array}{ccccccc} -1 & 0 & 0 & 0 & 0 & \leftarrow & I_1 \\ -1 & 0 & 0 & 0 & 0 & & \\ -1 & 0 & +1 & 0 & -1 & \leftarrow & I_2 \\ -1 & 0 & +1 & -1 & -1 & \leftarrow & I_3 \\ -1 & +1 & +1 & -1 & -1 & \leftarrow & I_4 \\ \hline -5 & +1 & +3 & -2 & -3 & \leftarrow & f \end{array}$$

Given $c$ a chain $I_1 \leqslant \cdots \leqslant I_k$ in $J(P)$, we will say $c$ is *P-compatible* if it satisfies the following condition: for $0 \leqslant i < k$, when we restrict $I_{i+1}$ to the set $S_i$ of coordinates where it differs from $I_i$ (setting $I_0 = \varnothing$), we get a vector in $\mathscr{A}(P_{S_i}; 1)$, where $P_{S_i}$ is the induced signed subposet of $P$ on $S_i$.

The next proposition gives the basic relation between $\mathscr{A}(P)$ and $J(P)$.

PROPOSITION 3.1. *A vector $f$ is in $\mathscr{A}(P)$ if and only if $c(f)$ is a P-compatible chain in $J(P) - \hat{0}$.*

*Proof.* A look at the histogram picture should convince one that $\langle \alpha, f \rangle \geqslant 0\ \forall \alpha \in P$ if and only if $\langle \alpha, I_i \rangle \geqslant 0\ \forall \alpha \in P$, $\forall i$, i.e., if and only if $c(f)$ is a chain in $J(P)$. It then remains to show that $\langle \alpha, f \rangle > 0\ \forall \alpha \in P \cap -\Phi^+$ if and only if $c(f)$ is $P$-compatible. One can check this for the various cases of $\alpha \in P \cap -\Phi^+$, i.e., $\alpha$ of the form $-e_j$, $-e_j - e_k$, or $+e_j - e_k$ for $j > k$. We illustrate this for the second case; the others are similar. If $\alpha = -e_j - e_k$ then $\langle \alpha, f \rangle > 0$ if and only if $f_j + f_k < 0$. This is equivalent to the condition that whenever $I_{i+1}$ and $I_i$ differ in coordinates $j$ and $k$ we have $I_{i+1}(j) = I_{i+1}(k) = -1$, which is one of the conditions for $c(f)$ to be $P$-compatible. ∎

COROLLARY 3.2.

$$F(P, x) = \sum \frac{x^{I_1} \cdots x^{I_k}}{(1 - x^{I_1}) \cdots (1 - x^{I_k})},$$

*where the above sum ranges over all P-compatible chains $I_1 < \cdots < I_k$ in $J(P) - \hat{0}$.*

*Proof.*

$$\begin{aligned} F(P, x) &= \sum_{f \in \mathscr{A}(P)} x^f \\ &= \sum_{\substack{P\text{-compatible} \\ I_1 < \cdots < I_k \in J(P)}} \ \sum_{\substack{f \in \mathbf{Z}^n \\ c(f) = I_1 < \cdots < I_k}} x^f \\ &= \sum_{\substack{P\text{-compatible} \\ I_1 < \cdots < I_k \in J(P)}} \frac{x^{I_1} \cdots x^{I_k}}{(1 - x^{I_1}) \cdots (1 - x^{I_k})}, \end{aligned}$$

where the second equality comes from Proposition 3.1. ∎

EXAMPLE. For $P = \{+e_2 - e_1, +e_2\}$ as before, the $P$-compatible chains in $J(P)$ are

$$(-1, 0), \qquad (0, +1), \qquad (-1, +1)$$

$$(-1, 0) < (-1, +1), \qquad (0, +1) < (-1, +1), \qquad (0, +1) < (+1, +1)$$

and thus according to the previous proposition,

$$F(P, x) = \frac{x_{-1}}{(1-x_{-1})} + \frac{x_2}{(1-x_2)} + \frac{x_{-1}x_2}{(1-x_{-1}x_2)}$$

$$+ \frac{x_{-1}\quad {}_{-1}x_2}{(1-x_{-1})(1-x_{-1}x_2)} + \frac{x_2 \cdot x_{-1}x_2}{(1-x_2)(1-x_{-1}x_2)}$$

$$+ \frac{x_2 \cdot x_1 x_2}{(1-x_2)(1-x_1x_2)}.$$

With a great deal of manipulation, one can check that this agrees with our earlier calculation of $F(P, x)$.

Most of our expressions for generating functions will follow from our next result, which is the analogue of the central result on $P$-partitions [St2, Theorem 6.2]. Although a more general statement was proven in [Re], we include the proof in this case for the sake of completeness.

DEFINITION. Given $P$ a signed poset (on $n$ elements), let its *Jordan–Hölder set* $\mathscr{L}(P)$ be the set $\{w \in B_n : P \subseteq w\Phi^+\}$.

THEOREM 3.3.

$$\mathscr{A}(P) = \coprod_{w \in \mathscr{L}(P)} \mathscr{A}(w\Phi^+).$$

*Proof.* (cf. [Ge, Theorem 1]). We proceed by induction on

$$r = \#\{\alpha \in \Phi^+ : \alpha \notin P, -\alpha \notin P\}.$$

$r = 0$: In this case, it suffices to show that $P = w\Phi^+$ for some $w \in B_n$ (since then one easily sees that $\mathscr{L}(P) = \{w\}$). Since $r = 0$, for each $i$, we have either $+e_i \in P$ or $-e_i \in P$. Thus we can find an element $w' \in B_n$ such that $+e_i \in P' = w'P$ for all $i$. Since $\overline{P'}^{\mathrm{PLC}} = P'$, this implies that $+e_i + e_j \in P'$ for all $i < j$, and hence $P' = Q^+$ for some poset $Q$ on $\{1, 2, ..., n\}$. Since either $+e_i - e_j \in P'$ or $+e_j - e_i \in P'$ for all $i < j$, $Q$ must be a total order on $\{1, 2, ..., n\}$. Thus we can find some permutation $w'' \in B_n$ such that $P'' = w''P'$ has $+e_i - e_j \in P''$ if and only if $i < j$. Then $P'' = \Phi^+$ and hence $P = (w''w')^{-1}\,\Phi^+$, as we wanted.

$r > 0$: Assume $\alpha$, $-\alpha \notin P$, and let $P_\alpha = \overline{P \cup \{\alpha\}}^{\mathrm{PLC}}$. We claim $P_\alpha$ is a signed poset, i.e., it also satisfies the "antisymmetry" axiom. To see this, suppose not, i.e., let $\beta$, $-\beta \in P_\alpha$. Then we must have

$$\beta = a\alpha + \sum a_i \alpha_i$$

$$-\beta = b\alpha + \sum b_i \alpha_i$$

for some $a_i$, $b_i \geqslant 0$, $a$, $b > 0$, and $\alpha_i \in P$. Adding these equations, and dividing by $a+b$, yields

$$-\alpha = \sum \frac{1}{a+b}(a_i + b_i)\alpha_i$$

and hence $-\alpha \in P$, a contradiction. Similarly we can form the signed poset $P_{-\alpha}$. We then have

$$\mathscr{A}(P) = \mathscr{A}(P_\alpha) \amalg \mathscr{A}(P_{-\alpha})$$

$$\mathscr{L}(P) = \mathscr{L}(P_\alpha) \amalg \mathscr{L}(P_{-\alpha}).$$

The first equality holds because any $f \in \mathscr{A}(P)$ satisfies either $\langle \alpha, f \rangle \geqslant 0$ or $\langle -\alpha, f \rangle > 0$. The second equality holds because any $w \in \mathscr{L}(P)$ satisfies either $\alpha \in w\Phi^+$ or $-\alpha \in w\Phi^+$. Thus by induction on $r$, we are done. ▮

In order to make use of this theorem, we need only understand the sets $\mathscr{A}(w\Phi^+)$ for $w \in B_n$ more fully. For this, we require the notion of a descent of $w$.

DEFINITION. Given

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & & w_n \end{pmatrix} \in B_n,$$

its (right) descent set $D(w)$ is defined as follows: numbering the elements of $\Pi$ as

$$\pi_1 = +e_1 - e_2,\ \pi_2 = +e_2 - e_3, ..., \pi_{n-1} = +e_{n-1} - e_n,\ \pi_n = +e_n,$$

we define $D(w) = \{i : w(\pi_i) \in -\Phi^+\}$.

For $1 \leqslant i \leqslant n$, let $\delta_i(w)$ be the characteristic vector of the signed subset $\{w_1, w_2, ..., w_i\}$ (i.e., $\delta_i(w)$ is the vector $(\varepsilon_1, ..., \varepsilon_n) \in \{+1, -1, 0\}^n$ with $\varepsilon_i = +1$, $-1$, or $0$ depending on whether $i$ appears in $\{w_1, w_2, ..., w_i\}$ with a $+$, with a $-$, or not at all). We then define two chains $c_R(w)$, $c_E(w)$ in $\{+1, -1, 0\}^n$ associated to $w$ as

$$c_R(w) = \delta_{i_1} < \cdots < \delta_{i_k}, \qquad \text{where} \quad D(w) = \{i_1 < \cdots < i_k\}$$

$$c_E(w) = \delta_1 < \cdots < \delta_n.$$

EXAMPLE. Let $w = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & +1 & -4 & +5 & -2 \end{smallmatrix})$. Then $D(w) = \{1, 3, 5\}$ and

$$c_R(w) = (0, 0, -1, 0, 0) < (+1, 0, -1, -1, 0) < (+1, -1, -1, -1, +1)$$
$$c_E(w) = (0, 0, -1, 0, 0) < (+1, 0, -1, 0, 0) < (+1, 0, -1, -1, 0)$$
$$< (+1, 0, -1, -1, +1) < (+1, -1, -1, -1, +1).$$

The following proposition tells us what we need to know about $\mathscr{A}(w\Phi^+)$. Its proof is a straightforward unravelling of the various definitions, which is left to the reader.

PROPOSITION 3.4. *Let*

$$w = \begin{pmatrix} 1 & \cdots & b \\ w_1 & & w_n \end{pmatrix} \in B_n$$

*and* $f \in \mathbf{Z}^n$. *Then the following are equivalent:*

1. $f \in \mathscr{A}(w\Phi^+)$.
2. $\operatorname{sgn}(w_i) = \operatorname{sgn}(f_{|w_i|})$ *and*

$$|f_{|w_1|}| \sim_1 \cdots \sim_{n-1} |f_{|w_n|}| \sim_n 0,$$

*where* "$\sim_i$" = "$\geqslant$" *if* $s_i \in D(w)$ *and* "$\sim_i$" = "$>$" *else.*

3. $c_R(w) \subseteq c(f) \subseteq c_E(w)$.

EXAMPLE. If $w = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & +1 & -4 & +5 & -2 \end{smallmatrix})$, then $D(w) = \{(12), (34), (\begin{smallmatrix} 5 \\ -5 \end{smallmatrix})\}$, and $f \in \mathscr{A}(w\Phi^+)$ if and only if $f_2, f_4 \geqslant 0$, $f_1, f_3, f_5 \leqslant 0$ and

$$|f_3| > |f_1| \geqslant |f_4| > |f_5| \geqslant |f_2| > 0.$$

PROPOSITION 3.5.

$$F(P, x) = \sum_{w \in \mathscr{L}(P)} \frac{\prod_{s_i \in D(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{i=1}^{n} (1 - x_{w_1} x_{w_2} \cdots x_{w_i})}$$

*and hence*

$$U(P, x) = \sum_{w \in \mathscr{L}(P)} \frac{x^{\operatorname{maj}(w)}}{(1-x)(1-x^2) \cdots (1-x^n)},$$

*where* $\operatorname{maj}(w) = \sum_{s_i \in D(w)} i$.

*Proof.* We have

$$F(P, x) = \sum_{f \in AP} x^f = \sum_{w \in \mathscr{L}(P)} \sum_{f \in \mathscr{A}(w\Phi^+)} x^f$$

and from the previous proposition, one can see that

$$\sum_{f \in \mathscr{A}(w\Phi^+)} x^f = \frac{\prod_{s_i \in D(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{i=1}^{n} (1 - x_{w_1} x_{w_2} \cdots x_{w_i})}. \quad \blacksquare$$

To write down expressions for the other generating functions, we need a little terminology.

DEFINITION. Let

$$W(P, x) = \sum_{w \in \mathscr{L}(P)} x^{\operatorname{maj}(w)}$$

be the numerator in the above expression for $U(P, x)$, and for $0 \leqslant s \leqslant n$ let

$$W_s(P, x) = \sum_{\substack{w \in \mathscr{L}(P) \\ \# D(w) = s}} x^{\operatorname{maj}(w)}$$

so that $W(P, x) = \sum_{s=0}^{n} W_s(P, x)$. The Gaussian coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_x$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{(1-x^n)(1-x^{n-1}) \cdots (1-x^{n-k+1})}{(1-x^k)(1-x^{k-1}) \cdots (1-x)}.$$

PROPOSITION 3.6.

1. $U_m(P, x) = \sum_{s=0}^{n} \begin{bmatrix} n+s \\ n \end{bmatrix}_x W_{m-s}(P, x),$

2. $\sum_{m \geqslant 0} U_m(P, x)\, q^m = \dfrac{\sum_{s=0}^{n} q^s W_s(P, x)}{(1-q)(1-qx) \cdots (1-qx^n)}.$

*Proof.*

1. We have

$$U_m(P, x) = \sum_{f \in \mathscr{A}(P;\, m)} x^{|f_1| + \cdots + |f_n|} = \sum_{w \in \mathscr{L}(P)} \sum_{f \in \mathscr{A}(w\Phi^+;\, m)} x^{|f_1| + \cdots + |f_n|}.$$

By Proposition 3.4, $f \in \mathscr{A}(w\Phi^+; m)$ if and only if $\operatorname{sgn}(w_i) = \operatorname{sgn}(f_{|w_i|})$ and

$$m \geqslant |f_{|w_1|}| \geqslant \cdots \geqslant |f_{|w_n|}| \geqslant 0$$

with strict inequalities at the descents of $w$. If we let

$$\lambda_i = |f_{|w_i|}| - \#(D(w) \cap \{s_i, s_{i+1}, ..., s_n\})$$

then we have

$$|f_1| + \cdots + |f_n| = \mathrm{maj}(w) + \lambda_1 + \cdots + \lambda_n$$

and

$$m - \#D(w) \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

Thus

$$\begin{aligned} U_m(P, x) &= \sum_{w \in \mathscr{L}(P)} x^{\mathrm{maj}(w)} \sum_{m - \#D(w) \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0} x^{\lambda_1 + \cdots + \lambda_n} \\ &= \sum_{s=0}^{m} \sum_{m - s \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0} x^{\lambda_1 + \cdots + \lambda_n} \sum_{\substack{w \in \mathscr{L}(P) \\ \#D(w) = s}} x^{\mathrm{maj}(w)} \\ &= \sum_{s=0}^{n} \begin{bmatrix} n+m-s \\ n \end{bmatrix}_x W_s(P, x), \end{aligned}$$

where the last equality follows from a result of Euler (see, e.g., [HW, Theorem 349]). Our result now follows upon replacing $s$ by $m-s$ and noting that $W_s(P, x) = 0$ for $s > n$.

2. From the last equation we have

$$\begin{aligned} \sum_{m \geqslant 0} U_m(P, x)\, q^m &= \sum_{m \geqslant 0} \sum_{s \leqslant m} \begin{bmatrix} n+m-s \\ n \end{bmatrix}_x W_s(P, x)\, q^m \\ &= \sum_{s \geqslant 0} q^s W_s(P, x) \sum_{m-s \geqslant 0} \begin{bmatrix} n+(m-s) \\ n \end{bmatrix}_x q^{m-s} \\ &= \frac{\sum_{s \geqslant 0} q^s W_s(P, x)}{(1-q)(1-qx) \cdots (1-qx^n)}, \end{aligned}$$

where the last equality is also Euler's result (*ibid.*). ∎

Example. For $P = \{+e_2 - e_1, +e_2\}$ as before, we have

$$\mathscr{L}(P) = \left\{ \begin{pmatrix} 1 & 2 \\ -1\cdot & +2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ +2\cdot & +1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ +2 & -1\cdot \end{pmatrix} \right\}$$

(with descents indicated by dots). Thus by Proposition 3.5, we have

$$\begin{aligned} F(P, x) = &\frac{x_{-1}}{(1-x_{-1})(1-x_2)} + \frac{x_2}{(1-x_2)(1-x_1 x_2)} \\ &+ \frac{x_{-1} x_2}{(1-x_2)(1-x_{-1} x_2)} \end{aligned}$$

and

$$U(P, x) = \frac{x + x + x^2}{(1-x)(1-x^2)},$$

both of which agree (after a little manipulation) with our previous calculations. By the previous proposition,

$$U_m(P, x) = \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_x (x^2 + 2x)$$

and

$$\sum_{m \geqslant 0} U_m(P, x)\, q^m = \frac{q(x^2+2x)}{(1-q)(1-qx)(1-qx^2)}.$$

The decomposition given by Theorem 3.3 also gives us a useful partitioning of the $P$-compatible chains.

PROPOSITION 3.7.

$$\{P\text{-compatible chains in } J(P) - \hat{0}\} = \coprod_{w \in \mathscr{L}(P)} \{\text{chains } c : c_R(w) \subseteq c \subseteq c_E(w)\}.$$

*Proof.* A chain $c$ in $J(P) - \hat{0}$ is $P$-compatible

$$\begin{aligned} &\Leftrightarrow c = c(f) \quad \text{for some} \quad f \in \mathscr{A}(P) \\ &\Leftrightarrow c = c(f) \quad \text{for some} \quad f \in \mathscr{A}(w\Phi^+),\ w \in \mathscr{L}(P) \\ &\Leftrightarrow c_R(w) \subseteq c \subseteq c_E(w). \end{aligned}$$

The first equivalence is due to Proposition 3.1. The second equivalence is due to Theorem 3.3. The third equivalence is due to Proposition 3.4. ∎

We can now give combinatorial interpretations to two invariants associated to $P$.

DEFINITION. Given a chain $c = I_1 < \cdots < I_k$ in $\{+1, -1, 0\}$, we define the *rank set* $S$ of $c$ to be the set $S = \{\#I_1, ..., \#I_k\}$ (where if $I_i = (\varepsilon_1, ..., \varepsilon_n)$, then $\#I_i = \#\{k : \varepsilon_k \neq 0\}$). For $S \subseteq \{1, 2, ..., n\}$, define

$$\alpha_S(P) = \#\{P\text{-compatible chains with rank set } S \text{ in } J(P) - \hat{0}\}$$

$$\beta_S(P) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_T(P).$$

$\beta_S(P)$ is sometimes called the *rank-selected Möbius invariant of P* (see [St1, Sect. 3.12]).

PROPOSITION 3.8.

$$\alpha_S(P) = \#\{w \in \mathscr{L}(P) : D(w) \subseteq S\}$$
$$\beta_S(P) = \#\{w \in \mathscr{L}(P) : D(w) = S\}.$$

*Proof.* By the previous proposition, we have

$$\begin{aligned}\alpha_S(P) &= \sum_{w \in \mathscr{L}(P)} \#\{\text{chains } c \text{ with rank set } S : c_r(w) \subseteq c \subseteq c_E(w)\}\\ &= \sum_{w \in \mathscr{L}(P)} \begin{Bmatrix} 1 & \text{if } D(w) \subseteq S \\ 0 & \text{else} \end{Bmatrix}\\ &= \#\{w \in \mathscr{L}(P) : D(w) \subseteq S\}.\end{aligned}$$

The second assertion follows by inclusion–exclusion. ∎

Our next result gives reciprocity formulas that hold between the various generating functions for $P$ and $-P = w_0 P$, where

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ -1 & -2 & & -n \end{pmatrix}$$

is the *longest element* of $B_n$ (see [Bo, Chap. VI, Sect. 1, Corollaire 3] for information about longest elements).

THEOREM 3.9 (Reciprocity). 1. $F(-P, x) = (-1)^n F(P, x)\,|_{x_i \to 1/x_{-i}}$.

2. $\beta_J(-P) = \beta_{S-J}(P)$.
3. $W_s(-P, x) = x^{\binom{n+1}{2}} W_{n-s}(P, x^{-1})$.
4. $W(-P, x) = x^{\binom{n+1}{2}} W(P, x^{-1})$.
5. $U(-P, x) = (-1)^n U(P, x^{-1})$.
6. $U_{-m}(-P, x) = (-1)^n U_{m-1}(P, x^{-1})$.

*Proof.* 1. From Proposition 3.5, we have that

$$\begin{aligned}F(P, x) &= \sum_{w \in \mathscr{L}(P)} \frac{\prod_{s_i \in D(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{i=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_i})}\\ &= \sum_{w \in \mathscr{L}(P)} \frac{x_{w_1}^{c_1(w)} \cdots x_{w_n}^{c_n(w)}}{\prod_{i=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_i})},\end{aligned}$$

where $c_i(w) = \# D(w) \cap \{s_i, s_{i+1}, ..., s_n\}$. Note that $\mathscr{L}(-P) = w_0 \mathscr{L}(P)$, and $c_i(w_0 w) = n + 1 - i - c_i(w)$. Thus we have

$$\begin{aligned} F(-P, x) &= \sum_{u \in \mathscr{L}(-P)} \frac{x_{u_1}^{c_1(u)} \cdots x_{u_n}^{c_n(u)}}{\prod_{i=1}^n (1 - x_{u_1} x_{u_2} \cdots x_{u_i})} \\ &= \sum_{w \in \mathscr{L}(P)} \frac{x_{(w_0 w)_1}^{c_1(w_0 w)} \cdots x_{(w_0 w)_n}^{c_n(w_0 w)}}{\prod_{i=1}^n (1 - x_{(w_0 w)_1} x_{(w_0 w)_2} \cdots x_{(w_0 w)_i})} \\ &= \sum_{w \in \mathscr{L}(P)} \frac{x_{-w_1}^{n - c_1(w)} x_{-w_2}^{n-1-c_2(w)} \cdots x_{-w_n}^{1 - c_n(w)}}{\prod_{i=1}^n (1 - x_{-w_1} x_{-w_2} \cdots x_{-w_i})}. \end{aligned}$$

Multiplying numerator and denominator above by $\prod_{i=1}^n (x_{-w_1} x_{-w_2} \cdots x_{-w_i})^{-1}$ gives

$$F(-P, x) = \sum_{w \in \mathscr{L}(P)} \frac{x_{-w_1}^{-c_1(w)} x_{-w_2}^{-c_2(w)} \cdots x_{-w_n}^{-c_n(w)}}{\prod_{i=1}^n (x_{-w_1}^{-1} x_{-w_2}^{-1} \cdots x_{-w_i}^{-1} - 1)} = (-1)^n F(P, x) \mid_{x_i \to 1/x_{-i}}.$$

2. This follows from the fact that $\beta_J(P) = \# \{w \in \mathscr{L}(P) : D(w) = J\}$, since it is easy to see that $\mathscr{L}(-P) = w_0 \mathscr{L}(P)$ and $D(w_0 w) = S - D(w)$.

3. Since $\# D(w_0 w) = \# (S - D(w)) = n - \# D(w)$, and $\operatorname{maj}(w_0 w) = \binom{n+1}{2} - \operatorname{maj}(w)$, we have

$$\begin{aligned} W_s(-P, x) &= \sum_{\substack{u \in \mathscr{L}(-P) \\ \# D(u) = s}} x^{\operatorname{maj}(u)} \\ &= \sum_{w \in \mathscr{L}(P)} x^{\operatorname{maj}(w_0 w)} \\ &= \sum_{\substack{w \in \mathscr{L}(P) \\ \# D(w) = n - s}} x^{\binom{n+1}{2} - \operatorname{maj}(w)} \\ &= x^{\binom{n+1}{2}} W_{n-s}(P, x^{-1}). \end{aligned}$$

4. This follows from 3 and the fact that $W(P, x) = \sum_{s=0}^n W_s(P, x)$.

5. This follows from 1 and the fact that $U(P, x) = F(P, x) \mid x_{\pm i} \to x$.

6. We have

$$\begin{aligned} U_{-m}(-P, x) &= \sum_{s=0}^n \begin{bmatrix} n - m - s \\ n \end{bmatrix}_x W_s(-P, x) \\ &= \sum_{s=0}^n \begin{bmatrix} n - m - s \\ n \end{bmatrix}_x x^{\binom{n+1}{2}} W_{n-s}(P, x^{-1}) \end{aligned}$$

$$= \sum_{s=0}^{n} (-1)^n \begin{bmatrix} m+s-1 \\ n \end{bmatrix}_{x^{-1}} W_{n-s}(P, x^{-1})$$
$$= (-1)^n \sum_{s=0}^{n} \begin{bmatrix} m+n-s-1 \\ n \end{bmatrix}_{x^{-1}} W_s(P, x^{-1})$$
$$= (-1)^n U_{m-1}(P, x^{-1}),$$

where the third equality comes from the easy-to-check fact that

$$\begin{bmatrix} n-m-s \\ n \end{bmatrix}_x x^{\binom{n+1}{2}} = (-1)^n \begin{bmatrix} m+s-1 \\ n \end{bmatrix}_{x^{-1}},$$

and the fourth equality comes from replacing $s$ by $n-s$. ∎

We will explore one further counting function of $P$.

DEFINITION. The *order polynomial* $\Omega(P; m)$ of $P$ is defined as

$$\Omega(P; m) = \#\mathscr{A}(P, m-1) = U_{m-1}(P; 1)$$

(its name anticipates the soon-to-be-proven fact that it is a polynomial in $m$). Define the *P-Eulerian numbers* $w_0(P), ..., w_n(P)$ by

$$w_s(P) = \#\{w \in \mathscr{L}(P) : \#D(w) = s\} = W_s(P, 1).$$

For $1 \leqslant j \leqslant n$, define the numbers $e_j(P)$, $e'_j(P)$ by

$$e_j(P) = \#\{f \in \mathscr{A}(P) : \{|f_i|\}_{i=1,...,n} = \{1, 2, ..., j\}\}$$
$$e'_j(P) = \#\{f \in \mathscr{A}(P) : \{|f_i|\}_{i=1,...,n} = \{0, 1, ..., j-1\}\}.$$

PROPOSITION 3.10. 1. $\Omega(P; m) = \sum_{j=1}^{n} (e_j(P)\binom{m-1}{j} + e'_j(P)\binom{m-1}{j-1})$ *and hence* $\Omega(P; m)$ *is a polynomial in* $m$ *of degree* $n$.

2. $\Omega(P; m) = \sum_{s=0}^{n} \binom{n+m-1-s}{n} w_s(P)$.
3. $\sum_{m \geqslant 0} q^m = (\sum_{s=0}^{n} w_s(P) q^{s+1})/(1-q)^{n+1}$.
4. (*Reciprocity*) $\Omega(-P; m) = (-1)^n \Omega(P; -m+1)$.

*Proof.* 1. We consider two classes of $f \in \mathscr{A}(P; m-1)$: those $f$ having $f_i \neq 0$ for all $i$, and those having some $f_i = 0$. For those $f$ in the first class, if we know the set $R(f) = \{|f_i|\}_{i=1,...,n}$ has cardinality $j$, then there is a unique $f' \in \mathscr{A}(P)$ such that $R(f') = \{1, 2, ..., j\}$ and $c(f) = c(f')$. Conversely, $f'$ and $R(f)$ completely determine $f$, so there are $\sum_{j=1}^{n} e_j(P)\binom{m-1}{j}$ elements in the first class. Similarly, for $f$ in the second class we must have $\{0\} \subseteq R(f) \subseteq \{0, 1, ..., m-1\}$, and hence if $\#R(f) = j$ then there is a unique $f' \in \mathscr{A}(P)$ with $R(f') = \{0, 1, ..., j-1\}$ and $c(f) = c(f')$. Again, $f'$

and $R(f)$ determine $f$, so there are $\sum_{j=1}^{n} e_j(P)\binom{m-1}{j-1}$ elements in the second class.

2. Plug $x=1$ into Proposition 3.6, part 1.
3. Plug $x=1$ into Proposition 3.6, part 2.
4. Plug $x=1$ into Theorem 3.9, part 6. ▮

EXAMPLE. Let $P=\{+e_2-e_1, +e_2\}$, so $-P=\{-e_2+e_1, -e_2\}$. One can check that

$$e_1(P)=1,\ e_2(P)=3, \qquad e'_1(P)=e'_2(P)=0$$

and

$$w_0(P)=w_2(P)=0, \qquad w_1(P)=3.$$

Thus by the first part of the previous proposition, we have

$$\Omega(P;m)=1\binom{m-1}{1}+3\binom{m-1}{2}+0\binom{m-1}{0}+2\binom{m-1}{1}=3\binom{m}{2}$$

or by the second part of the previous proposition, we have

$$\Omega(P;m)=\binom{2+m-1-0}{2}0+\binom{2+m-1-1}{2}3+\binom{2+m-1-2}{2}0$$
$$=3\binom{m}{2},$$

so the two agree. To check a case of the third (reciprocity) part of the proposition, note that $w_0(-P)=w_2(-P)=0$, $w_1(-P)=3$, and hence $\Omega(-P;m)=3\binom{m}{2}$ also. Therefore we have

$$(-1)^2\,\Omega(P,-m+1)=3\binom{-m+1}{2}=\frac{3(-m+1)(-m)}{2}$$
$$=\frac{3m(m-1)}{2}=\Omega(P;m),$$

as expected.

## 4. THE LATTICES $\hat{J}(P)$

In this section, we take a closer look at the posets $J(P)$. Our goal is to show that they give a $B_n$-analogue of distributive lattices, by proving an analogue of the *Fundamental Theorem of Distributive Lattices* [St1, Theorem 3.4.1], or *Birkhoff's theorem.*

DEFINITION. Let $\hat{J}(P)$ be the poset obtained from $J(P)$ by adjoining a (new) greatest element $\hat{1}$.

PROPOSITION 4.1. 1. *$\hat{J}(P)$ is a sublattice of the lattice* $\{+1, -1, 0\}^n$.

2. *$\hat{J}(P)$ depends (up to lattice-isomorphism) only on the isomorphism class of $P$.*

*Proof.* We prove 2 first. If $P \cong P'$, then $P = wP'$ for some $w \in B_n$ and hence we have

$$\langle f, \alpha \rangle \geqslant 0 \ \forall \alpha \in P \Leftrightarrow \langle w(f), w(\alpha) \rangle \geqslant 0 \ \forall \alpha \in P \Leftrightarrow \langle w(f), \beta \rangle \geqslant 0 \ \forall \beta \in P'.$$

So $w$ maps ideals of $P$ onto ideals of $P'$, and since $w$ is an automorphism of the order on $\{+1, -1, 0\}^n$, $w$ is an order-isomorphism of $J(P)$ onto $J(P')$.

To prove 1, let $\wedge$, $\vee$ denote meet and join in the lattice $\{+1, -1, 0\}^n$. We must show that if $g, f \in \hat{J}(P)$, then $g \wedge f$, $g \vee f \in \hat{J}(P)$. Clearly we may assume $g$, $f$, $g \wedge f$, $g \vee f$ are all unequal to $\hat{1}$. Given $\alpha \in P$, we want to show that $\langle \alpha, g \rangle \geqslant 0$ imply that $\langle \alpha, g \wedge f \rangle \geqslant 0$ and $\langle \alpha, g \vee f \rangle \geqslant 0$. We may assume $\alpha$ is of the form $+e_i$ or $+e_i - e_j$, since otherwise we could apply an element $w$ of $B_n$ to make it of this form, and use assertion 2.

If $\alpha = +e_i$, then $g_i, f_i \in \{+1, 0\}$, and hence $(g \wedge f)_i$, $(g \vee f)_i \in \{+1, 0\}$. Thus we have $\langle \alpha, g \wedge f \rangle \geqslant 0$ and $\langle \alpha, g \vee f \rangle \geqslant 0$, as desired.

If $\alpha = +e_i - e_j$, then the possibilities for $(g_i, g_j)$, $(f_i, f_j)$ are exactly the vectors shown in Fig. 5. One can see that the vectors in Fig. 5 are closed under meets, and also closed under joins whenever their join is unequal to $\hat{1}$. Hence as long as $g \vee f \neq \hat{1}$ we have $\langle \alpha, g \wedge f \rangle \geqslant 0$ and $\langle \alpha, g \vee f \rangle \geqslant 0$, as desired. ∎

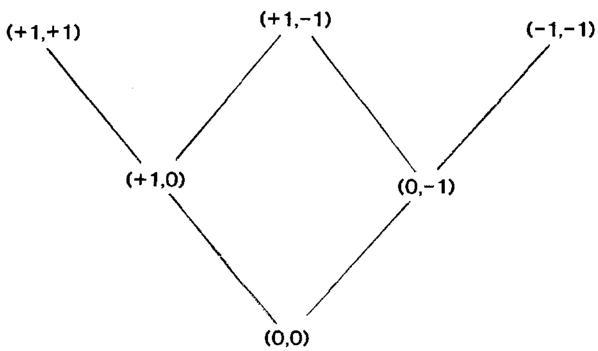



FIG. 5. Possibilities for $(g_i, g_j)$ if $\langle +e_i - e_j, g \rangle \geqslant 0$.

We now assemble some properties of the lattices $\hat{J}(P)$ that will help us to characterize them intrinsically.

PROPOSITION 4.2. *$J(P)$ is* locally distributive, *i.e., any interval $[x, y]$ in $J(P)$ is distributive.*

*Proof.* It is easy to see that any interval in $\{+1, -1, 0\}^n$ is a Boolean algebra and hence distributive. Since an interval $[x, y]$ in $J(P)$ is a sublattice of an interval in $\{+1, -1, 0\}^n$ (Proposition 4.1), it must also be distributive. ∎

DEFINITION. Given a finite lattice $L$, let $G(L)$ be the graph whose vertex set is the maximal elements of $L - \hat{1}$, and having an edge between two vertices $m_1$ and $m_2$ if $m_1$, $m_2$ both cover $m_1 \wedge m_2$ in $L$.

PROPOSITION 4.3. *$G(\hat{J}(P))$ is connected.*

*Proof.* First, we claim that $f \in \hat{J}(P) - \hat{1} = J(P)$ is maximal if and only if every coordinate $f_i \neq 0$. To see this, assume $f_i = 0$ for some $i$. We may assume $P \subseteq \Phi^+$ by applying some element $w^{-1}$ with $w \in \mathscr{L}(P)$. Then $f \in J(P)$ implies $f \in \mathscr{A}(P)$, so $f \in \mathscr{A}(w\Phi^+)$ for some $w \in \mathscr{L}(P)$ by Proposition 3.3. Let $f' = f + w(+e_{|w^{-1}(i)|})$, and note that $w(+e_{|w^{-1}(i)|}) = \pm e_i$, so $f'_i \in \{+1, -1\}$ and hence $f < f'$ in the order on $\{+1, -1, 0\}$. Furthermore, if $\alpha \in \Phi^+$, then

$$\begin{aligned} \langle w(\alpha), f' \rangle &= \langle w(\alpha), f \rangle + \langle w(\alpha), w(+e_{|w^{-1}(i)|}) \rangle \\ &= \langle w(\alpha), f \rangle + \langle \alpha, +e_{|w^{-1}(i)|} \rangle \geqslant 0. \end{aligned}$$

So $f' \in J(P)$ (since $P \subseteq w\Phi^+$), contradicting the maximality of $f$.

Now suppose $f$, $g$ are two distinct maximal elements of $J(P)$, and we will show that there is a path in $G(\hat{J}(P))$ connecting them. By restricting attention to the coordinates where they differ, we can assume $f_i \neq g_i$ for $i = 1, ..., n$, and by applying an element $w \in B_n$, we can assume $f = (+1, +1, ..., +1)$, $g = (-1, -1, ..., -1)$. This implies that $P$ can only contain roots of the form $+e_i - e_j$, and thus $P$ corresponds to a *poset* on $\{1, 2, ..., n\}$ (in which $i <_P j$ when $+e_i - e_j \in P$). Let $i$ be minimal in this poset, and let $g'$ be the vector with all $-1$'s except for a $+1$ in the $i$th coordinate. Then $g$ and $g'$ cover $g \wedge g'$, and we have that $f$ and $g'$ differ in one fewer coordinate than $f$ and $g$ did. So by induction we can find such a path. ∎

DEFINITION. An element $f \in J(P)$ is said to be *join-irreducible* (written $f \in \operatorname{Irr}(J(P))$) if $f \neq \mathbf{0}$, and $f = x \vee y$ implies either $f = x$ or $f = y$. For

$1 \leqslant i \leqslant n$, if $-e_i \notin P$, let $I^{+i}$ denote the least element $f \in J(P)$ having $f_i = +1$ (i.e., $I^{+1} = \bigwedge \{f \in J(P) : f_i = +1\}$). Define $I^{-i}$ similarly.

PROPOSITION 4.4. *$f \in \mathrm{Irr}(J(P)) \Leftrightarrow f = I^{+i}$ or $I^{-i}$ for some $i$.*

*Proof.* ($\Leftarrow$) Suppose $f = I^{+i}$ for some $i$ (the $f = I^{-i}$ case is identical, or apply $w = (\,^{i}_{-i})$). Then if $f = x \vee y$, either $x_i = +1$ or $y_i = +1$, so either $x \geqslant I^{+i} = f$ or $y \geqslant I^{+i} = f$.

($\Rightarrow$) Suppose $f \in \mathrm{Irr}(J(P))$. Let

$$T = \{+i : f_i = +1\} \cup \{-i : f_i = -1\}.$$

Clearly, $f \geqslant I^t \, \forall t \in T$, and $f \leqslant \bigvee_{t \in T} I^t$. Using the fact that $f$ is join-irreducible, and induction, we have $f = I^t$ for some $t \in T$. ∎

DEFINITION. Given a finite lattice $L$, and $I_1$, $I_2$ two join-irreducibles in $L - \hat{1}$, we will say $I_1 \sim I_2$ if there exist two maximal elements $m_1$, $m_2$ in $L - \hat{1}$ that are adjacent in $G(L)$ and satisfy $I_i \leqslant m_i$, but $I_i \nleqslant m_1 \wedge m_2$ for $i = 1, 2$.

PROPOSITION 4.5. *Let $I_1$, $I_2 \in \mathrm{Irr}(J(P))$. Then $I_1 \sim I_2$ if and only if for some $i \in \{\pm 1, ..., \pm n\}$ we have $I_1 = I^i$ and $I_2 = I^{-i}$.*

*Proof.* ($\Rightarrow$) Given $I_1 \sim I_2$ and $m_1$, $m_2$ as in the previous definition, by applying some element $w \in B_n$, we may assume $m_1 = (+1, +1, +1, ..., +1)$, $m_2 = (-1, +1, +1, ..., +1)$. Then the conditions that $I_i \leqslant m_i$ but $I_i \nleqslant m_1 \wedge m_2$ imply $I_1 = I^{+1}$, $I_2 = I^{-1}$.

($\Leftarrow$) Given that $I^i$, $I^{-i}$ both exist in $J(P)$, we must exhibit $m_1$, $m_2$ as in the above definition. Let

$$M_1 = \{m \in J(P) : m \text{ maximal, and } m \geqslant I^i\}$$
$$M_2 = \{m \in J(P) : m \text{ maximal, and } m \geqslant I^{-i}\}.$$

Since we saw (in the proof of Proposition 4.3) that every maximal element $m$ in $J(P)$ has all non-zero coordinates, these two sets $M_1$, $M_2$ disjointly cover all the maximal elements of $J(P)$. Since $G(\hat{J}(P))$ is connected, there must exist a pair of elements $m_1 \in M_1$, $m_2 \in M_2$ such that $m_1$, $m_2$ are adjacent in $G(\hat{J}(P))$. It is easy to see that these $m_1$, $m_2$ satisfy the conditions of the definition for $I^i \sim I^{-i}$. ∎

PROPOSITION 4.6. *Suppose $I_1$, $I_2$, $I_3$, $I_4 \in \mathrm{Irr}(J(P))$ satisfy $I_1 \sim I_2$ and $I_3 \sim I_4$. Then*

$$I_1 \leqslant I_3 \Leftrightarrow I_2 \geqslant I_4.$$

*Proof.* From the previous proposition, we have $I_1 = I^i$, $I_2 = I^{-i}$, $I_3 = I^j$, $I_4 = I^{-j}$ for some $i, j \in \{\pm 1, ..., \pm n\}$. But

$$I^i \leqslant I^j \Leftrightarrow -\operatorname{sgn}(j)\, e_{|j|} + \operatorname{sgn}(i)\, e_{|i|} \in P \Leftrightarrow I^{-i} \geqslant I^{-j}$$

so the result follows. ∎

PROPOSITION 4.7. *Let $\{I_i\}_{i=1,...,m} \in \operatorname{Irr}(J(P))$. Then $\bigvee_{i=1}^m I_i = \hat{1}$ if and only if there exist some $k \in \{1, 2, ..., n\}$ and $r, s \in \{1, ..., m\}$ such that $I^{+k} \leqslant I_r$ and $I^{-k} \leqslant I_s$.*

*Proof.* $\bigvee_{i=1}^m I_i = \hat{1}$ if and only if there exists some $k \in \{1, 2, ..., n\}$ such that $I_1(k), ..., I_m(k)$ have no upper bound in the partial order $+1 > 0$, $-1 > 0$. This is equivalent to saying that there exist $r, s \in \{1, ..., m\}$ with $I_r(k) = +1$, $I_s(k) = -1$, which is the same as $I^{+k} \leqslant I_r$, $I^{-k} \leqslant I_s$. ∎

It turns out that Propositions 4.2, 4.3, 4.6, 4.7 characterize the lattices $\hat{J}(P)$.

DEFINITION. We will say that a finite lattice $L$ is *$B_n$-distributive* if it satisfies the following four conditions:

1. $L - \hat{1}$ is locally distributive and
2. $G(L)$ is connected.
3. If $I_1, I_2, I_3, I_4 \in \operatorname{Irr}(L - \hat{1})$ satisfy $I_1 \sim I_2$ and $I_3 \sim I_4$, then we have

$$I_1 \leqslant I_3 \Leftrightarrow I_2 \geqslant I_4.$$

4. If $\{I_i\}_{i=1,...,m} \subseteq \operatorname{Irr}(L - \hat{1})$ and $\bigvee_{i=1,...,m} I_i = \hat{1}$, then there exist $I_0$, $I'_0 \in \operatorname{Irr}(L - \hat{1})$ and $r, s \in \{1, ..., m\}$ such that $I_0 \sim I'_0$ and $I_0 \leqslant I_r$, $I'_0 \leqslant I_s$.

We have not mentioned how the number $n$ (in the name $B_n$-distributive) enters the picture. However, it is easy to see that conditions 1 and 2 together imply that $L$ is ranked, and then we require that $n$ be equal to the rank of $L$.

THEOREM 4.8 (Signed Birkhoff's Theorem). *A finite lattice $L$ is isomorphic to $\hat{J}(P)$ for some signed posed $P$ if and only if $L$ is $B_n$-distributive. Furthermore, $P$ is determined by $L$ up to isomorphism as a signed poset.*

*Proof.* $(\Rightarrow)$ This is the content of Propositions 4.2, 4.3, 4.6, 4.7.

$(\Leftarrow)$ Assume $L$ is $B_n$-distributive. We give a procedure to extract a signed poset $P$ from $L$ with the property that $L \cong \hat{J}(P)$.

Let $m_1, m_2, ..., m_M$ be an ordering of the maximal elements in $L - \hat{1}$ such that for all $k \geqslant 1$, there exists an $l < k$ for which $m_{k+1}$ is adjacent to $m_l$ in

$G(L)$ (such an ordering exists since $G(L)$ is connected by condition 2). We will construct $P$ by a sequence $P_1, P_2, ..., P_M$ of approximations.

The first approximation $P_1$ is defined as follows. Let $+e_i$, $+e_i+e_j \in P_1$ for all $1 \leqslant i < j \leqslant n$. We will "think" of $m_1$ as being the ideal $(+1, +1, ..., +1)$ and label the elements of $\mathrm{Irr}(L-\hat{1})$ underneath $m_1$ arbitrarily as $I^{+1}, ..., I^{+n}$. Then we add $+e_i-e_j$ to $P_1$ if and only if $I^{+i} \leqslant I^{+j}$. This completes the construction of $P_1$.

Having gotten to stage $k$ and constructed $P_k$, we proceed inductively as follows. Let $m_l$ be adjacent to $m_{k+1}$ in $G(L)$. Since $m_l$ and $m_{k+1}$ are incomparable, there exist at least one element $I \in \mathrm{Irr}(L-\hat{1})$ satisfying $I \leqslant m_l$ but $I \nleqslant m_{k+1}$ and at least one $I' \in \mathrm{Irr}(L-\hat{1})$ satisfying $I' \leqslant m_{k+1}$ but $I' \nleqslant m_l$. Any two such $I$, $I'$ will have $I \sim I'$ by definition. But condition 3 implies that a given join-irreducible $J$ can have $J \sim J'$ for at most one element $J'$: if $J \sim J'$ and $J \sim J''$, then we have $J \leqslant J \Rightarrow J' \geqslant J''$ and vice versa, so $J' = J''$. Thus there is a *unique* pair of join-irreducibles $I$, $I'$ such that $I \sim I'$, $I \leqslant m_l$, $I' \leqslant m_{k+1}$. Since $I \leqslant m_l$, by induction, $I$ has already been labelled $I^i$ for some $i \in \{\pm 1, ..., \pm n\}$. We then label $I'$ as $I^{-i}$, and "think" of $m_{k+1}$ as the ideal that differs from $m_l$ exactly in the $i$th coordinate and nowhere else. We produce $P_{k+1}$ from $P_k$ by removing $+e_i$, and then removing $+e_i+e_j$ if and only if $I^{-i} \ngeqslant I^j$ for some previously labelled join-irreducible $I^j$. This completes the construction of $P_k$. Proceeding through all of the elements $m_1, m_2, ..., m_M$ yields our final approximation $P_M = P$.

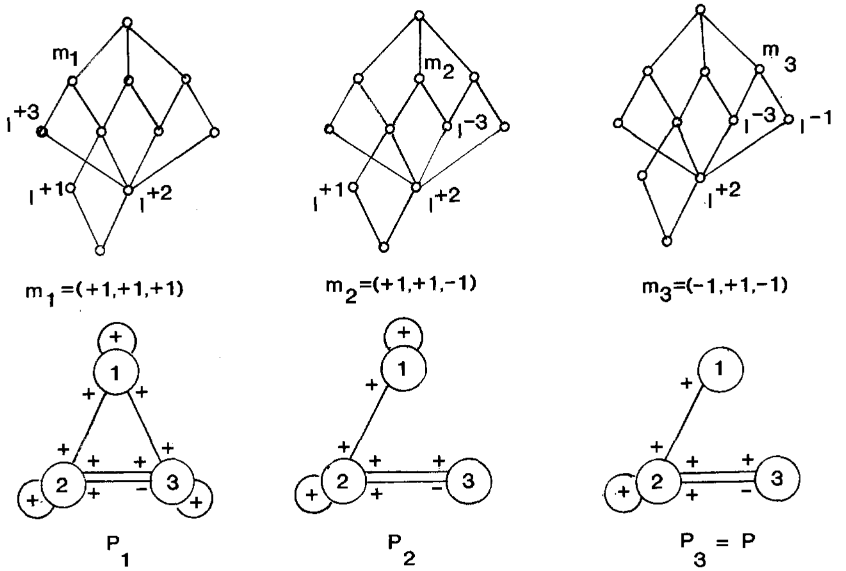



FIG. 6. Recovering a signed poset from a $B_n$-distributive lattice.

We want to show that $L \cong \hat{J}(P)$. The labelling of join-irreducibles during the above procedure gives a map $\phi$: $\mathrm{Irr}(L-\hat{1}) \to \mathrm{Irr}(J(P))$. A little thought shows that $\phi$ is a bijection (because of the way we removed roots of the form $+e_i$). Also, $\phi$ is a poset isomorphism, because of the way we included roots of the form $+e_i-e_j$ in $P_1$ (along with condition 3), and the way we removed roots of the form $+e_i+e_j$. We now use this bijection $\phi$ to define two maps $\tilde{\phi}: L \to \hat{J}(P)$ and $\tilde{\psi}: \hat{J}(P) \to L$. We let $\tilde{\phi}(\hat{1})=\hat{1}$, and $\tilde{\phi}(x)=\bigvee_i \phi(I_i)$ if $x \neq \hat{1}$ and $x=\bigvee_i I_i$ is the unique irredundant decomposition of $x$ into join-irreducibles (assured by the fact that the $[\hat{0}, x]$ is distributive). Similarly, we let $\tilde{\psi}(\hat{1})=\hat{1}$, and $\tilde{\psi}(x)=\bigvee_i \phi^{-1}(I^i)$ if $f \neq \hat{1}$ and $f=\bigvee_i I^i$ is the unique irredundant decomposition of $f$. One can easily check that condition 4 implies that $\tilde{\phi}(x)=\hat{1}$ if and only if $x=\hat{1}$, and that $\tilde{\psi}(f)=\hat{1}$ if and only if $f=\hat{1}$. Also, since $\phi$ is a poset-isomorphism, $\tilde{\phi}$ and $\tilde{\psi}$ are inverse poset-isomorphisms. Hence $L \cong \hat{J}(P)$.

If $L \cong J(Q)$ for some other signed poset $Q$, we can produce an element $w \in B_n$ such that $wP=Q$ as follows. For $1 \leqslant i \leqslant n$, a given join-irreducible of $L$ labelled $I^{+i}$ during the above procedure must correspond to some join-irreducible $I$ of $J(Q)$, and we know that $I$ must in fact be of the form $I^{w_i}$ for some $w_i \in \{\pm 1, ..., \pm n\}$. Let $w=\binom{1 \; \cdots \; n}{w_1 \cdots w_n}$, and it is not hard to see that $wP=Q$. ∎

An example of the procedure in the preceding proof is shown in Fig. 6.

## 5. More about $\hat{J}(P)$

In this section, we investigate the interval structure of $\hat{J}(P)$ and compute its Möbius function and characteristic polynomial. We also give an EL-labelling (and hence a shelling) of a larger class of lattices which are hyperoctahedral analogues of *upper-semimodular* lattices.

Proposition 5.1. *Let $[x, y]$ be an interval in $\hat{J}(P)$.*

1. *If $y=\hat{1}$, then $[x, y] \cong \hat{J}(P')$ for some signed poset $P'$.*

2. *If $y \neq 1$, then $[x, y] \cong J(P'')$ for some poset $Q$ (where here $J(P'')=J(P''^{+})$ is the usual lattice of order ideals in $P''$).*

*Proof.* 1. Given $[x, \hat{1}]$, let $T=\{i : x_i=0\}$, and let $P'$ be the induced signed subposet of $P$ on $T$. Then the map from $[x, \hat{1}]$ to $\hat{J}(P')$ which ignores all coordinates outside $T$ is clearly an isomorphism.

2. Given $[x, y]$, let $T=\{i : x_i \neq y_i\}$. Then by applying an element of $B_n$, we can make the restrictions of $x$ and $y$ to $T$ look like $(0, ..., 0)$ and $(+1, ..., +1)$, respectively. Let $P''$ be the partial order on the numbers in $T$ determined by $i <_{P''} j$ if $+e_i-e_j \in P$ and $i, j \in T$. Again, the map from

$[x, y]$ to $J(P'')$ which ignores all coordinates outside $T$ is clearly an isomorphism. ▮

In light of the previous proposition, rather than looking at intervals, we can concentrate our attention on the structure of the *whole* distributive lattice $J(P)$ for posets $P$ and the *whole* $B_n$-distributive lattice $\hat{J}(P)$ for signed posets $P$.

EXAMPLE. Let $P = \{+e_2 - e_1, +e_2 + e_3, +e_3\}$. Then the interval $[(0, 0, +1), \hat{1}]$ in $\hat{J}(P)$, along with $\hat{J}(P')$ (where $P' = \{+e_2 - e_1\}$), is shown in Fig. 7. The interval $[(0, 0, 0), (+1, +1, +1)]$, along with $J(P'')$, where $P''$ is the poset determined by $\{+e_2 - e_1\}$, is also shown in Fig. 7.

DEFINITION. Let $L$ be a lattice with a least element $\hat{0}$, and a greatest element $\hat{1}$. $L$ is *complemented* if $\forall x \in L\ \exists y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$ ($y$ is called a *complemented* of $x$). A minimal element of $L - \hat{0}$ is called an *atom*. $L$ is called *atomic* if $\hat{1} = \bigvee_{\text{atoms } x} x$.

It is well known (see, e.g., [St1, remarks after Proposition 3.4.4]), that for posets $P$ on $n$ elements, the following are equivalent:

1. $J(P)$ is complemented.
2. $J(P)$ is atomic.
3. $J(P)$ is the Boolean algebra $\{1, 0\}^n$.

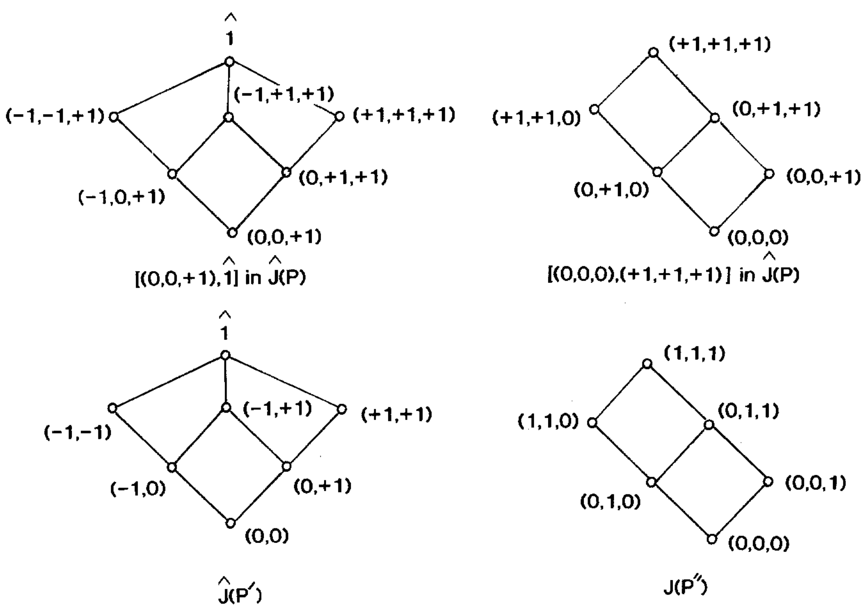



FIG. 7. Some examples of intervals in $\hat{J}(P)$.

PROPOSITION 5.2. *Let $P$ be a signed poset. Then*

1. *$\hat{J}(P)$ is complemented if and only if $\hat{J}(P)$ is the lattice $\{+1, -1, 0\}^n$, i.e., $P = \varnothing$.*

2. *$\hat{J}(P)$ is atomic if and only if some coordinate $i \in \{1, 2, ..., n\}$ is vacuous in $P$, i.e., every $\alpha \in P$ has zero $i^{th}$ coordinate.*

*Proof.* 1. Clearly $\{+1, -1, 0\}^n$ is complemented, since the complement of $f$ is given by $-f$. We must show then that $P \neq \varnothing$ implies $\hat{J}(P)$ is not complemented. Let $\alpha \in P$. Since $\hat{J}(P)$ only depends up to lattice-isomorphism on the isomorphism class of $P$, we can assume $P \subseteq \Phi^+$, so $\alpha \in \Phi^+$. If $\alpha = +e_i$, then one can check that $I^{+i}$ has no complement in $\hat{J}(P)$. If $\alpha = +e_i + e_j$ or $+e_i - e_j$ then one can check that $I^{+i} \vee I^{+j}$ or $I^{+i} \vee I^{-j}$ has no complement in $\hat{J}(P)$, respectively.

2. By Proposition 4.7, $\hat{J}(P)$ is atomic if and only if for some $i \in \{1, 2, ..., n\}$, both $I^{+i}$ and $I^{-i}$ are atoms. One can check that this means that $i$ is vacuous in $P$. ∎

DEFINITION. The *Möbius function* $\mu_Q$ of a poset $Q$ is the map from the intervals of $Q$ to $\mathbf{Z}$ defined recursively as

$$\mu_Q(x, x) = 1 \qquad \forall x \in Q,$$

$$\mu_Q(x, y) = - \sum_{z:\, x \leqslant z < y} \mu_Q(x, z).$$

If $Q$ is ranked with rank function $r$ and has a least element $\hat{0}$, then the *characteristic polynomial* $\chi(Q, \lambda)$ is defined by

$$\chi(Q, \lambda) = \sum_{x \in Q} \mu_Q(\hat{0}, x)\, \lambda^{r(Q) - r(x)}.$$

See [Ro] for more on these definitions.

It is known [Cr, Corollary to Theorem 3] that for a finite lattice $L$, $\mu_L(\hat{0}, \hat{1}) = 0$ unless $L$ is complemented. Hence for a poset $P$, we have

$$\mu_{J(P)}(\hat{0}, \hat{1}) = \begin{cases} (-1)^n & \text{if } P \text{ is an antichain} \\ 0 & \text{else.} \end{cases}$$

PROPOSITION 5.3. *If $P$ is a signed poset, then*

$$\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = \begin{cases} (-1)^n & \textit{if } P = \varnothing \\ 0 & \textit{else.} \end{cases}$$

*Proof.* If $P \neq \varnothing$, then $P$ is not complemented by Proposition 5.2, so $\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = 0$. If $P = \varnothing$, then $\mu_{\hat{J}(P)}(\hat{0}, \hat{1}) = (-1)^{n-1}$, since $\hat{J}(P)$ is the poset of faces of the $n$-dimensional *cross-polytope* (see [St1, Proposition 3.8.9]). Alternatively, one could compute this directly. ∎

PROPOSITION 5.4. *Let $P$ be a signed poset.*

1. *If $P=\varnothing$, then*

$$\chi(\hat{J}(P), \lambda)=\lambda(\lambda-2)^n+(-1)^{n+1}.$$

2. *If $P\neq\varnothing$, let $k$ be the number of coordinates $i$ which are vacuous in $P$, and let $a$ be the number of atoms of $\hat{J}(P)$. Then*

$$\chi(\hat{J}(P), \lambda)=\lambda^{n+1-a+k}(\lambda-1)^{a-2k}\,(\lambda-2)^k.$$

*Proof.* 1. Since $\hat{J}(P)=\{+1,-1,0\}^n$, we can just compute directly. For any $x\in J(P)$ we have $\mu(\hat{0},x)=(-1)^{r(x)}$, since $[\hat{0},x]$ is a Boolean algebra of rank $r(x)$. There are $\binom{n}{i}2^i$ elements of rank $i$ in $\{+1,-1,0\}$, and thus

$$\begin{aligned}\chi(\hat{J}(P), \lambda)&=\sum_{x\in\hat{J}(P)}\mu(\hat{0},x)\,\lambda^{n+1-r(x)}\\&=(-1)^{n+1}+\sum_{i=0}^{n}(-1)^i\binom{n}{i}2^i\lambda^{n+1-i}\\&=\lambda(\lambda-2)^n+(-1)^{n+1}.\end{aligned}$$

2. Our first observation is that

$$\chi(\hat{J}(P), \lambda)=\mu_{\hat{J}(P)}(\hat{0},\hat{1})+\lambda\cdot\chi(J(P),\lambda).$$

But $\mu_{\hat{J}(P)}(\hat{0},\hat{1})=0$ by the previous proposition, so $\chi(\hat{J}(P),\lambda)=\lambda\cdot\chi(J(P),\lambda)$.

Next we note that $J(P)$ factors as a direct product of posets in the following manner. Let $P'$ be the induced signed subposet of $P$ on the set of $n-k$ non-vacuous coordinates. Then one easily sees that

$$J(P)=\{+1,-1,0\}^k\times J(P').$$

Since it is easy to show that the characteristic polynomial satisfies

$$\chi(Q_1\times Q_2,\lambda)=\chi(Q_1,\lambda)\,\chi(Q_2,\lambda),$$

we have

$$\begin{aligned}\chi(\hat{J}(P),\lambda)&=\lambda\chi(\{+1,-1,0\}^k,\lambda)\,\chi(J(P'),\lambda)\\&=\lambda\chi(\{+1,-1,0\},\lambda)^k\,\chi(J(P'),\lambda)\\&=\lambda(\lambda-2)^k\,\chi(J(P'),\lambda).\end{aligned}$$

It remains only for us to calculate $\chi(J(P'), \lambda)$. Since $P'$ has no vacuous coordinates by construction, $\hat{J}(P')$ is not atomic (by Proposition 5.2). Hence if we let $z = \bigvee_{\text{atoms } x \in \hat{J}(P)} x$, then $z \neq \hat{1}$ so $z \in J(P)$. Since any interval $[\hat{0}, x]$ is distributive, $\mu(\hat{0}, x) = 0$ unless $x$ is a join of atoms, i.e., unless $x \in [\hat{0}, z]$. Thus

$$\begin{aligned}
\chi(J(P), \lambda) &= \sum_{x \in J(P')} \mu(\hat{0}, x)\, \lambda^{r(J(P')) - r(x)} \\
&= \sum_{x \in [\hat{0}, z]} \mu(\hat{0}, x)\, \lambda^{n-k-r(x)} \\
&= \lambda^{n-k-r(z)} \sum_{x \in [\hat{0}, z]} \mu(\hat{0}, x)\, \lambda^{r(z)-r(x)} \\
&= \lambda^{n-k-r(z)} \chi([\hat{0}, z], \lambda) \\
&= \lambda^{n-a+k} (\lambda - 1)^{a-2k},
\end{aligned}$$

where the last equality holds because $[\hat{0}, z]$ is a Boolean algebra of rank $a - 2k$. Thus, we have

$$\chi(\hat{J}(P), \lambda) = \lambda^{n+1-a+k} (\lambda - 1)^{a-2k} (\lambda - 2)^k. \quad \blacksquare$$

EXAMPLE. Let $n = 3$, and $P = \{+e_2 - e_3\}$. Then 1 is vacuous in $P$, so $k = 1$, and $a = \#\{(+1, 0, 0), (-1, 0, 0), (0, +1, 0), (0, 0, -1)\} = 4$. Thus by the previous proposition we have

$$\chi(\hat{J}(P), \lambda) = \lambda(\lambda - 1)^2 (\lambda - 2).$$

Figure 8 shows $\hat{J}(P)$ labelled with the values $\mu(\hat{0}, x)$, and the factorization $J(P) = \{+1, -1, 0\}^a \times J(P')$.

For a signed poset $P$, there is a condition on $P$ which allows us to give the numbers $\beta_S(P)$ for $S \subseteq \{1, 2, ..., n\}$ a Möbius function interpretation.

DEFINITION. We will say a signed poset $P$ is *natural* if $P \subseteq \Phi^+$. For $S \subseteq \{1, 2, ..., n\}$, let $\hat{J}(P)_S$ be the subposet of $\hat{J}(P)$ consisting of $\hat{0}$, $\hat{1}$, and all ideals $I$ such that $\# I \in S$.

PROPOSITION 5.5. *For $P$ a natural signed poset and $S \subseteq \{1, 2, ..., n\}$, we have*

$$\beta_S(P) = (-1)^{\# S} \mu_{\hat{J}(P)_S}(\hat{0}, \hat{1}).$$

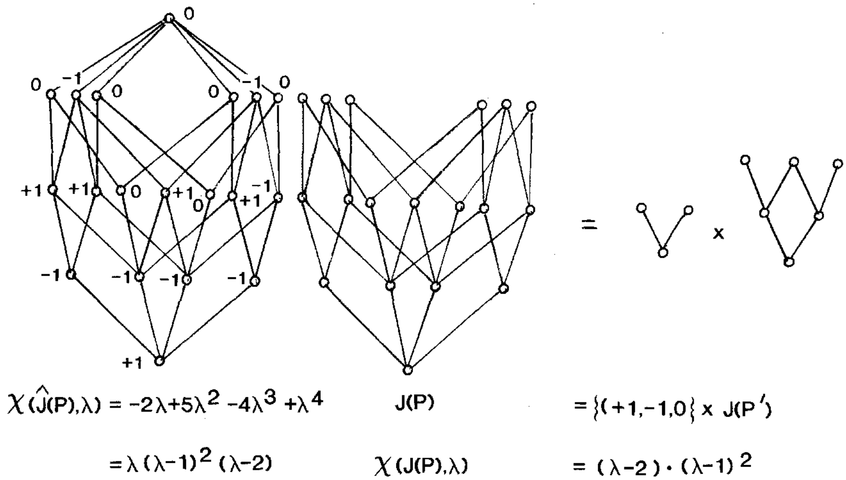



FIG. 8. An example of $\chi(j(P), \lambda)$.

*Proof.* We have that

$$\begin{aligned}\alpha_S(P) &= \#\{P\text{-compatible chains in } J(P)-\hat{0} \text{ with rank set } S\}\\ &= \#\{\text{chains in } J(P)-\hat{0} \text{ with rank set } S\}\end{aligned}$$

since $P$ is natural. Hence

$$\begin{aligned}\beta_S(P) &= \sum_{T\subseteq S} (-1)^{\#(S-T)}\,\alpha_L(P)\\ &= \sum_{T\subseteq S} (-1)^{\#(S-T)}\ \#\{\text{chains in } J(P)-\hat{0} \text{ with rank set } T\}\\ &= (-1)^{\#S} \sum_{\text{chains } c\in J(P)_S} (-1)^{\#c}\\ &= (-1)^{\#S}\,\mu_{\hat{J}(P)_S}(\hat{0},\hat{1}),\end{aligned}$$

where the last equality is by P. Hall's theorem ([Ro, Proposition 6], [St1, Proposition 3.8.5]). ∎

COROLLARY 5.6. $(-1)^{\#S}\mu_{\hat{J}(P)_S}(\hat{0},\hat{1}) = \#\{w\in B_n : D(w)=S\}$ *and hence is non-negative, for all* $S\subseteq\{1,2,...,n\}$.

*Proof.* Combine the previous proposition with Proposition 3.8. ∎

The previous corollary is sometimes phrased as follows: the Möbius function of $\hat{J}(P)_S$ *alternates in sign.* We now show that there is an even larger class of posets (containing all $B_n$-distributive lattices) with this property.

DEFINITION. We will say a finite lattice $L$ is *$B_n$-semimodular* if $L$ satisfies conditions 2, 3, and 4 in the definition of $B_n$-distributive, along with the following condition (which is weaker than the condition of local distributivity): every interval in $L-\hat{1}$ is *(upper)-semimodular* (a lattice is *upper-semimodular* if whenever $x$ covers $x \wedge y$ we have that $x \vee y$ covers $y$).

We will make use of the notion of an EL-labelling [Bj].

DEFINITION. Let $Q$ be a ranked poset. Write $x < \cdot y$ if $y$ covers $x$ in $Q$. We say $Q$ is *edgewise-lexicographically labellable* or *EL-labellable* if we can label the edges $E = \{(x, y) : x < \cdot y\}$ in the Hasse diagram using a map $\lambda \colon E \to \Lambda$ to a linearly ordered set $\Lambda$ satisfying:

1. For any interval $[x, y] \in Q$, there is a unique maximal chain

$$c[x, y] : x = x_0 < \cdot x_1 < \cdot \cdots \cdot < \cdot x_{k-1} < \cdot x_k = y,$$

for which the sequence of labels

$$(\lambda(x_0, x_1), \lambda(x_1, x_2), ..., \lambda(x_{k-1}, x_k))$$

is (weakly) increasing in $\Lambda$.

2. $c[x, y]$ is the least among all maximal chains of $[x, y]$ when we order them by their label sequences, using the lexicographic extension of $\Lambda$ to $\Lambda^k$.

In [Bj], Björner shows that when $Q$ is EL-labellable and has $\hat{0}$, $\hat{1}$, for any subset $S$ of the rank set of $Q$, the Möbius function of $Q_S$ alternates in sign for the following reason:

$$(-1)^{\# S} \mu_{Q_S}(\hat{0}, \hat{1}) = \# \{\text{maximal chains in } Q \text{ whose label set}$$
$$\text{decreases exactly after the ranks in } K\}.$$

It is known [Ga, Sect. 5] that all semimodular lattices are EL-labellable. We now prove a hyperoctahedral analogue of this result.

THEOREM 5.7. *If a finite lattice $L$ is $B_n$-semimodular, then $L$ is EL-labellable.*

*Proof.* We do the hyperoctahedral analogue of the proof of Theorem 5.1 in [Ga].

First we describe the edge-labelling $\lambda$. Let $\Lambda$ be the following linear order on $\mathbf{Z} \cup \{\infty\} - \{0\}$:

$$+1 <_\Lambda +2 <_\Lambda +3 <_\Lambda \cdots <_\Lambda \infty <_\Lambda \cdots <_\Lambda -3 <_\Lambda -2 <_\Lambda -1.$$

Now pick a maximal element $m$ in $L - \hat{1}$, and label the elements of $\mathrm{Irr}(L - \hat{1})$ which lie under $m$ by $I^{+1}, I^{+2}, ..., I^{+k}$ in such a way that $I^{+i} \leqslant_L I^{+j}$ implies $i \leqslant j$. We extend this to all join-irreducibles as in the proof of Theorem 4.8, i.e., if $I \in \mathrm{Irr}(L - \hat{1})$ and $I \sim I^{+i}$ for some $i$, then label $I$ as $I^{-i}$ (it is not hard to check that *all* join-irreducibles are labelled in this way). Now given $x < \cdot y$ in $L$, we label the edge $(x, y)$ in the Hasse diagram with $\lambda(x, y)$ defined as

$$\lambda(x, y) = \begin{cases} \infty & \text{if } y = \hat{1} \\ \min_\Lambda \{i : x \vee I^i = y\} & \text{else.} \end{cases}$$

Before we show that this is an EL-labelling, we note one property of our labelling of $\mathrm{Irr}(L - \hat{1})$: if $I^i \leqslant_L I^j$ then $i \leqslant_L j$. To see this, we check cases:

*Case* 1. $i, j$ both positive. Then $I^i \leqslant_L I^j$ implies $i \leqslant_\Lambda j$ by construction.

*Case* 2. $i, j$ both negative. Then $I^i \leqslant_L I^j$ implies $I^{-i} \geqslant_L I^{-j}$ by condition 3 of $B_n$-semimodularity, which implies $-i \geqslant -j$ and hence $i \leqslant_\Lambda j$.

*Case* 3. $i$ positive, $j$ negative. Then $i \leqslant_\Lambda j$ anyway.

*Case* 4. $i$ negative, $j$ positive. Then $I^i \leqslant I^j$ implies $I^i \leqslant m$, which contradicts the construction, so this case never happens.

Now we show that it is an EL-labelling. Let $x \leqslant y$ in $L$. We must exhibit $c[x, y]$, and show that it satisfies the two properties in the definition. If $x < \cdot y$, then $c[x, y]$ is just $x < \cdot y$, which trivially satisfies the definition. Otherwise, we will show how to construct $c[x, y]$ by induction on the length of the interval $[x, y]$.

Let

$$i = \min_\Lambda \{j : I^j \nleqslant x,\ I^j \leqslant y, \text{ and } I^j \vee x \neq \hat{1}\}$$

(if this set is empty, then $y = \hat{1}$ and $x < \cdot y$). We claim that $I^i \wedge x < \cdot I^i$. To see this, assume not, i.e., let $I^k$ satisfy

$$I^i \wedge x < I^k \vee (I^i \wedge x) < I^i.$$

Then $k \in \{j : I^j \nleqslant x,\ I^j \leqslant y, \text{ and } I^j \vee x \neq \hat{1}\}$, and $I^k < I^i$ implies $k <_\Lambda i$, contradicting the minimality of $i$. Thus $I^i \wedge x < \cdot I^i$, and using the fact that the interval $[\hat{0}, I^i \vee x]$ is semimodular, we conclude that $x < \cdot x \vee I^i$. Thus if we start our chain $c[x, y]$ with $x < \cdot x \vee I^i$, we can then continue by induction (replacing $x$ by $x \vee I^i$), and $c[x, y]$ will certainly be the lexicographically smallest maximal chain from $x$ to $y$.

We must check that this $c[x, y]$ has increasing labels. This is clearly true by construction if $y \neq \hat{1}$, since at each stage, the edge $x < \cdot x \vee I^i$ is labelled $i$. If $y = \hat{1}$, then the labels are certainly increasing, until the last step, which is labelled $\infty$. Thus it would suffice to show that $i$ is always positive. To see this, suppose not, i.e., $i < 0$. Then $I^i \vee x \neq \hat{1}$ implies that $I^{-i} \nleqslant x$. We also can infer that $I^{-i} \vee x \neq \hat{1}$, else by condition 4 of $B_n$-semimodularity there would be some $l$ for which $I^{-l} \leqslant I^{-i}$ and $I^{+l} \leqslant x$ and we would get the contradiction $I^i \leqslant I^{+l} \leqslant x$. Thus $-i$ is also in the set $\{j : I^j \nleqslant x,\ I^j \leqslant y$, and $I^j \vee x \neq \hat{1}\}$, and we have $-i <_A i$, contradicting the minimality of $i$.

Thus we have exhibited the lexicographically smallest chain from $x$ to $y$ and have shown that it has increasing labels. Now suppose $c$ is some other maximal chain from $x$ to $y$ with increasing labels. It remains only to show that $c = c[x, y]$, which we will do by induction on the length of $c$. Let $c$ be $x = x_0 < \cdot x_1 < \cdot \cdots < \cdot x_t = y$, and let $j$ be the unique index satisfying $I^i \nleqslant x_j$, but $I^i \leqslant x_{j+1}$.

*Case* 1. $x_{j+1} \neq \hat{1}$. Then $x_j \vee I^i = x_{j+1}$. Thus by minimality of $i$, this edge of $c$ must be labelled $i$. In order for $c$ to have increasing labels, this must be the first edge of $c$, i.e., $j = 0$, $x = x_j$, $x_{j+1} = x \vee I^i$. So $c$ and $c[x, y]$ agree in their first step, and we can apply induction on the length of $c$.

*Case* 2. $x_{j+1} = \hat{1}$. We will show that one of the labels on $c$ between $x$ and $x_j$ is negative, and hence $c$ is not increasing (since the last label on $c$ is $\infty$).

To see this, let the labels on $c$ between $x$ and $x_j$ be $i_1, ..., i_k$. Let $x = I^{i_{k+1}} \vee \cdots \vee I^{i_l}$ be an irredundant decomposition of $x$. Then we have

$$\begin{aligned} \hat{1} &= I^i \vee x_j = I^i \vee x \vee I^{i_1} \vee \cdots \vee I^{i_k} \\ &= I^i \vee I^{i_1} \vee \cdots \vee I^{i_l}. \end{aligned}$$

This implies (by condition 4 of $B_n$-semimodularity) that for some $r$, $s$ we have $I^s \leqslant I^i$ and $I^{-s} \leqslant I^{i_r}$. Hence (by condition 3 of $B_n$-semimodularity) we have $I^{-i} \leqslant I^{-s} \leqslant I^{i_r}$, as long as $I^{-i}$ exists in $L$. But $I^{-i}$ *must* exist, since $x_j$ is a maximal element of $L - \hat{1}$ which does not lie above $I^i$, so it must lie above $I^{-i}$ (it is easy to see from condition 1 of $B_n$-semimodularity that every maximal element of $L - \hat{1}$ must lie above either $I^i$ or $I^{-i}$). Now, if $r \geqslant k+1$ then $I^{-i} \leqslant x$, contradicting the fact that $I^i \vee x \neq \hat{1}$. If $r \leqslant k$, then $I^{-i} \leqslant I^{i_r}$ implies that $i_r$ is negative, as we desired. ∎

*Remark.* One can check that for a natural signed poset $P$ and $L = \hat{J}(P)$, if we choose $m$ in the above proof to be the ideal $(+1, +1, ..., +1) \in \hat{J}(P)$, and label each of the join-irreducibles $I^{+i}$ as themselves (i.e., label the least

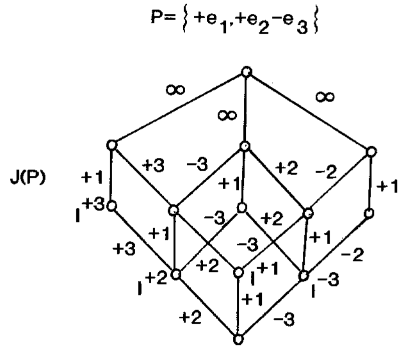



FIG. 9. An example of an EL-labelling for a lattice $J(P)$.

ideal having $+1$ in the $i$th coordinate as $I^{+i}$), then the label sequences of the maximal chains in $J(P)$ are exactly the same as $\mathscr{L}(P)$ (where we identify $w \in \mathscr{L}(P)$ with a sequence of numbers in $\{\pm 1, ..., \pm n\}$).

EXAMPLE. Let $L = \hat{J}(P)$ for $P = \{+e_1, +e_2 - e_3\}$. Figure 9 shows an EL-labelling as in the proof above, and a listing of $\mathscr{L}(P)$.

A corollary of the previous theorem is the fact that finite $B_n$-semimodular lattices are *shellable*, and hence *Cohen–Macaulay* (see [Bj] for the definitions and significance of these two conditions).

## 6. APPLICATIONS

Signed posets and their $P$-partitions may be used to derive the distributions of signed permutation statistics in the same way that posets and $P$-partitions are used to derive the distributions of permutation statistics (see [GG]). As an example, we give here a quick derivation of the generating function counting signed permutations $w \in B_n$ by the *major index* statistic $\text{maj}(w)$ (defined in Proposition 3.5). This may be viewed as an analogue of a special case of MacMahon's calculation of the generating function for permutations by their major index [Ma, Vol. I, No. 105; Vol. II, No. 453].

PROPOSITION 6.1. $\sum_{w \in B_n} x^{\text{maj}(w)} = (1+x)^n \, [n]!_x$, *where*

$$[n]!_x = \frac{1-x}{1-x} \frac{1-x^2}{1-x} \cdots \frac{1-x^n}{1-x}.$$

*Proof.* Let $P=\varnothing$, the empty signed poset. We will count $U(P, x)$ in two ways. On the one hand, by Proposition 3.5,

$$U(P, x)=\frac{\sum_{w \in B_n} x^{\operatorname{maj}(w)}}{(1-x)(1-x^2)\cdots(1-x^n)}.$$

On the other hand,

$$\begin{aligned} U(P, x) &= \sum_{f \in \mathbf{Z}^n} x^{|f_1|+\cdots+|f_n|} \\ &= \left(\sum_{f \in \mathbf{Z}} x^{|f|}\right)^n \\ &= (\cdots x^3+x^2+x+1+x+x^2+x^3+\cdots)^n \\ &= \left(\frac{1+x}{1-x}\right)^n. \end{aligned}$$

Setting these two expressions for $U(P, x)$ equal to each other, we conclude that

$$\sum_{w \in B_n} x^{\operatorname{maj}(w)}=(1-x)(1-x^2)\cdots(1-x^n)\left(\frac{1+x}{1-x}\right)^n=(1+x)^n\,[n]!_x. \quad \blacksquare$$

Another application of signed posets arises in a certain class of problems from invariant theory, which we will now describe. Let $R$ be the ring $\mathbf{Q}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ of finite Laurent polynomials in $n$ variables with rational coefficients, and let $B_n$ act on $R$ in the following way: if $w=\binom{1\ \cdots\ n}{w_1\ \cdots\ w_n}$ then

$$w(x_i)=\begin{cases} x_{w_i} & \text{if } \ w_i>0 \\ 1/x_{w_i} & \text{if } \ w_i<0. \end{cases}$$

Given any subgroup $G \subseteq B_n$, one can ask for a description of the subring $R^G$ of invariants under the action of $G$. Using the results in [Re, Sects. 4.1, 6.5], one can show that when $G$ is a *reflection subgroup* of $G$, Theorem 5.7 may be used to write down an explicit basis for $R^G$ as a free module over $R^{B_n}$. We refer the interested reader to [Re] for more details.

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