NOTE ON 1-CROSSING PARTITIONS

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ABSTRACT. It is shown that there are $\binom{2n-r-1}{n-r}$ noncrossing partitions of an n-set together with a distinguished block of size r, and $\binom{n}{k-1}\binom{n-r-1}{k-2}$ of these have k blocks, generalizing a result of Bóna on partitions with one crossing. Furthermore, specializing natural q-analogues of these formulae with q equal to certain d^{th} roots-of-unity gives the number of such objects having d-fold rotational symmetry.

Given a partition π of the set $[n] := \{1, 2, \ldots, n\}$, a *crossing* in π is a quadruple of integers (a, b, c, d) with $1 \leq a < b < c < d \leq n$ for which a, c are together in a block, and b, d are together in a different block. It is well-known [10, Exercise 6.19(pp)],[4] that the number of *noncrossing partitions* of $[n]$ (that is, those with no crossings) is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and the number of noncrossing partitions of $[n]$ into k blocks is the Narayana number $\frac{1}{n}\binom{n}{k-1}\binom{n}{k}$.

Our starting point is the more recent observation of Bóna [2, Theorem 1] that the number of partitions of $[n]$ having *exactly one* crossing has the even simpler formula $\binom{2n-5}{n-4}$. Bóna's proof utilizes the fact that C_n is also well-known to count triangulations of a convex $(n+2)$-gon; this allows him to biject 1-crossing partitions of $[n]$ to dissections of an n-gon that use exactly $n-4$ diagonals. The proof is then completed by plugging $d = n-4$ into the formula $\frac{1}{d+1}\binom{n+d-1}{d}\binom{n-3}{d}$ of Kirkman (first proven by Cayley; see [7]) for the number of dissections of an n-gon using d diagonals.

The goal here is to generalize Bóna's result to count 1-crossing partitions by their number of blocks, and also to examine a natural q-analogue with regard to the *cyclic sieving phenomenon* shown in [8] for certain q-Catalan and q-Narayana numbers. The crux is the observation that 1-crossing partitions of $[n]$ biject naturally with noncrossing partitions of $[n]$ having a distinguished 4-element block: replace the crossing pair of

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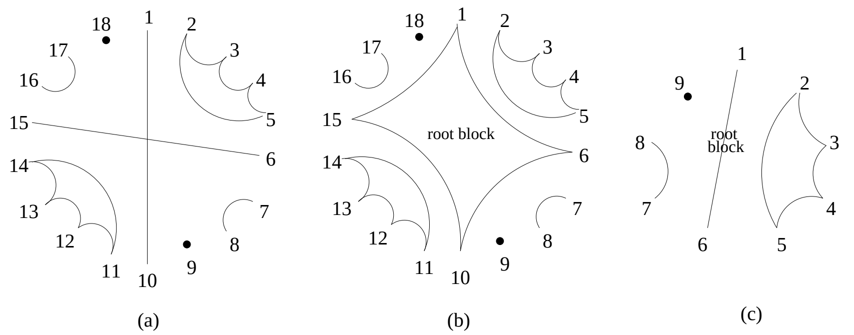

FIGURE 1. (a) A 1-crossing partition of the set [18]. (b) Its corresponding 4-rooted noncrossing partition of [18], which has 2-fold rotational symmetry. (c) The corresponding 2-rooted noncrossing partition of the set [9].

blocks $\{a,c\},\{b,d\}$ with a single distinguished *root block* $\{a,b,c,d\}$. Figure 1(a) gives an example of the 1-crossing partition of [18] having blocks $\{1,10\},\{2,3,4,5\},\{6,15\},\{7,8\},\{9\},\{11,12,13,14\},\{16,17\},\{18\}$, shown in its *circular representation*, with the two blocks $\{1,10\},\{6,15\}$ responsible for the unique crossing pair. Figure 1(b) shows the corresponding noncrossing partition of $[n]=[18]$ with distinguished 4-element root block $\{a,b,c,d\}=\{1,6,10,15\}$ that replaced the crossing pair of blocks.

Thus one is motivated to count the following more general objects.

Definition 1. An *r-rooted noncrossing partition of* $[n]$ is a pair (π,B) of a noncrossing partition π together with a distinguished r-element block B of π, which we will call the *root* block.

Note that the notion of a crossing in a partition is invariant under cyclic rotations $i \mapsto i+1 \mod n$ of the set $[n]$. Consequently the cyclic group $C=\mathbb{Z}_n$ acts on the set of r-rooted noncrossing partitions of $[n]$, preserving the number of blocks. For the sake of stating our result, define these standard q-analogues:

$$[n]_q := \frac{1-q^n}{1-q}$$
$$[n]!_q := [n]_q[n-1]_q\cdots[2]_q[1]_q$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}.$$

Theorem 1. *The number of r-rooted noncrossing partitions of $[n]$, and the number of such partitions with exactly k blocks, are given by the formulae*

$$
\begin{aligned}
a(n,r) &:= \binom{2n-r-1}{n-r}, \\
a(n,k,r) &:= \binom{n}{k-1}\binom{n-r-1}{k-2}.
\end{aligned}
\tag{1}
$$

Furthermore, for any d dividing n, the number of r-rooted noncrossing partitions of $[n]$ fixed under a d-fold cyclic rotation, and the number of such partitions having exactly k blocks, are obtained by plugging in any primitive d^{th} root-of-unity for q in these q-analogues:

$$
\begin{aligned}
a_q(n,r) &:= \begin{bmatrix} 2n-r-1 \\ n-r \end{bmatrix}_q, \\
a_q(n,k,r) &:= q^{(k-1)(k-2)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ k-2 \end{bmatrix}_q.
\end{aligned}
\tag{2}
$$

Note that taking $r = 4$ and replacing k by $k-1$ in (1), one finds agreement with Bóna's count of $\binom{2n-5}{n-4}$, as well as the (new) formula $\binom{n}{k-2}\binom{n-5}{k-3}$ for the number of 1-crossing partitions with k blocks.

Proof. (of Theorem 1) Note that the formula for $a(n,k)$ follows from the one for $a(n,k,r)$:

$$
\begin{aligned}
a(n,r) &= \sum_{k=1}^{n} a(n,r,k) \\
&= \sum_{k=1}^{n} \binom{n}{k-1}\binom{n-r-1}{k-2} \\
&= \sum_{k=1}^{n} \binom{n}{k-1}\binom{n-r-1}{n-r-k+1} \\
&= \sum_{i+j=n-r}^{n} \binom{n}{i}\binom{n-r-1}{j} \\
&= \binom{2n-r-1}{n-r}
\end{aligned}
$$

where the last equality follows from the Chu-Vandermonde summation formula $\binom{M+N}{\ell} = \sum_{i+j=\ell}\binom{M}{i}\binom{N}{j}$ specialized to

$$M := n,\ N := n-r-1,\ \ell := n-r.$$

To prove the formula for $a(n,k,r)$, consider three related sets. Let $A(n,k,r)$ denote the set of r-rooted noncrossing partitions of $[n]$ with k blocks, which we wish to count. Let $B(n,k,r)$ denote the set of triples

(π, B, i) in which π is a noncrossing partition of $[n]$ with k blocks, i is a chosen element of $[n]$, and B is an r-element block of π, with $i \in B$. Let $C(n,k,r)$ denote the set of noncrossing partitions of $[n]$ in which the element 1 lies in an r-element block.

Counting $|B(n,k,r)|$ in two ways, one finds

$$r \cdot |A(n,k,r)| = |B(n,k,r)| = n \cdot |C(n,k,r)|,$$

and hence

$$a(n,k,r) = |A(n,k,r)| = \frac{n}{r}|C(n,k,r)|. \tag{3}$$

To count $|C(n,k,r)|$, note that Dershowitz and Zaks [4] give a bijection between noncrossing partitions and ordered trees, which restricts to a bijection between $C(n,k,r)$ and the set $D(n,k,r)$ of all ordered trees having n edges, root degree r, and k internal nodes. On the other hand, the set $D(n,k,r)$ has been enumerated multiple times in the literature via generating functions and Lagrange inversion (e.g. in [3, 5]), and can also be done semi-bijectively (see [1]):

$$|D(n,k,r)| = \frac{r}{n}\binom{n}{k-1}\binom{n-r-1}{k-2}.$$

Thus the formula for $a(n,k,r)$ follows from combining this with (3):

$$a(n,k,r) = \frac{n}{r}|C(n,k,r)| = \frac{n}{r}|D(n,k,r)| = \binom{n}{k-1}\binom{n-r-1}{k-2}.$$

For the assertion of the theorem about q-analogues, we first deal with the case of $a_q(n,k,r)$. Note that for any d dividing n, an r-rooted noncrossing partition of $[n]$ having k blocks has no chance of being d-fold symmetric unless r is divisible by d and k is congruent to 1 mod d. Furthermore, when these congruences hold, if one defines $n' := \frac{n}{d}, r' := \frac{r}{d}, k' := \frac{k-1}{d}$, then the map $[n] \cong \mathbb{Z}_n \to \mathbb{Z}_{n'} \cong [n']$ which reduces modulo n' gives a natural bijection between d-fold rotationally symmetric r-rooted noncrossing partitions of $[n]$ with k blocks, and r'-rooted noncrossing partitions of $[n']$ with $k'+1$ blocks. For example, in Figure 1(b), one has such a d-fold rotationally symmetric r-rooted noncrossing partition with $d = 2, n = 18, r = 4, k = 7$, and Figure 1(c) depicts the corresponding r'-rooted noncrossing partition of $[n']$ with $n' = 9, r' = 2, k' = 3$.

Hence by the first part of the theorem, there are exactly $\binom{n'}{k'}\binom{n'-r'-1}{k'-1}$ such d-fold rotationally symmetric r-rooted noncrossing partitions of $[n]$ having k blocks in this case.

On the other hand, one can easily evaluate $a_q(n,k,r)$ when q is a primitive d^{th} root-of-unity for d dividing n, using the *q-Lucas theorem* (Lemma 2 below). One finds that it vanishes unless r is divisible by d and k is congruent to 1 mod d, in which case it equals $\binom{n'}{k'}\binom{n'-r'-1}{k'-1}$, as desired.

For the assertion about $a_q(n,r)$, one can either argue similarly, or use the identity $\begin{bmatrix} 2n-r-1 \\ n-r \end{bmatrix}_q = \sum_{k=1}^{n} q^{(k-1)(k-2)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ k-2 \end{bmatrix}_q$, which follows from setting $M := n$, $N := n-r-1$, $\ell := n-r$ in the *q-Chu-Vandermonde summation* (see e.g. [6, (7.6)]):

$$\begin{bmatrix} M+N \\ \ell \end{bmatrix}_q = \sum_{i+j=\ell} q^{j(M-i)} \begin{bmatrix} M \\ i \end{bmatrix}_q \begin{bmatrix} N \\ j \end{bmatrix}_q .$$

□

The following straightforward lemma used in the above proof has been rediscovered many times; see [9, Theorem 2.2] for a proof and some history.

Lemma 2. *(q-Lucas theorem) Given nonnegative integers n, k, d, with $1 \leq d \leq n$, uniquely write $n = n'd + n''$ and $k = k'd + k''$ with $0 \leq n'', k'' < d$. If q is a primitve d^{th} root-of-unity, then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n'}{k'} \begin{bmatrix} n'' \\ k'' \end{bmatrix}_q .$$

Lastly we remark that one can derive an explicit formula for the number of 2-crossing partitions of $[n]$, but it is much messier than $a(n,r)$ above, and appears to have no q-analogue with good behavior. However, Bóna [2] does show that for each fixed k, the generating function counting k-crossing partitions of $[n]$ is a rational function of x and $\sqrt{1-4x}$.

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