FIBER POLYTOPES FOR THE PROJECTIONS BETWEEN CYCLIC POLYTOPES

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Abstract. The cyclic polytope $C(n,d)$ is the convex hull of any n points on the moment curve $\{(t,t^2,\dots,t^d) : t \in \mathbb{R}\}$ in $\mathbb{R}^d$. For $d' > d$, we consider the fiber polytope (in the sense of Billera and Sturmfels) associated to the natural projection of cyclic polytopes $\pi : C(n,d') \to C(n,d)$ which "forgets" the last $d'-d$ coordinates. It is known that this fiber polytope has face lattice indexed by the coherent polytopal subdivisions of $C(n,d)$ which are induced by the map π. Our main result characterizes the triples (n,d,d') for which the fiber polytope is canonical in either of the following two senses:

- *all* polytopal subdivisions induced by π are coherent,
- the structure of the fiber polytope does not depend upon the choice of points on the moment curve.

We also discuss a new instance with a positive answer to the Generalized Baues Problem, namely that of a projection $\pi : P \to Q$ where Q has only regular subdivisions and P has two more vertices than its dimension.

1. Introduction

The cyclic d-polytope with n vertices is the convex hull of any n points on the moment curve $\{(t,t^2,\dots,t^d) : t \in \mathbb{R}\}$ in $\mathbb{R}^d$. Historically, the cyclic polytopes played an important role in polytope theory because they provide the upper bound for the number of faces of a d-polytope with n vertices [31, Chapter 8],[19, §4.7]. Although the cyclic polytope itself depends upon the choice of these n points, much of its combinatorics, such as the structure of its lattice of faces or its set of triangulations, is well-known to be independent of this choice (see [31, 14, 24]). For this reason, we will often abuse notation and refer to the cyclic d-polytope with n vertices as $C(n,d)$, making reference to the choice of points only when necessary.

The cyclic polytopes come equipped with a natural family of maps between them: fixing a pair of dimensions $d' > d$, the map $\pi : \mathbb{R}^{d'} \to \mathbb{R}^d$ which forgets the last $d'-d$ coordinates restricts to a surjection $\pi : C(n,d') \to C(n,d)$. Here we are implicitly assuming that if the points on the moment curve in $\mathbb{R}^{d'}$ chosen to define $C(n,d')$ have first coordinates $t_1 < \cdots < t_n$, then the same is true for the points in $\mathbb{R}^d$ chosen to define $C(n,d)$.

Our starting point is that these maps $\pi : C(n,d') \to C(n,d)$ provide interesting and natural examples for Billera and Sturmfels' theory of *fiber polytopes* [5]. Given an affine surjection of polytopes $\pi : P \to Q$, the *fiber polytope* $\Sigma(P \xrightarrow{\pi} Q)$ is a polytope of dimension $\dim(P) - \dim(Q)$ which is (in a well-defined sense; see [5]) the "average" fiber of the map π. The face poset of $\Sigma(P \xrightarrow{\pi} Q)$ has a beautiful combinatorial-geometric interpretation: it is the refinement ordering on the set of all polytopal subdivisions of Q which are induced by the projection π from P in a certain combinatorial sense, and also *π-coherent* in a geometric sense - see Section 2. This face

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poset sits inside the larger *Baues poset* $\omega(P \xrightarrow{\pi} Q)$, which is the refinement ordering on all polytopal subdivisions of Q which are induced by π, or π*-induced*.

For the case of cyclic polytopes, the Baues poset $\omega(C(n,d') \xrightarrow{\pi} C(n,d))$ of all π-induced subdivisions does not depend on the choice of points along the moment curve. On the other hand, a π-induced subdivision may be π-coherent or not depending on the choice of points. The main question addressed by this paper is: "How canonical is the fiber polytope $\Sigma(C(n,d') \xrightarrow{\pi} C(n,d))$, i.e. to what extent does its combinatorial structure vary with the choice of points on the moment curve?". There are at least two ways in which $\Sigma(C(n,d') \xrightarrow{\pi} C(n,d))$ can be canonical:

- If all π-induced subdivisions of $C(n,d)$ are π-coherent (for a certain choice of points, and hence for all by Lemma 4.2) then the face lattice of $\Sigma(C(n,d') \xrightarrow{\pi} C(n,d))$ coincides with the Baues poset $\omega(C(n,d') \xrightarrow{\pi} C(n,d))$.
- Even if there exist π-induced subdivisions of Q which are not π-coherent, it is possible that the identity of the π-coherent subdivisions (and, in particular, the face lattice of the fiber polytope) is independent of the choice of points.

Our main result characterizes exactly for which values of n, d and d' each of these two situations occurs.

Theorem 1.1. *Consider the map* $\pi : C(n,d') \to C(n,d)$.

(1) *If* $d = 1$ *then the set of* π*-coherent polytopal subdivisions of* $C(n,1)$*, and hence the face lattice of the fiber polytope* $\Sigma(C(n,d') \xrightarrow{\pi} C(n,1))$*, is independent of the choice of points on the moment curve. In fact, the face lattice of* $\Sigma(C(n,d') \xrightarrow{\pi} C(n,1))$ *coincides with that of the cyclic* $(d'-1)$*-zonotope having* $n-2$ *zones. Furthermore, all* π*-induced polytopal subdivisions of* $C(n,1)$ *are* π*-coherent if and only if* $d' = n-1$ *or* $d' = 2$.

(2) *If* $n - d' = 1$ *then:*
- *If either* $d \leq 2$ *or* $n - d \leq 3$ *or* $(n,d) \in \{(8,4),(8,3),(7,3)\}$ *then all* π*-induced subdivisions of* $C(n,d)$ *are* π*-coherent.*
- *In all other cases with* $n - d' = 1$*, there exists a* π*-induced subdivision of* $C(n,d)$ *whose* π*-coherence varies with the choice of points on the moment curve and for every choice of points there is some* π*-induced but not* π*-coherent subdivision.*

(3) *If* $d' - d = 1$ *then there are exactly two proper* π*-induced subdivisions, both of them* π*-coherent in every choice of points.*

(4) *If* $n - d' \geq 2$*,* $d' - d \geq 2$ *and* $d \geq 2$ *then there exists a* π*-induced subdivision of* $C(n,d)$ *whose* π*-coherence varies with the choice of points on the moment curve and for every choice of points there is some* π*-induced but not* π*-coherent subdivision.*

Part (1) is proved in Section 3. The π-induced subdivisions in this case are the so-called *cellular strings* [7] and the finest ones (the atoms in the Baues poset) the *monotone edge paths*. The fiber polytope in this case is the so-called *monotone path polytope* [5, 7]. The *cyclic zonotope* $Z(n,d)$, which appears in the statement, is the Minkowski sum of line segments in the directions of any n points on the moment curve. Like the cyclic polytope $C(n,d)$, its combinatorial structure (face lattice) does not depend upon the choice of points on the moment curve.

In the case of part (2) all subdivisions of $C(n,d)$ are π-induced and the fiber polytope is the *secondary polytope* of $C(n,d)$, introduced by Gel'fand et al. [17]. The same authors [18] and Lee [21, 22] proved that in the cases $d = 1$ and $d = 2$ or $n \leq d+3$, respectively, all (π-induced) subdivisions are *regular* for an arbitrary polytope Q. Even more, it can be deduced from their work that the secondary polytope of $C(n,d)$ is an $(n-2)$-cube for $d=1$, an $(n-3)$-dimensional associahedron for $d = 2$ and an n-gon for $n = d+3$. We prove the rest of part (2) in Section 4.

Part (3) is trivial and is included only for completeness. For any surjection $\pi : P \to Q$ of a $(d+1)$-polytope P onto a d-polytope Q there are only two π-induced proper polytopal subdivisions, both π-coherent: the subdivisions of Q induced by the "upper" and "lower" faces of P with respect to the projection π.

Part (4) is proved in Section 5.

Section 6 deals with an instance of the Generalized Baues Problem (GBP) posed in [7]. The GBP asks, in some sense, how close topologically the Baues poset $\omega(P \xrightarrow{\pi} Q)$ is to the face poset of $\Sigma(P \xrightarrow{\pi} Q)$ inside it. The proper part of this face poset is the face poset of the $(\dim(P)-\dim(Q)-1)$-dimensional sphere which is the boundary of $\Sigma(P \xrightarrow{\pi} Q)$. The GBP asks whether the proper part of the Baues poset (suitably topologized [9]) is homotopy equivalent to a $(\dim(P) - \dim(Q) - 1)$-dimensional sphere. This is known to be true when $\dim(Q) = 1$ [7] and when $\dim(P)-\dim(Q) \leq 2$, but false in general [25, 27]. In previous work on cyclic polytopes ([26] and [15] for $d \leq 3$) it was shown to be true for $C(n, n-1) \xrightarrow{\pi} C(n, d)$. We prove the following result, which in particular answers the question positively for $C(n, n-2) \xrightarrow{\pi} C(n, 2)$.

Theorem 1.2. *Let $\pi : P \to Q$ have the property that*

- *P has $\dim(P) + 2$ vertices and*
- *the point configuration $\mathcal{A}$ which is the image of the set of vertices of P under π has only coherent subdivisions.*

Then the GBP has a positive answer for $\pi : P \to Q$.

One might be tempted to conjecture the following extension of Theorem 1.2: the GBP has a positive answer if $\mathcal{A}$ has only regular subdivisions, no matter what P might be. However, one of the counterexamples to the GBP given in [27] disproves this extension. In that counterexample, $\mathcal{A}$ is planar and its 10 elements are three copies of the vertices of a triangle, together with a point inside.

2. Background on fiber polytopes

The fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$, introduced in [5], is a polytope naturally associated to any linear projection of polytopes $\pi : P \to Q$. An introduction to fiber polytopes may also be found in [31, Chapter 9]. In this section, we review the definitions given in these two sources and discuss some reformulations and further theory which we will need later.

Let P be a d'-dimensional polytope with n vertices in $\mathbb{R}^{d'}$, Q a d-dimensional polytope in $\mathbb{R}^d$ and $\pi : \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d$ a linear map with $\pi(P) = Q$. A *polytopal subdivision* of Q is a polytopal complex which subdivides Q. A polytopal subdivision of Q is *π-induced* if

(i) it is of the form $\{\pi(F) : F \in \mathcal{F}\}$ for some specified collection $\mathcal{F}$ of faces of P and
(ii) $\pi(F) \subseteq \pi(F')$ implies $F = F' \cap \pi^{-1}(\pi(F))$, thus in particular $F \subseteq F'$.

It is possible that different collections $\mathcal{F}$ of faces of P project to the same subdivision $\{\pi(F) : F \in \mathcal{F}\}$ of Q, so we distinguish these subdivisions by labelling them with the family $\mathcal{F}$. Note that condition (ii) is superfluous for the family of projections $C(n, d') \to C(n, d)$. We partially order the π-induced subdivisions of Q by $\mathcal{F}_1 \leq \mathcal{F}_2$ if and only if $\bigcup \mathcal{F}_1 \subseteq \bigcup \mathcal{F}_2$. The resulting partially ordered set is denoted by $\omega(P \xrightarrow{\pi} Q)$ and called the *Baues poset*. The minimal elements in this poset are the *tight* subdivisions, i.e. those for which F and $\pi(F)$ have the same dimension for all F in $\mathcal{F}$.

There are a number of ways to define π-coherent subdivisions of Q. We start with the original definition from [5]. Choose a linear functional $f \in (\mathbb{R}^{d'})^*$. For each point q in Q, the fiber $\pi^{-1}(q)$ is a convex polytope which has a unique face $\overline{F}_q$ on which the value of f is minimized. This face lies in the relative interior of a unique face F_q of P and the collection of faces $\mathcal{F} = \{F_q\}_{q \in Q}$ projects under π to a subdivision of Q. Subdivisions of Q which arise from a functional f in this fashion are called *π-coherent.*

It is worth mentioning here a slight variant of this description of the π-coherent subdivision induced by f (see also the paragraph after the proof of Theorem 2.1 in [6]). Note that the inclusion $ker(\pi) \hookrightarrow \mathbb{R}^{d'}$ induces a surjection $(\mathbb{R}^{d'})^* \twoheadrightarrow ker(\pi)^*$. A little thought shows that two functionals $f, f' \in (\mathbb{R}^{d'})^*$ having the same image under this surjection will induce the same π-coherent subdivision. As a consequence, we may assume that the functional f lies in $ker(\pi)^*$. From this point of view, the following lemma should be clear.

Lemma 2.1. *A face F of P belongs to the π-coherent subdivision of Q induced by $f \in ker(\pi)^*$ if and only if its normal cone $\mathcal{N}(F) \subseteq (\mathbb{R}^{d'})^*$ has the property that its image under the surjection $(\mathbb{R}^{d'})^* \twoheadrightarrow ker(\pi)^*$ contains f.*

In [31, §9.1], Ziegler defines π-coherent subdivisions in the following equivalent fashion. Having chosen the functional $f \in (\mathbb{R}^{d'})^*$ as above, form the graph of the linear map $\hat{\pi} : P \longrightarrow \mathbb{R}^{d+1}$ given by $p \mapsto (\pi(p), f(p))$. The image of this map is a polytope $\hat{Q}$ in $\mathbb{R}^{d+1}$ which maps onto Q under the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ which forgets the last coordinate. Therefore, the set of *lower faces* of $\hat{Q}$ (those faces whose normal cone contains a vector with negative last coordinate) form a polytopal subdivision of Q. We identify this subdivision of Q with the family of faces $\mathcal{F} = \{F\}$ in P which are the inverse images under $\hat{\pi}$ of the lower faces of $\hat{Q}$. Under this identification, it is easy to check that the subdivision of Q is exactly the same as the π-coherent subdivision induced by f, described earlier. Let $\omega_{coh}(P \xrightarrow{\pi} Q)$ denote the induced subposet of $\omega(P \xrightarrow{\pi} Q)$ consisting of all π-coherent subdivisions of Q.

Theorem 2.2. [5] *The poset $\omega_{coh}(P \xrightarrow{\pi} Q)$ is the face lattice of a $(d'-d)$-dimensional polytope, the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$.*

It will be useful for us later to have a reformulation of these definitions using *affine functionals*, *Gale transforms* and *secondary polytopes*. For this purpose, given our previous situation of a linear map of polytopes $\pi : P \to Q$, define a map $\phi_P : \mathbb{R}^n \to \mathbb{R}^{d'+1}$ by the $(d'+1) \times n$ matrix having the vertices p_i of P as its columns and an extra row on top consisting of all ones. Let $q_i = \pi(p_i) \in Q$ and define similarly the map $\phi_Q : \mathbb{R}^n \longrightarrow \mathbb{R}^{d+1}$. Then π extends to a map $\pi : \mathbb{R}^{d'+1} \longrightarrow \mathbb{R}^{d+1}$ such that $\pi \circ \phi_P = \phi_Q$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\phi_P"] \arrow[dr, "\phi_Q"] & \mathbb{R}^{d'+1} \arrow[d, "\pi"] \\ & \mathbb{R}^{d+1} \end{tikzcd} \tag{1}$$

Consider the map $\phi_Q : \mathbb{R}^n \longrightarrow \mathbb{R}^{d+1}$ as a projection onto Q of the $(n-1)$-simplex Δ^{n-1} whose vertices are the standard basis vectors in $\mathbb{R}^n$. Given a linear functional $f \in (\mathbb{R}^n)^*$, we can interpret the ϕ_Q-coherent subdivision of Q induced by f in the following fashion, using Ziegler's description: Write $f(\mathbf{x}) = \sum_i w_i x_i$ and lift the i^{th} vertex in Q (i.e. the image under ϕ_Q of the i^{th} standard basis vector) into $\mathbb{R}^{d+1}$ with last coordinate w_i. Then take the convex hull of these points to form a polytope $\hat{Q}$. The lower faces of $\hat{Q}$ form the desired ϕ_Q-coherent subdivision, which is sometimes referred to as the *regular subdivision* induced by the heights w_i.

We wish to describe when two functionals f, f' induce the same regular subdivision. As before, this will certainly be true whenever they have the same image under the surjection $(\mathbb{R}^n)^* \twoheadrightarrow ker(\phi_Q)^*$ and therefore one may consider them as elements of $ker(\phi_Q)^*$. Let G_Q be any $(n-d-1) \times n$

matrix whose rows form a basis for $ker(\phi_Q)$. The *Gale transform* Q^* is defined to be the vector configuration $q_1^*, \dots, q_n^*$ given by the columns of G_Q. Note that by construction, the row space $Row(G_Q)$ is identified with $ker(\phi_Q)$ and since there is a canonical identification of the dual of the row space with the column space, we have that f is a vector in the column space $Col(G_Q)$, i.e. the space containing the Gale transform points $q_1^*, \dots, q_n^*$. The following lemma is a form of oriented matroid duality (see [4, Lemma 3.2],[3, Section 4]).

Lemma 2.3. *A subset $S \subseteq \{q_1, \dots, q_n\}$ spans a subpolytope of Q which appears in the regular subdivision induced by f if and only if f lies in the relative interior of the positive cone spanned by $S^c := \{q_i^* : q_i \notin S\}$.*

Consequently, two functionals $f, f' \in ker(\phi_Q)^* = Col(G_Q)$ induce the same regular subdivision if and only if they lie in the same face of the *chamber complex* of Q^*, which is the common refinement of all positive cones spanned by subsets of the Gale transform $Q^* = \{q_1^*, \dots, q_n^*\}$. The chamber complex of Q^* turns out to be the normal fan of the *secondary polytope* $\Sigma(Q) := \Sigma(\Delta^{n-1} \stackrel{\phi_Q}{\to} Q)$ defined by Gel'fand, Kapranov and Zelevinsky [17].

It is also possible to determine in this picture when the regular subdivision of Q induced by $f \in (\mathbb{R}^n)^*$ is also π-induced (see [5, Theorem 2.4]). The surjection $\phi_P : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^{d'+1}$ induces an inclusion $\phi_P^* : (\mathbb{R}^{d'+1})^* \hookrightarrow (\mathbb{R}^n)^*$. It then follows immediately from Ziegler's description [31, §9.1] that the π-induced subdivision of Q induced by some functional $f \in (\mathbb{R}^{d'+1})^*$ is the same as the regular subdivision of Q induced by the functional $\phi_P^*(f) = f \circ \phi_P \in (\mathbb{R}^n)^*$. In other words, a set of heights $(w_1, \dots, w_n)$ induces a regular subdivision of Q which is also π-induced if and only if $\sum_i w_i x_i \in im(\phi_P^*)$. Since the vectors of $im(\phi_P^*)$ are characterized by the fact that they vanish on $ker(\phi_P)$ ("row-space is orthogonal to null-space"), to check $\sum_i w_i x_i \in im(\phi_P^*)$ in practice one only needs to verify that every affine dependence $\sum_i c_i p_i = 0$ of the vertices of P satisfies $\sum_i c_i w_i = 0$. We state this as a lemma for later use.

Lemma 2.4. *For the projection $\pi : P \to Q$, a regular subdivision of Q is also π-coherent if and only if it can be induced by a functional $f(\mathbf{x}) = \sum_i w_i x_i \in (\mathbb{R}^n)^*$ which vanishes on $ker(\phi_P)$ or, equivalently, which satisfies $\sum_i c_i w_i = 0$ for every affine dependence $\sum_i c_i p_i = 0$ of the vertices of P.*

Furthermore, we can identify the normal fan $\mathcal{N}(\Sigma(P \stackrel{\pi}{\to} Q))$ to the fiber polytope as a subspace intersected with the chamber complex of Q^*. Composing the embedding $(\mathbb{R}^{d'+1})^* \hookrightarrow (\mathbb{R}^n)^*$ with our earlier surjection $(\mathbb{R}^n)^* \twoheadrightarrow ker(\phi_Q)^*$ gives a map $\phi_{P,Q}^*$ whose image $im(\phi_{P,Q}^*)$ is a subspace in $ker(\phi_Q)^* = Col(G_Q)$. The next proposition then follows immediately from our previous discussion.

Theorem 2.5. *A regular subdivision of Q is π-coherent if and only if the relative interior of its corresponding cone in the chamber complex of Q^* in $Col(G_Q)$ contains a vector in the subspace $im(\phi_{P,Q}^*)$. The normal fan $\mathcal{N}(\Sigma(P \stackrel{\pi}{\to} Q))$ is identified with the cone complex obtained by intersecting the chamber complex of Q^* in $Col(G_Q)$ with the subspace $im(\phi_{P,Q}^*)$.*

Because the cyclic polytopes $C(n,d)$, $C(n+1,d+1)$ are related by a single element lifting, it will later be necessary for us to recall how the chamber complex and Gale transform behave with respect to such liftings (see [4, §3]). Given a d-polytope Q in $\mathbb{R}^d$ with n vertices $q_1, \dots, q_n$, we say that a $(d+1)$-polytope $\hat{Q}$ in $\mathbb{R}^{d+1}$ is a *single element lifting* of Q if it has $n+1$ vertices $\hat{q}_1, \dots, \hat{q}_n, \hat{q}_{n+1}$ and there is surjection $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ satisfying $f(\hat{q}_{n+1}) = 0$ and $f(\hat{q}_i) = c_i q_i$ for $i \leq n$ and some positive scalars $c_i \in \mathbb{R}$. For the case of $Q = C(n,d)$, $\hat{Q} = C(n+1, d+1)$, if we assume that the parameters for the points on the moment curve are chosen so that $t_1 < \cdots < t_n < t_{n+1} = 0$, the map f is the one which ignores the first coordinate and reverses the signs of the rest and the constant c_i

is $-t_i$. When we have a single element lifting $\hat{Q}$ of Q, let $\tau : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the map which sends e_{n+1} to 0 and e_i to $c_i e_i$ for $i \leq n$. One can check that this definition makes the diagram

$$\begin{array}{ccc} \mathbb{R}^{n+1} & \xrightarrow{\phi_{\hat{Q}}} & \mathbb{R}^{d+2} \\ \tau\downarrow & & \downarrow f \\ \mathbb{R}^n & \xrightarrow{\phi_Q} & \mathbb{R}^{d+1} \end{array}$$

commute, where $\phi_Q, \phi_{\hat{Q}}$ were defined earlier and the map f has been extended to $\mathbb{R}^{d+2} \to \mathbb{R}^{d+1}$ by mapping $e_1 \mapsto e_1$. One can check that under these hypotheses, τ restricts to an isomorphism $\tau : ker(\phi_{\hat{Q}}) \to ker(\phi_Q)$ and hence its dual τ^* gives an isomorphism $ker(\phi_Q)^* \to ker(\phi_{\hat{Q}})^*$ between the spaces containing the Gale transforms $Q^*, \hat{Q}^*$.

Lemma 2.6. [4, Lemma 3.4] *When $\hat{Q}$ is a single-element lifting of Q, one can choose the matrices $G_Q, G_{\hat{Q}}$ whose columns give the Gale transform points $Q^*, \hat{Q}^*$ in such a way that the isomorphism τ^* maps q_i^* to $\hat{q}_i^*$ for $i \leq n$.*

In other words, the Gale transform $\hat{Q}^*$ is a single-element extension of the Gale transform Q^*.

We also wish to deal with the relation between the fiber polytopes for the natural projections $\pi : C(n, d') \to C(n, d)$ and $\hat{\pi} : C(n+1, d'+1) \to C(n+1, d+1)$. More generally, given two single-element liftings $\hat{P}$ of P, with map f_P, and $\hat{Q}$ of Q, with map f_Q, we say that two linear surjections $\pi : P \to Q$ and $\hat{\pi} : \hat{P} \to \hat{Q}$ are *compatible with the liftings* if $f_Q \circ \hat{\pi} = \pi \circ f_P$. Since $\phi_Q = \pi \circ \phi_P$ and $\phi_{\hat{Q}} = \hat{\pi} \circ \phi_{\hat{P}}$, it easily follows that $\tau_P = \tau_Q$ and the following diagram commutes:

$$\begin{array}{ccccccc} \mathbb{R}^{n+1} & \xrightarrow{\phi_{\hat{P}}} & \mathbb{R}^{d'+2} & \xrightarrow{\hat{\pi}} & \mathbb{R}^{d+2} & \xleftarrow{\phi_{\hat{Q}}} & \mathbb{R}^{n+1} \\ \downarrow \tau_P & & \downarrow f_P & & \downarrow f_Q & & \downarrow \tau_Q \\ \mathbb{R}^n & \xrightarrow{\phi_P} & \mathbb{R}^{d'+1} & \xrightarrow{\pi} & \mathbb{R}^{d+1} & \xleftarrow{\phi_Q} & \mathbb{R}^n \end{array} \tag{2}$$

One can easily check that this is the case for $\pi : C(n, d') \to C(n, d)$ and $\hat{\pi} : C(n+1, d'+1) \to C(n+1, d+1)$.

Lemma 2.7. *Assume as above that $\hat{P}, \hat{Q}$ are single-element liftings of P, Q and that $\pi, \hat{\pi}$ are compatible projections. Then the isomorphism $\tau^* : ker(\phi_Q) \to ker(\phi_{\hat{Q}})$ restricts to an isomorphism of the subspaces $im(\phi^*_{P,Q})$, $im(\phi^*_{\hat{P},\hat{Q}})$, which contain the normal fans of the fiber polytopes $\Sigma(P \xrightarrow{\pi} Q)$, $\Sigma(\hat{P} \xrightarrow{\hat{\pi}} \hat{Q})$.*

Proof. As a preliminary step, we note that the subspace $im(\phi^*_{P,Q})$ in $ker(\phi_Q)^*$ can be characterized slightly differently. Consider the short exact sequence

$$0 \to ker(\phi_P) \xrightarrow{i_{P,Q}} ker(\phi_Q) \xrightarrow{\phi_{P,Q}} \mathbb{R}^{d'+1} \to 0,$$

where $\phi_{P,Q}$ is the composite $ker(\phi_Q) \hookrightarrow \mathbb{R}^n \overset{\pi}{\twoheadrightarrow} \mathbb{R}^{d'+1}$ and $i_{P,Q}$ is the inclusion map. Dualizing this says that $im(\phi^*_{P,Q}) = ker(i^*_{P,Q})$, which is the characterization we will need.

The commutative diagram in (2) gives rise to the following commutative square

$$\begin{array}{ccc} ker(\phi_{\hat{P}}) & \xrightarrow{i_{\hat{P},\hat{Q}}} & ker(\phi_{\hat{Q}}) \\ \tau\downarrow & & \downarrow\tau \\ ker(\phi_P) & \xrightarrow{i_{P,Q}} & ker(\phi_Q) \end{array}$$

Dualizing this square gives a square in which the kernels of the horizontal maps can be added:

$$\begin{array}{ccc} ker(i^*_{\hat{P},\hat{Q}}) \hookrightarrow ker(\phi_{\hat{Q}})^* & \xrightarrow{i^*_{\hat{P},\hat{Q}}} & ker(\phi_{\hat{P}})^* \\ \tau^* \uparrow & & \uparrow \tau^* \\ ker(i^*_{P,Q}) \hookrightarrow ker(\phi_Q)^* & \xrightarrow{i^*_{P,Q}} & ker(\phi_P)^* \end{array}$$

One then checks that the vertical map τ^* restricts to an isomorphism $ker(i^*_{P,Q}) \to ker(i^*_{\hat{P},\hat{Q}})$. We combine this with the fact that $ker(i^*_{P,Q}) = im(\phi^*_{P,Q})$ to get the assertion. □

3. The case $d = 1$: monotone path polytopes

In this section we restrict our attention to the natural projection $\pi : C(n, d') \to C(n, 1)$ and prove the assertions of Theorem 1.1 concerning the case $d = 1$. We recall and separate out these assertions in the following theorem where, for ease of notation, we have replaced d' by d.

Theorem 3.1. *For the natural projection $\pi : C(n, d) \to C(n, 1)$, the set of π-coherent polytopal subdivisions of $C(n, 1)$, and hence the face lattice of the fiber polytope $\Sigma(C(n, d) \xrightarrow{\pi} C(n, 1))$, is independent of the choice of points on the moment curve. In fact, the face lattice of $\Sigma(C(n, d) \xrightarrow{\pi} C(n, 1))$ coincides with that of the cyclic $(d-1)$-zonotope having $n-2$ zones. Furthermore, all π-induced polytopal subdivisions of $C(n, 1)$ are π-coherent if and only if $d = 2$ or $d = n-1$.*

In the case of $\pi : P \to Q$ with $\dim(Q) = 1$, tight π-coherent subdivisions $\mathcal{F}$ correspond to certain *monotone* edge paths on P. The fiber polytope in this case is called the *monotone path polytope* (see [5, §5] [31, §9.2] [2] for examples). Before proving Theorem 3.1, we recall the definitions of cyclic polytopes and zonotopes, and describe explicitly the face lattice of the cyclic zonotopes.

The *cyclic d-polytope with n vertices* $C(n, d)$ is the convex hull of the points

$$\{v_i\} = \{(t_i, t_i^2, \ldots, t_i^d)\}_{i=1}^n$$

in $\mathbb{R}^d$, where $t_1 < t_2 < \cdots < t_n$. Similarly, the *cyclic d-zonotope with n zones* $Z(n, d)$ is the d-zonotope generated by the vectors

$$\{u_i\} = \{(1, s_i, \ldots, s_i^{d-1})\}_{i=1}^n$$

in $\mathbb{R}^d$, where $s_1 < s_2 < \cdots < s_n$, i.e. $Z(n, d)$ is the set of all linear combinations $\sum_i c_i u_i$ with $0 \le c_i \le 1$.

Let $\Lambda_n = \{0, +, -\}^n$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda_n$. An *even gap* of λ is a pair of indices $i < j$ such that λ_i, λ_j are nonzero entries of *opposite* sign which are separated by an even number of zeros, i.e. $\lambda_r = 0$ for all $i < r < j$ and $j - i - 1$ is even, possibly zero. An *odd gap* is a pair of indices $i < j$ such that λ_i, λ_j are nonzero entries of the *same* sign which are separated by an odd number of zeros. We define the quantity $m(\lambda)$ to be the sum of the number of even gaps, the number of odd gaps and the number of zero entries of λ. For example, if $\lambda = (+, +, 0, -, -, 0, -, -, +, +, -)$ then $m(\lambda) = 2 + 1 + 2 = 5$, accounting for empty gaps as well.

Partially order the set Λ_n by extending componentwise the partial order on $\{0, +, -\}$ defined by the relations $+ < 0$ and $- < 0$.

Proposition 3.2. *The poset of proper faces of $Z(n, d)$ is isomorphic to the induced subposet of Λ_n which consists of the n-tuples λ satisfying $m(\lambda) \le d - 1$.*

Proof. Recall that the face poset of $Z(n,d)$ is anti-isomorphic to the poset of covectors of the polar hyperplane arrangement [31, §7.3]. These covectors are all possible n-tuples of the form

$$\lambda = (\mathrm{sign} f(s_1), \mathrm{sign} f(s_2), \ldots, \mathrm{sign} f(s_n)), \tag{3}$$

where f is a polynomial of degree at most $d-1$. It follows from elementary properties of polynomials that f has at least $m(\lambda)$ zeros, counting multiplicities, so $m(\lambda) \leq d-1$ unless λ is the zero vector. Conversely, given λ, one can construct a polynomial f of degree $m(\lambda)$ that satisfies (3) by locating its zeros and choosing the sign of the leading coefficient appropriately. □

We now turn to the combinatorics of the monotone path polytope $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$. Recall from Theorem 2.2 that the face poset of $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$ is isomorphic to the poset of π-coherent subdivisions of $C(n,1)$. The π-induced subdivisions in this case correspond to the *cellular strings* [7] on $C(n,d)$ with respect to π. These are sequences $\sigma = (F_1, F_2, \ldots, F_k)$ of faces of $C(n,d)$ having the property that $v_1 \in F_1, v_n \in F_k$ and $\max(\pi(F_i)) = \min(\pi(F_{i+1}))$ for $1 \leq i < k$. Such a σ gives rise to a vector $\lambda = \lambda_\sigma = (\lambda_2, \ldots, \lambda_{n-1}) \in \Lambda_{n-2}$ as follows: For $2 \leq i \leq n-1$ let $\lambda_i = +$, respectively $-$, if the vertex v_i of $C(n,d)$ does not appear in σ, respectively is an initial or terminal vertex, with respect to π, of some face F_r of σ and $\lambda_i = 0$ otherwise. For example, if $n = 10$ and the faces of σ have vertex sets $\{v_1, v_3, v_4\}$, $\{v_4, v_7\}$ and $\{v_7, v_8, v_{10}\}$ then $\lambda_\sigma = (\lambda_2, \ldots, \lambda_{n-1}) = (+, 0, -, +, +, -, 0, +)$.

Recall that $\sigma_1 \leq \sigma_2$ in the Baues poset if and only if the union of the faces of σ_1 is contained in the union of the faces of σ_2. For cellular strings on $C(n,d)$, this happens if and only if $\lambda_{\sigma_1} \leq \lambda_{\sigma_2}$ in Λ_{n-2}. It follows that the face poset of $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$ is isomorphic to the induced subposet of Λ_{n-2} which consists of the tuples of the form λ_σ for all coherent cellular strings σ on $C(n,d)$.

The following lemma will be used in the proof of Theorem 3.1, and is closely related to Lemma 2.3 of [1] (incidentally, [1] also contains very interesting enumerative aspects of projections of polytopes polar to the cyclic polytopes).

Lemma 3.3. *Let P, P' be polytopes with face posets L, L' respectively. Suppose that* $\dim(P) \geq \dim(P')$ *and that $\phi : L \longrightarrow L'$ satisfies $\phi(x) \leq \phi(y)$ if and only if $x \leq y$ for all $x, y \in L$. Then ϕ is an isomorphism, i.e. a combinatorial equivalence between P and P'.*

Proof. The hypothesis on ϕ implies that ϕ is injective and sends chains of L to chains of L'. Thus, ϕ induces a simplicial injective map from the order complex of L into that of L'. These order complexes are isomorphic to the barycentric subdivisions of the boundary complexes of the polytopes P and P', respectively. Injectivity then implies that $dim(P) \leq dim(P')$. Since a topological sphere cannot properly contain another topological sphere of the same dimension (see e.g. [23, Theorem 6.6. and Exercise 6.9, pp. 67-68]), the simplicial map is bijective, hence an isomorphism of simplicial complexes, and hence ϕ is an isomorphism of posets. □

Proof of Theorem 3.1. Recall that the face posets of both $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$ and $Z(n-2,d-1)$ are isomorphic to certain induced subposets of Λ_{n-2}. It suffices to show that if σ is a coherent cellular string on $C(n,d)$ with respect to π then $m(\lambda_\sigma) \leq d-2$. Indeed, it then follows that there is a well defined map $\phi = \phi_{n,d}$ from the face poset of $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$ to that of $Z(n-2,d-1)$ that satisfies the hypothesis of Lemma 3.3. The face of $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$ defined by the coherent cellular string σ is mapped under ϕ to the face of $Z(n-2,d-1)$ which corresponds to λ_σ under the isomorphism of Proposition 2.1. Since both polytopes have dimension $d-1$, the lemma completes the proof.

So suppose that σ is coherent and let $\lambda_\sigma = \lambda = (\lambda_2, \ldots, \lambda_{n-1})$, $\lambda_1 = \lambda_n = -$. By Ziegler's definition of π-coherence [31, §9.1], there is a polynomial f of degree at most d such that the polygon $\hat{Q}_f := \text{conv}\{(t_i, f(t_i)) : 1 \leq i \leq n\}$ has the following property: the points $(t_i, f(t_i))$ for which $\lambda_i = -$ are the lower vertices of $\hat{Q}_f$, the ones for which $\lambda_i = 0$ lie on its lower edges and the ones for which $\lambda_i = +$ lie above. In the rest of the proof we show that f has degree at least $m(\lambda) + 2$, so that $m(\lambda) \leq d - 2$.

We assume that there is no $2 \leq i < n-1$ such that $\lambda_i = \lambda_{i+1} = +$, since otherwise we can drop any of the two indices i or $i+1$ without decreasing the value of $m(\lambda)$. Let l_i be the line segment joining the points $(t_i, f(t_i))$ and $(t_{i+1}, f(t_{i+1}))$, for $1 \leq i < n$. We construct the $(n-2)$-tuple $\lambda'' = (\lambda''_2, \dots, \lambda''_{n-1})$ as follows: λ''_i equals $+$, 0 or $-$ depending on whether the slope of the segment l_i is smaller, equal or greater than the slope of the segment l_{i-1}.

It is easy to verify that $\lambda_i \in \{+, -\}$ implies $\lambda''_i = \lambda_i$. On the other hand, $\lambda_i = 0$ implies that $\lambda''_i = 0$ unless at least one of λ_{i-1} or λ_{i+1} equals $+$, in which case $\lambda''_i = -$. In other words, the $(n-2)$-tuple λ'' is obtained from λ by changing every pair of consecutive entries $(+, 0)$ to $(+, -)$ and every pair $(0, +)$ to $(-, +)$. This implies that $m(\lambda'') = m(\lambda)$. For $1 \leq i < n$ let θ_i be such that $t_i < \theta_i < t_{i+1}$ and

$$\frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i} = f'(\theta_i).$$

Observe that $f'(\theta_i)$ equals the slope of the segment l_i. For $2 \leq i \leq n-1$ let μ_i be such that $\theta_{i-1} < \mu_i < \theta_i$ and

$$\frac{f'(\theta_i) - f'(\theta_{i-1})}{\theta_i - \theta_{i-1}} = f''(\mu_i).$$

Then $\text{sign} f''(\mu_i) = \text{sign}(f'(\theta_i) - f'(\theta_{i-1})) = -\lambda''_i$. This implies that the degree of f'' is at least $m(\lambda'')$ and finishes the proof. □

The combinatorial equivalence $\phi_{n,d}$, described in the proof of Theorem 3.1, together with Proposition 3.2 implies the following corollary.

Corollary 3.4. *A cellular string σ for $\pi : C(n,d) \to C(n,1)$ is π-coherent if and only if $m(\lambda_\sigma) \leq d-2$. In particular, whether σ is coherent or not does not depend on the choice of $t_1, \dots, t_n$ used to define $C(n,d)$.*

A cellular string $\sigma = (F_1, F_2, \dots, F_k)$ on $C(n,d)$ with respect to π is a tight π-induced subdivision if all its faces F_r are edges $v_{i_{r-1}} v_{i_r}$ of $C(n,d)$, where $1 = i_0 < i_1 < \cdots < i_k = n$. Equivalently, σ is tight if λ_σ contains no zeros. The vertices v_i of σ correspond to the indices i for which $\lambda_i = -$, together with the indices 1 and n. The tight cellular strings are the *monotone edge paths.*

As a corollary, we characterize and enumerate the monotone edge paths on $C(n,d)$ which are π-coherent for $\pi : C(n,d) \to C(n,1)$. For $\lambda \in \{+, -\}^{n-2}$ let $c(\lambda)$ be the number of maximal strings of successive $+$ signs or successive $-$ signs in λ. Note that $m(\lambda) = c(\lambda) - 1$.

Corollary 3.5. *For $\lambda \in \{+, -\}^{n-2}$ we have $\lambda = \lambda_\sigma$ for a π-coherent monotone edge path σ in $C(n,d)$ if and only if $c(\lambda) \leq d-1$. The number of π-coherent monotone edge paths in $C(n,d)$ is*

$$2 \sum_{j=0}^{d-2} \binom{n-3}{j}.$$

Proof. The first statement follows from Corollary 3.4. For the second statement, note that there are $2\binom{n-3}{j}$ tuples $\lambda \in \{+, -\}^{n-2}$ with $c(\lambda) = j+1$. □

Remark 3.6.

If $d \geq 4$, any two vertices of $C(n,d)$ are connected by an edge and the total number of monotone edge paths on $C(n,d)$ is 2^{n-2}. Hence when d is fixed and n gets large, the fraction of coherent paths approaches zero. Similar behavior is exhibited in Proposition 5.10 of [12], where it is proved that the cyclic polytope $C(n, n-4)$ has $\Omega(2^n)$ triangulations but only $O(n^4)$ regular ones. Another example of this behavior with regard to monotone paths for non-cyclic polytopes appears in [2].

Remark 3.7.
One can rephrase Theorem 3.1 as saying that for the linear functional $f(\mathbf{x}) = x_1$ mapping $C(n,d)$ onto a 1-dimensional polytope $f(C(n,d))$, the monotone path polytope

$$\Sigma(C(n,d) \xrightarrow{f} f(C(n,d)))$$

has face lattice independent of the choice of $t_1, \ldots, t_n$. One might ask whether this is true for all linear functionals. It turns out that this is not the case. Consider the linear functional $f(\mathbf{x}) = x_1 + x_3$ on the polytope $C(5,3)$. Since the functional is monotone along the moment curve, it will produce the same monotone paths (actually the same ones as the standard functional x_1) for any choice of parameters $t_1 < \cdots < t_5$. However, different choices of parameters can change the set of coherent monotone edge paths.

Let us fix $t_3 = 0$ and $t_2 = -t_4$, $t_1 = -t_5$, so that we only have two free parameters $0 < t_4 < t_5$ and the cyclic polytope $C(5,3)$ has a symmetry $[x_1 \to -x_1; x_3 \to -x_3]$ which exchanges the vertices i and $6-i$. We want to find out when the monotone path consisting of the edges 13 and 35 is coherent. The normal vectors to the faces 235 and 345 are $(t_4t_5, -t_4-t_5, 1)$ and $(-t_4t_5, t_4-t_5, 1)$, respectively, so these two vectors generate the two boundary rays in the normal cone to the segment 35. Thus, the projection of the normal cone of 35 to $ker(f)$ is bounded by the vectors $(\frac{t_4t_5-1}{2}, -t_4-t_5, \frac{1-t_4t_5}{2})$ and $(\frac{-t_4t_5-1}{2}, t_4-t_5, \frac{1+t_4t_5}{2})$. By symmetry, the projection to $ker(f)$ of the normal cone to the segment 13 is bounded by $(\frac{1-t_4t_5}{2}, -t_4-t_5, \frac{t_4t_5-1}{2})$ and $(\frac{t_4t_5+1}{2}, t_4-t_5, \frac{-1-t_4t_5}{2})$. If $t_4t_5 > 1$ then the relative interiors of these two cones in $ker(f)$ intersect (in the vector $(0,-1,0)$ for example). If $t_4t_5 \le 1$ then they do not intersect since the first one only contains vectors with $x_1 < 0$ and the second one vectors with $x_1 > 0$. Thus, the path containing the segments 13 and 35 is coherent if and only if $t_4t_5 > 1$.

Remark 3.8.
Recall that the Upper Bound Theorem [31, §8.4], [19, §4.7] states that the cyclic polytope $C(n,d)$ has the most boundary i-faces among all d-polytopes with n vertices for all i. We have also seen that the facial structure of the monotone path polytope $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$ is independent of the choice of points on the moment curve. These two facts might tempt one to make the following "Upper Bound Conjecture (UBC) for monotone path polytopes": For all d-polytopes with n-vertices and linear functionals f, the monotone path polytope $\Sigma(P \xrightarrow{f} f(P))$ has no more boundary i-faces than $\Sigma(C(n,d) \xrightarrow{\pi} C(n,1))$. However, this turns out to be false, as demonstrated by the example of a non-neighborly simplicial 4-polytope with eight vertices whose monotone path polytope with respect to the projection to the first coordinate has two more coherent paths than C(8,4). The vertex coordinates of this 4-polytope are given by the columns of the following matrix:

$$\begin{bmatrix} -84 & -36 & -35 & 11 & 90 & 31 & 47 & -50 \\ -54 & 71 & -71 & -17 & 65 & -34 & 60 & 99 \\ 48 & 36 & 73 & -40 & 50 & 54 & 24 & 65 \\ 6 & -65 & 52 & 100 & -39 & 49 & -76 & -15 \end{bmatrix}$$

This raises the following question.

Question 3.9. *Is there some natural family of polytope projections $P \xrightarrow{\pi} Q$ indexed by (n, d', d) with $dim(P) = d', dim(Q) = d$, such that P has n vertices and the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$ has more i-faces than any other fiber polytope of an n-vertex d'-polytope projecting onto a d-polytope?*

For the case $d = 0$, the Upper Bound Theorem says that the family of projections $C(n, d') \xrightarrow{\pi} C(n, d)$ provides the answer, but the above counterexample shows that it does not already for $d = 1$. However, one could still ask whether this family provides the answer asymptotically. For simplicity, we restrict our attention to the case $d = 1$ of monotone path polytopes and the number of vertices of $\Sigma(P \xrightarrow{\pi} Q)$. We also replace d' by d, as in the beginning of this section. If d is fixed, Corollary 3.5 implies that the number of vertices of $\Sigma(C(n, d) \xrightarrow{\pi} C(n, 1))$ is a polynomial in n of degree $d - 2$.

Let $r_d(n)$ denote the maximum number of vertices that a monotone path polytope of an n-vertex d-polytope projecting onto a line segments can have.

Question 3.10. *With d fixed, what is the asymptotic behaviour of $r_d(n)$ as $n \to \infty$? In particular, is $r_d(n)$ bounded above by a polynomial in n of degree $d - 2$?*

We close this section by giving a polynomial upper bound for $r_d(n)$ of degree $3d - 6$. Let

$$q_d(m) = \sum_{j=0}^{d} \binom{m}{j}.$$

Proposition 3.11. *We have $r_d(n) \leq 2\, q_{d-2}(\binom{n}{3} - 1)$, a polynomial in n of degree $3d - 6$.*

Proof. Let $P \subseteq \mathbb{R}^d$ be d-dimensional with vertices $p_1, p_2, \ldots, p_n$ and $\pi : \mathbb{R}^d \to \mathbb{R}$ be a linear map. Let $a_i = \pi(p_i)$ for $1 \leq i \leq n$.

We use Ziegler's definition [31, §9.1] of a π-coherent monotone edge path, described in Section 2. Let $f \in ker(\pi)^*$ be a generic linear functional and $\hat{Q}$ be the convex hull of the points $\hat{a}_i := (a_i, f(p_i))$ in $\mathbb{R}^2$, for $1 \leq i \leq n$. The set of lower edges of $\hat{Q}$ is determined by the oriented matroid of the point configuration $\mathcal{A} := \{\hat{a}_i\}_{i=1}^n$. Equivalently, it is determined by the data which record for each triple $1 \leq i < j < k \leq n$ which of the two halfplanes determined by the line though $\hat{a}_i$ and $\hat{a}_k$ the point $\hat{a}_j$ lies on. This is equivalent to recording which side of a certain linear hyperplane in $ker(\pi)^*$, depending on (i, j, k), the functional f lies on. Hence the number of π-coherent monotone edge paths on P is at most the number of regions into which some $\binom{n}{3}$ linear hyperplanes dissect $ker(\pi)^*$. This number is at most the number of regions into which a *generic* arrangement of $\binom{n}{3}$ linear hyperplanes dissects $\mathbb{R}^{d-1}$, which is $2\, q_{d-2}(\binom{n}{3} - 1)$ (see [30] for more on counting regions in hyperplane arrangements). ☐

4. The case $d' = n - 1$: triangulations and secondary polytopes

In this section we restrict our attention to the natural projection $\pi : C(n, n-1) \to C(n, d)$ and prove the assertions of Theorem 1.1 concerning the case $d' = n - 1$. In this case, since $C(n, n-1)$ is an $(n-1)$-simplex Δ^{n-1} (and since all $(n-1)$-simplices are affinely equivalent), the fiber polytope $\Sigma(C(n, n-1) \xrightarrow{\pi} C(n, d))$ coincides with the *secondary polytope* $\Sigma(C(n, d))$ (see Section 2 or [17, §7]), whose vertices correspond to the regular triangulations of $C(n, d)$. The question of the existence of non-regular triangulations of $C(n, d)$ was first raised in [20, Remark 3.5]. Billera, Gel'fand and Sturmfels first constructed such a triangulation for $C(12, 8)$ in [4, §4]. Our results show that this example is far from minimal and provide a complete characterization of the values of n and d for which $C(n, d)$ has non-regular triangulations.

We recall and separate out the assertions of Theorem 1.1 which deal with secondary polytopes. In this context, we use the terms "coherent subdivision" and "regular subdivision" interchangeably, as both occur in the literature.

Theorem 4.1. *All polytopal subdivisions of $C(n,d)$ are coherent if and only if either*

- $d \leq 2$ *or*
- $n - d \leq 3$ *or*
- $(n,d) \in \{(8,4),(8,3),(7,3)\}$.

In all other cases, there exists a subdivision of $C(n,d)$ whose coherence varies with the choice of points on the moment curve, and for every choice of points there is some incoherent subdivision.

The proof of this result occupies the remainder of this section. We begin by showing that in all of the cases asserted above, all polytopal subdivisions are coherent. For $d = 1$ this is easy and for $d = 2$ and $n - d \leq 3$ this was shown by Lee [21, 22]. In fact, these references show that all subdivisions in these cases are *placing* subdivisions (see definition in [22]). It therefore remains to show that all subdivisions of $C(n,d)$ are coherent for (n,d) equal to $(8,4),(8,3)$ or $(7,3)$.

Our task is simplified somewhat by the following fact.

Lemma 4.2. *Suppose that for a certain choice of points along the moment curve, the canonical projection $C(n,d') \xrightarrow{\pi} C(n,d)$ has the property that every π-induced subdivision is π-coherent. Then, the same happens for every other choice of points along the moment curve.*

Proof. In every choice of points the poset of π-coherent subdivisions is isomorphic to the face poset of the corresponding fiber polytope, which is a polytope of dimension $d' - d$. The hypothesis of the lemma implies that, for a certain choice of points, the face poset of the fiber polytope is the whole Baues poset $\omega(C(n,d') \xrightarrow{\pi} C(n,d))$. Since the Baues poset is independent of the choice of points, Lemma 4.4 of [6] implies that in every other choice of points the Baues poset coincides with the face poset of the fiber polytope. □

Remark 4.3.
Lemma 4.2 is true in a more general situation, namely, whenever we have two projections of polytopes $P \xrightarrow{\pi} Q$ and $P' \xrightarrow{\pi'} Q'$ and there is a bijection ϕ between the vertices of P and P' which induces an isomorphism between the oriented matroids of affine dependencies of P and P', as well as those of Q and Q'. This is so because these assumptions imply that the two Baues posets are isomorphic.

On the other hand, it is not enough to assume that ϕ induces only a combinatorial equivalence for P and P' and for Q and Q'. For example, if P and P' are two 5-simplices projecting in the natural way onto two combinatorial octahedra Q and Q' with different oriented matroid (i.e. different affine dependence structure) then both Baues posets contain all the polytopal subdivisions of Q and Q' respectively, but they are different and the proof of Lemma 4.2 is not valid. This is relevant to the situation with cyclic polytopes $C(n,d)$ since there exist polytopes with the same face lattice as $C(n,d)$ but whose vertices have different affine dependence structure [8].

Remark 4.4.
We also note that Lemma 4.2 shows that the last assertions in parts (2) and (4) of Theorem 1.1 follow from the assertions preceding them. To be precise, if (n,d,d') are such that there exists some π-induced subdivision of $C(n,d)$ whose π-coherence depends upon the choice of points on the moment curve, then Lemma 4.2 implies that there cannot exist a choice of points for which every π-induced subdivision of $C(n,d)$ is π-coherent.

It was recently shown by Rambau [24] that all triangulations of $C(n,d)$ are connected by bistellar flips. Hence one can rely on this fact to enumerate all triangulations in small instances (see Table 4). The program PUNTOS is an implementation of this algorithmic procedure and can be obtained via anonymous ftp at `ftp://geom.umn.edu, directory /priv/deloera` (see [13] for details). In Lemma 4.6 we will use the information given by PUNTOS for the three cases which interest us to

triangulations of $C(7,3)$ modulo symmetries	good links at:
2356,1234,4567,3467,2345,2367,1256,3456,1267,1245	1,7
2456,2346,1234,4567,3467,2367,1256,1267,1245	1,7
2356,1234,2345,2367,1256,1267,1245,3457,3567	1
2346,1234,4567,3467,2367,1267,1456,1246	5,7
2356,2367,1256,1267,1235,1345,3457,3567	4
1234,2345,1256,1267,1245,3457,2567,2357	1
2367,1267,1345,3457,3567,1236,1356	4
4567,3467,3456,1345,1356,1237,1367	2
4567,3467,2367,1267,1456,1236,1346	5,6,7
4567,3467,1456,1237,1367,1346	2,5
1345,3457,3567,1356,1237,1367	2,3,4
1345,3457,1237,1357,1567	2,4,6
1237,1567,1457,1347	1,2,6,7
1234,2347,1567,1457,1247	3,6
1234,4567,1456,2347,1247,1467	3,4,5
1234,4567,1267,1456,1246,2347,2467	3,5

TABLE 1. Triangulations of $C(7,3)$ are placing triangulations

prove that all the subdivisions are regular in a certain choice of points along the moment curve, and hence in all choices by Lemma 4.2.

The following lemma is a direct proof of the fact that all the triangulations are regular, in every choice of points, for the three cases. The lemma also clearly follows from Lemma 4.6, but we find the proof below of independent interest.

Lemma 4.5. *All triangulations of $C(7,3)$ and $C(8,4)$ are placing. All triangulations of $C(8,3)$ are regular.*

Proof. We know from the results of [22] that all triangulations of $C(6,3)$ and $C(7,4)$ are placing. It is enough to check that each triangulation of $C(7,3)$ and $C(8,4)$ has at least one vertex whose link is contained in the boundary complex of $C(6,3)$ and $C(7,4)$ respectively. The polytope $C(7,3)$ has a single symmetry that maps i to $7-i+1$. The original 25 triangulations are divided into 16 distinct symmetry classes. For $C(8,4)$, which has the dihedral group of order 16 as its group of symmetries, the original 40 triangulations are divided into only 4 symmetry classes. In Tables 1 and 2 we show, for each of the 16 symmetrically distinct triangulations of $C(7,3)$ and each of the 4 symmetrically distinct triangulations of $C(8,4)$, that the above "good link" property is indeed satisfied at least at one vertex.

In the case of $C(8,3)$ one can verify that, modulo symmetries, only the following five triangulations are not placing triangulations (if symmetries are not considered, these are 8 triangulations out of a total of 138):

$2378, 2356, 2367, 1267, 3456, 3478, 3467, 1256, 1278, 1345, 1235, 4568, 4678$
$2378, 2367, 1267, 3456, 3478, 3467, 1278, 1345, 4568, 4678, 1236, 1356$
$2356, 1267, 3456, 1256, 1278, 1345, 1235, 4568, 3468, 2678, 2368$
$1267, 3456, 1278, 1345, 4568, 1236, 1356, 3468, 2678, 2368$

triangulations of $C(8,4)$ modulo symmetries	good links at:
23678,23458,12568,12458,45678,23568,12678,12348,34568,34678	1,7,8
24568,23456,23678,12568,12458,45678,12678,12348,34678,23468	1,7
23678,12568,45678,23568,12678,34568,34678,13458,12358,12345	7
23678,45678,12678,34568,34678,13458,12345,12368,12356,13568	7

TABLE 2. Triangulations of $C(8,4)$ are placing triangulations

$2378, 2367, 1267, 3456, 1278, 1345, 4568, 1236, 1356, 3678, 3468.$

One can check that each of the first four triangulations has a neighbor which is a placing triangulation having only 4 bistellar flips. This implies that if any of the four were to disappear from the 4-dimensional secondary polytope $\Sigma(C(8,3))$ for some choice of points on the moment curve then the placing triangulation in question would be left with three neighbors, which is impossible in a 4-dimensional polytope.

This leaves only the fifth triangulation on the above list whose regularity must be checked directly. Regularity can be determined via the feasibility of a certain system of linear inequalities. The variables of this system are the heights w_i. The inequalities are determined by pairs of maximum dimensional simplices and points. Each inequality establishes the fact that a given point lies "above" a certain hyperplane after using the heights w_i for a lifting. The coefficients of the inequalities can be interpreted as oriented volumes. These inequalities form a system $B\bar{\lambda} < 0$ which will be feasible precisely when the triangulation K is regular.

Farkas' theorem [29] indicates that exactly one of the following holds: either the system of inequalities $B\bar{\lambda} < 0$ defined by the triangulation is consistent, or there exists $y \in \mathbb{R}^m$ such that $y^T B = 0, \quad \sum y_i > 0$ and $y \geq 0$. The support of such a vector y labels an inconsistent subset of inequalities in $B\bar{\lambda} < 0$. An explicit proof of non-regularity is then the impossible inequality $0 = (y^T B)\bar{\lambda} = y^T(B\bar{\lambda}) < 0$. Notice that such a vector y lies in the kernel of the transpose of B.

One can set up the following matrix B for the last triangulation of the list. The coefficients are simply Vandermonde determinants because they determine the volume of the simplices in the triangulation.

$$\begin{bmatrix} -vol(2367) & vol(1367) & -vol(1267) & 0 & 0 & vol(1237) & -vol(1236) & 0 \\ 0 & 0 & -vol(4568) & vol(3568) & -vol(3468) & vol(3458) & 0 & -vol(3456) \\ 0 & -vol(3678) & vol(2678) & 0 & 0 & -vol(2378) & vol(2368) & -vol(2367) \\ -vol(2378) & vol(1378) & -vol(1278) & 0 & 0 & 0 & vol(1238) & -vol(1237) \\ -vol(2678) & vol(1678) & 0 & 0 & 0 & -vol(1278) & vol(1268) & -vol(1267) \\ vol(2356) & -vol(1356) & vol(1256) & 0 & -vol(1236) & vol(1235) & 0 & 0 \\ -vol(3456) & 0 & vol(1456) & -vol(1356) & vol(1346) & -vol(1345) & 0 & 0 \\ 0 & 0 & vol(4678) & -vol(3678) & 0 & vol(3478) & -vol(3468) & vol(3467) \end{bmatrix}$$

For example, the first row of the matrix B corresponds to the inequality that indicates that point 1 is above the lifted plane 2367. The kernel of the transpose of the above matrix is 4-dimensional and spanned by the columns of the following matrix, where t_{ij} denotes the difference $t_i - t_j$ for $i < j$.

$$\begin{bmatrix} -\frac{(t_{78}t_{38}t_{28})}{(t_{67}t_{36}t_{26})} & -\frac{(t_{78}t_{68}t_{28})}{(t_{37}t_{36}t_{23})} & \frac{(t_{56}t_{35}t_{25})}{(t_{67}t_{37}t_{27})} & -\frac{(t_{56}t_{35}t_{46}t_{45}t_{34})}{(t_{67}t_{37}t_{27}t_{26}t_{23})} \\ 0 & 0 & -\frac{(t_{26}t_{23}t_{16}t_{13}t_{12})}{(t_{68}t_{38}t_{48}t_{46}t_{34})} & \frac{(t_{16}t_{13}t_{14})}{(t_{68}t_{38}t_{48})} \\ -\frac{(t_{17}t_{13}t_{12})}{(t_{67}t_{36}t_{26})} & -\frac{(t_{17}t_{16}t_{12})}{(t_{37}t_{36}t_{23})} & -\frac{(t_{56}t_{35}t_{16}t_{13}t_{57}t_{12})}{(t_{78}t_{68}t_{67}t_{38}t_{37}t_{27})} & -\frac{(t_{56}t_{35}t_{46}t_{45}t_{34}t_{16}t_{13}t_{17})}{(t_{78}t_{68}t_{67}t_{38}t_{37}t_{27}t_{26}t_{23})} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(t_{56}t_{35}t_{26}t_{23}t_{16}t_{13}t_{12}t_{58})}{(t_{78}t_{68}t_{67}t_{38}t_{37}t_{48}t_{46}t_{34})} & -\frac{(t_{56}t_{35}t_{16}t_{13}t_{18}t_{45})}{(t_{78}t_{68}t_{67}t_{38}t_{37}t_{48})} \end{bmatrix}$$

The third row of the above matrix is negative regardless of the values of the parameters t_i. Since this matrix also contains a 4×4 identity submatrix, there is no non-zero vector y in its column space that has all its elements nonnegative. The system defined by the matrix B is always feasible, making the fifth triangulation regular regardless of the values of t_i. $\square$

We define the *ranking* of a polytopal subdivision of $C(n,d)$ to be the sum of the dimensions of the secondary polytopes of their disjoint cells that are not simplices. For instance, for $C(8,4)$ it is possible to have a polytopal subdivision with two copies of $C(6,4)$ as the only cells that are not simplices; the ranking of such subdivisions is 2. We do not use the word rank as the poset of polytopal subdivisions of $C(n,d)$ is not necessarily graded.

Lemma 4.6. *All polytopal subdivisions of* $C(7,3)$, $C(8,3)$ *and* $C(8,4)$ *are coherent for every choice of parameters.*

Proof. Because of Lemma 4.2 we only need to prove that all the subdivisions are coherent in one choice of parameters. So we fix the parameters to be $t_i = i$. Also, it suffices to prove the result for $C(8,3)$ and $C(8,4)$ because $C(7,3)$ is a subpolytope of $C(8,3)$.

We describe a procedure to enumerate all polytopal subdivisions of $C(n,d)$ of ranking k for small values of n. We say that a polytopal subdivision S of is of *type* $[r_1C(s_1,d), r_2C(s_2,d), \dots, r_mC(s_r,d)]$ if the total number of cells used in S which are not simplices is $r_1 + r_2 + \cdots + r_m$ and S contains precisely r_i disjoint isomorphic copies of $C(s_i,d)$ with $s_i > d+1$ ($C(s_i,d)$ is a cell which is not a simplex). For example, there are 18 polytopal subdivisions of $C(8,4)$ of type $[2C(6,4)]$. Clearly, all subdivisions of the same type have the same ranking. The possible isomorphic classes of subpolytopes of $C(n,d)$ which are not simplices are $C(d+2,d), C(d+3,d), \dots, C(n-1,d)$. Their secondary polytopes have dimensions $1, 2, \dots, n-d-2$ respectively.

Given that we have a complete list of triangulations of $C(n,d)$, to count the polytopal subdivisions of $C(n,d)$ of type $[r_1C(s_1,d),\ r_2C(s_2,d), \dots, r_mC(s_m,d)]$ we fix a triangulation for each $C(s_i,d)$ and form all possible $(r_1 + r_2 + \cdots + r_m)$-tuples of disjoint triangulated copies of $C(s_1,d)$, $C(s_2,d), \dots, C(s_m,d)$. Because the copies have been triangulated, it suffices to count the triangulations of $C(n,d)$ which complete the different tuples of triangulated subpolytopes. It is easy to list all types of polytopal subdivisions of $C(8,4)$ and $C(8,3)$ and their cardinalities.

We computed these numbers using a MAPLE implementation of the above criteria. We present the main information in Table 3. We disregard subdivisions of ranking one since these are exactly the bistellar flips that we can compute with PUNTOS [13]. Once we get zero subdivisions for all types of a certain ranking i, we do not need to compute the number for any ranking $j > i$ since any subdivision of ranking j could be refined into one of ranking i. Also, if a subdivision contains a cell $C(n-1,d)$ then all the other cells are simplicial and the subdivision is the one obtained from the trivial subdivision by *pushing* the vertex which is not in the cell $C(n-1,d)$ (see [10, page 411]). Thus, the number of subdivisions of type $[C(n-1,d)]$ equals n and we do not need to compute any other type containing a cell $C(n-1,d)$.

Using the program PUNTOS [13] we have computed all triangulations of $C(8,3)$ and $C(8,4)$ (in the standard choice of parameters) and checked that they are all regular. This also implies that all the ranking one subdivisions are regular, since ranking one subdivisions correspond to bistellar flips between triangulations and bistellar flips between regular triangulations correspond to edges of the secondary polytope. Thus for $C(8,4)$, whose secondary polytope is 3-dimensional, it only remains to check that the number of subdivisions of ranking at least two coincides with the number of facets of the secondary polytope. Since we have 40 triangulations and 64 bistellar flips computed by PUNTOS, the number of facets of the secondary polytope is given by Euler's formula and turns out to be 26. This coincides with the results of Table 3 and hence all subdivisions are regular.

In the case of $C(8,3)$, PUNTOS tells us that there are 138 triangulations and 302 ranking one subdivisions (flips), all of them regular (in the standard choice of parameters). The calculation of the secondary polytope using PORTA indicates that for the usual parameters, the secondary polytope for $C(8,3)$ is a 4-dimensional polytope which has indeed 50 facets. Then Euler's formula gives a number of 214 for the number of 2-dimensional faces. Since these numbers coincide with the numbers of ranking three and ranking two subdivisions in Table 3, all subdivisions are regular.

□

Remark 4.7.
In [28] it is proved that any non-regular subdivision of a polytope can be refined to a non-regular triangulation. With this, our last lemma also follows from the fact that in the standard choice of parameters all triangulations of $C(8,3)$, $C(8,4)$ and $C(7,3)$ are regular, with no need to analyze non-simplicial subdivisions.

The next lemma shows that in order to complete the proof of Theorem 4.1 we need only to exhibit certain minimal counterexamples.

type (case of C(8,4))	cardinality
$[C(7,4)]$	8
$[2C(6,4)]$	18
$[3C(6,4)]$	0
type (case of C(8,3))	cardinality
$[2C(5,3)]$	162
$[C(6,3)]$	52
$[3C(5,3)]$	18
$[C(6,3),C(5,3)]$	24
$[C(7,3)]$	8
$[2C(6,3)]$	0
$[C(6,3),2C(5,3)]$	0
$[4C(5,3)]$	0

TABLE 3. Polytopal subdivisions of $C(8,4)$ and $C(8,3)$ by type

Lemma 4.8. *Suppose there exists a triangulation T of the cyclic polytope $C(n,d)$ which is regular or non-regular, depending on the choice of points along the moment curve. Then such a triangulation also exists for $C(n+1,d)$ and $C(n+1,d+1)$.*

Proof. For the first statement, given such a triangulation T of $C(n,d)$, extend the triangulation by placing (see definition in [22]) the extra point $n+1$, that is to say, by joining $n+1$ to all the facets of T which are visible from it. This produces a triangulation T' of $C(n+1,d)$ and it is easy to see that T' is regular for a choice $t_1 < \cdots < t_n < t_{n+1}$ of parameters on the moment curve if and only if T is regular for the choice $t_1 < \cdots < t_n$.

For the second statement we use the fact that $C(n+1,d+1)$ is a single element lifting of $C(n,d)$, so that Lemma 2.6 applies. As was discussed in Section 2, we cannot guarantee that $C(n+1,d+1)$ is a single element lifting of $C(n,d)$ unless the parameters $t_1 < \cdots < t_n < t_{n+1}$ are chosen so that $t_{n+1} = 0$. However, this presents no problem since an affine transformation $t_i \mapsto at_i + b$ of the parameters produces an affine transformation of the points v_i and thus preserves regularity of triangulations.

By our hypotheses, there exist two choices of parameters for the cyclic polytope $C(n,d)$ which produce different chamber complexes in the dual. By Lemma 2.6, the Gale transform of $C(n+1,d+1)$ is obtained from that of $C(n,d)$ by adding a single point. It is impossible to add a new point to the Gale transforms and make them equal (in a labeled sense). Thus, there are choices of parameters for $C(n+1,d+1)$ which produce different chamber complexes in the Gale transform and which, in particular, produce different collections of regular triangulations. □

In light of the preceding lemma, the minimal counterexamples necessary to complete the proof of Theorem 4.1 are provided in our next result.

Lemma 4.9. *Each of the polytopes $C(9,3)$, $C(9,4)$ and $C(9,5)$ has a triangulation and two suitable choices of points along the moment curve that make the triangulation regular and non-regular, respectively.*

Proof. We exhibit explicit triangulations of $C(9,3)$, $C(9,4)$ and $C(9,5)$ that are regular and non-regular, depending upon the choice of parameters.

In the case of $C(9,5)$ there is a triangulation which is regular or non-regular for the parameters $[0,6,7,8,9,10,11,12,30]$ and $[1,2,3,4,5,6,7,8,9]$, respectively:

$$125689, 126789, 345679, 125678, 123489, 124578, 123478, 124589, 123457, 123567$$
$$134567, 256789, 235679, 234579, 234789, 245789.$$

For the typical parameters $[1,2,3,4,5,6,7,8,9]$ the polytope $C(9,4)$ has 4 non-regular triangulations (out of 357), one of which is given by the simplices

$$34789, 23789, 12789, 12345, 46789, 45678, 45689, 12356,$$
$$12379, 12367, 13479, 13456, 13467, 14679, 14569.$$

On the other hand, for the parameters $[0,1/20,1/3,4,50,60,67,68,69]$ the same triangulation becomes regular. Finally, in the case of $C(9,3)$, the triangulation

$$2578, 1345, 1256, 1267, 1278, 4589, 3489, 2389, 1289, 2567, 5789, 3458, 2358, 5679, 1235$$

is non-regular for the standard parameters $[1,2,3,4,5,6,7,8,9]$ but becomes regular for the parameters $[1,2,3,10/3,23/6,13/3,14/3,5,6]$. □

Remark 4.10.
One might ask whether there is a subdivision of $C(n,d)$ that is non-regular (i.e. a subdivision which is π-induced but not π-coherent for the projection $\pi : C(n, n-1) \to C(n,d)$) for all choices of the parameters $t_1 < \cdots < t_n$. Such an example was recently provided by J. Rambau (see [26]), who found 4 triangulations of $C(11,5)$ each having the property that it is adjacent to only 4 other triangulations by bistellar operations. Since a triangulation which is regular for some choice of parameters would have at least $n-1-d = 11-1-5 = 5$ other regular neighboring vertices in the secondary polytope, such a triangulation can never be regular. This is particularly interesting because of a recent result [26] stating that all triangulations of $C(n,d)$ are *lifting* triangulations (see [10, §9.6] for a definition). Any triangulation of $C(n,d)$ which is regular for some choice of points on the moment curve is automatically a lifting triangulation, but Rambau's examples show that the converse does not hold.

We close our discussion by presenting in Table 4 the numbers of triangulations of cyclic polytopes known to us. Those marked with * have been computed by Jörg Rambau.

number of points:	3	4	5	6	7	8	9	10	11	12
dimension 2	1	2	5	14	42	132	429	1430	4862	16796
dimension 3		1	2	6	25	138	972	8477	89405*	1119280*
dimension 4			1	2	7	40	357	4824	96426	2800212*
dimension 5				1	2	8	67	1233	51676*	5049932*
dimension 6					1	2	9	102	3278	340560*
dimension 7						1	2	10	165	12589
dimension 8							1	2	11	244
dimension 9								1	2	12
dimension 10									1	2

TABLE 4. The number of triangulations of $C(n,d)$ for $n \leq 12$.

5. The case $d, d'-d, n-d' \geq 2$

So far we have proved all the assertions of Theorem 1.1 in the cases $d = 1$ (Theorem 3.1) and $n - d' = 1$ (Theorem 4.1). Since the case $d' - d = 1$ is trivial (see Section 1) it only remains to deal with the case where $d, d'-d, n-d'$ are all at least 2. We collect the assertions of Theorem 1.1 which cover this case in the following result.

Theorem 5.1. *If $d, d'-d, n-d' \geq 2$ then for the natural projection $\pi : C(n,d') \to C(n,d)$ there exists a π-induced polytopal subdivision of $C(n,d)$ whose π-coherence depends upon the choice of points on the moment curve, and for every choice of points there exists some π-induced but not π-coherent subdivision.*

Our proof of Theorem 5.1 proceeds in three steps:

STEP 1. We show that for $\pi : C(n, n-2) \to C(n, 2)$ with $n \geq 6$, there is a particular π-induced subdivision of $C(n,2)$ whose π-coherence depends upon the choice of parameters.

STEP 2. We use the subdivision from Step 1 to produce a subdivision with the same property for $\pi : C(n,d') \to C(n,2)$ whenever $d'-2, n-d' \geq 2$.

STEP 3. We use the subdivision from Step 2 to produce a subdivision with the same property for $\pi : C(n,d') \to C(n,d)$ whenever $d, d'-d, n-d' \geq 2$.

Before we proceed, we review some facts about the facial structure of the cyclic polytopes $C(n,d)$. From now on, we will refer to a vertex $v_i = (t_i, t_i^2, \ldots, t_i^d)$ by its index i, so that a subset $S \subseteq [n] := \{1, 2, \ldots, n\}$ may or may not span a boundary face of $C(n,d)$. Gale's famous Evenness Criterion [31, Theorem 0.7],[19, §4.7] tells us exactly when this happens. The criterion is based on the unique decomposition of $S = Y_1 \cup X_1 \cup X_2 \cup \cdots \cup X_t \cup Y_2$ of S in which all X_i, Y_j are contiguous segments of integers and only Y_1, Y_2 may contain $1, n$, respectively (so that Y_1, Y_2 may be empty).

Theorem 5.2. *(Gale's Evenness Criterion) A subset $S \subseteq [n]$ spans a boundary $(|S|-1)$-face of $C(n,d)$ if and only if in the above decomposition of S, the number of interior components X_i with odd length is at most $d - |S|$.*

STEP 1. The following Lemma achieves Step 1.

Lemma 5.3. *For $n \geq 6$, the subdivision T of $C(n,2)$ into the polygons $P_1 = \{2,3,4,5\}$ and $P_2 = \{1,2,5,6,7,\ldots,n-1,n\}$ is π-induced for $\pi : C(n,n-2) \to C(n,2)$ but its π-coherence depends upon the choice of parameters.*

Proof. Let $p_1, \ldots, p_n$ denote the vertices of $C(n,n-2)$ and $q_1, \ldots, q_n$ the corresponding vertices of $C(n,2)$. It is clear that T is a polygonal subdivision of $C(n,2)$ and one can check from Gale's Evenness Criterion that it is π-induced from $C(n,n-2)$, if $n \geq 6$.

To prove that the π-coherence of T depends upon the parameters we use Lemma 2.4. Since $C(n,n-2)$ has only two more vertices than its dimension, there is a unique (up to scaling) affine dependence $\sum_i c_i p_i = 0$ among its vertices whose coefficients c_i are given by the formula

$$c_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{t_j - t_i}$$

as functions of the parameters $t_1 < \cdots < t_n$. For the purpose of exhibiting parameters which make T either π-coherent or π-incoherent, we fix the parameters $t_2 = 2, t_3 = 3, t_4 = 4, t_5 = 5$ and vary the rest.

If T is π-coherent, the functional $f(\mathbf{x}) = \sum_i w_i x_i$ exhibiting its π-coherence has the property that all the lifted points $(q_i, w_i) \in \mathbb{R}^3$ for i in the polygon P_2 are coplanar. We wish to show that we can furthermore assume that $w_i = 0$ for $i \in P_2$. To argue this, note that coplanarity implies that there is an affine functional h on $\mathbb{R}^2$ with the property that $h(q_i) = w_i$ for all $i \in P_2$. Hence the functional $f' = \sum_i w'_i x_i$ with $w'_i = w_i - h(q_i)$ will induce the same subdivision T and will have the property that $w'_i = 0$ for all $i \in P_2$. We further claim that f' also exhibits the π-coherence of T. Since f does so, $f \in im(\phi_P^*)$, where we are using the notation of Section 2 with $P = C(n,n-2)$ and $Q = C(n,2)$. Moreover, the funcional $\sum_i h(q_i) x_i$ is the composition $h \circ \pi \circ \rho \circ \phi_P \in im(\phi_P^*)$, where $\rho : \mathbb{R}^{n-1} \to \mathbb{R}^{n-2}$ simply forgets the first coordinate. Hence $f' \in im(\phi_P^*)$ as well and so f' exhibits the π-coherence of T.

Now notice that not only does $(w'_1, \ldots, w'_n)$ have the property that $w'_i = 0$ for all $i \in P_2$ but also, since our choice of t_2, t_3, t_4, t_5 makes the quadrilateral P_1 into a trapezoid and since the lifted points (q_i, w_i) are coplanar, we have $w'_3 = w'_4 > 0$. By Lemma 2.4, T is π-coherent if and only if $\sum_i c_i w_i = 0$, which becomes $c_3 + c_4 = 0$, or in other words $|c_3| = |c_4|$.

It now suffices to show that by suitable choices of the remaining parameters $t_1, t_6, t_7, \ldots, t_n$ we can make the equation $|c_3| = |c_4|$ valid or invalid. Let K be a very large positive number and ϵ a very small positive number. Consider the following two situations:

- Choosing $t_1 = -K$ and t_i close to $5 + \epsilon$ for $i \geq 6$, one has that
$$c_3 \text{ is approximately } \frac{1}{2 \cdot (2+\epsilon)^{n-5}(K+3)},$$
$$c_4 \text{ is approximately } -\frac{1}{2 \cdot (1+\epsilon)^{n-5}(K+4)}$$
and so $\frac{|c_4|}{|c_3|}$ is approximately 2^{n-5}.
- Choosing $t_1 = 2 - \epsilon$ and t_i close to K for $i \geq 6$, one has that
$$c_3 \text{ is approximately } \frac{1}{2 \cdot (K-3)^{n-5}(1+\epsilon)},$$
$$c_4 \text{ is approximately } -\frac{1}{2 \cdot (K-4)^{n-5}(2+\epsilon)}$$
and so $\frac{|c_4|}{|c_3|}$ is approximately $\frac{1}{2}$.

Since $\frac{|c_4|}{|c_3|}$ is greater than 1 in the first case but less than 1 in the second case, we conclude that for the choice of parameters in either of these two situations, T is not π-coherent. However if one varies $t_1, t_6, t_7, \ldots, t_n$ continuously, there must be some choice for which $\frac{|c_4|}{|c_3|} = 1$. This choice makes T π-coherent. ☐

STEP 2. For Step 2, we wish to classify the boundary faces F of $C(n, d')$ into three types: *upper, lower* and *contour faces* according to whether the normal cone to F in $(\mathbb{R}^{d'})^*$ only contains functionals $f(\mathbf{x}) = \sum_i w_i x_i$ in which $w_{d'}$ is positive or only those with $w_{d'}$ negative or both kinds, respectively. Equivalently, contour faces of $C(n, d')$ are those faces whose normal cone contains a functional $f(\mathbf{x}) = \sum_i w_i x_i$ with $w_{d'} = 0$, which is to say that they are the faces which project to boundary faces of $C(n, d' - 1)$.

Observe that the surjection $\mathbb{R}^{d'} \to ker(\pi)^*$ associated to the projection $\pi : C(n, d') \to C(n, d)$ is the one which forgets the first d coordinates. In particular, the projections to $ker(\pi)^*$ of the normal cones of an upper face and a lower face of $C(n, d')$ do not intersect and, as a consequence of Lemma 2.1, any π-induced subdivision of $C(n, d)$ having a lower and an upper face of $C(n, d')$ has to be π-incoherent.

One can explicitly say which *facets* (maximal faces) of $C(n, d')$ are lower and which are upper (they are never contour faces). Gale's Evenness Criterion tells us that a subset $S \subset [n]$ forms a facet of $C(n, d)$ if and only if all of its internal contiguous segments X_i are of even length in the decomposition $S = Y_1 \cup X_1 \cup X_2 \cup \cdots \cup X_t \cup Y_2$. One can then check that the upper (respectively lower) facets are those in which Y_2 has odd (respectively even) length. A non-maximal boundary face F is then upper (respectively lower) if and only if it lies only in upper (respectively lower) facets. Otherwise F is a contour face.

The point of introducing this terminology is the following Lemma.

Lemma 5.4. *Let T be a π-induced polytopal subdivision for $\pi : C(n, d') \to C(n, d)$ and some choice of parameters $t_1 < \cdots < t_n$. Assume T contains no face $\pi(F)$, where F is an upper face of $C(n, d')$. Let T' be the subdivision of $C(n + 1, d)$ with parameters $t_1 < \cdots < t_n < t_{n+1}$ obtained by adding* *to the faces of T all simplices obtained by adjoining the vertex $n+1$ to the upper facets of $C(n,d)$. Then*

(i) *T' is π-induced and contains no face $\pi(F)$ for any upper faces F of $C(n+1,d')$.*
(ii) *If T is π-coherent and induced by a functional $f(\mathbf{x}) = \sum_{i=1}^{d'} w_i x_i$ with $w_{d'} < 0$ then T' is π-coherent and induced by the same f for any sufficiently large choice of the parameter t_{n+1}.*
(iii) *If T is not π-coherent then neither is T'.*

Proof. (i) This follows from the fact that every lower facet of $C(n,d')$ is a lower facet of $C(n+1,d')$ as well and that when one adjoins the vertex $n+1$ to an upper facet of $C(n,d)$ one obtains a (lower) facet of $C(n+1,d+1)$ and, thus, a contour face of $C(n+1,d')$ for every $d' \geq d+2$.

(ii) According to Lemma 2.1, saying that T is π-coherent and induced by $f(\mathbf{x}) = \sum_{i=1}^{d'} w_i x_i$ is equivalent to saying that for every cell F in T, the functional $f(0,\dots,0,x_{d+1},x_{d+2},\dots,x_{d'})$ is in the projection under $(\mathbb{R}^{d'})^* \twoheadrightarrow ker(\pi)^*$ of the normal cone to the face of $C(n,d')$ corresponding to F.

If t_{n+1} is chosen large, the lower faces of $C(n+1,d')$ that use the point $n+1$, in particular the lower faces of $C(n+1,d')$ that occur in T' but were not already in T, are "almost vertical". As a consequence, the vectors with d'-th coordinate negative lying in the normal cones to these new lower facets are "almost horizontal". Since $w_{d'} < 0$, given any cell in T, we can choose t_{n+1} so large that $f(0,\dots,0,x_{d+1},x_{d+2},\dots,x_{d'})$ still lies in the projection of the normal cone of the face of $C(n+1,d')$ which corresponds to that cell. Thus, the coherent subdivision of $C(n+1,d)$ induced by f will contain all the cells of T, so it must be precisely T'.

(iii) Adding the extra point $n+1$ makes the normal cones of the faces of $C(n,d')$ corresponding to cells of T smaller. Since T' contains all the cells of T, Lemma 2.1 implies that if T were π-incoherent then T' would also be π-incoherent. $\square$

The next corollary is immediate from the previous Lemma.

Corollary 5.5. *Let T be a subdivision of $C(n,2)$ which is π-induced for $\pi : C(n,d') \to C(n,2)$ with $d' \geq 4$. Assume T has some cell which corresponds to a lower face and no cell which corresponds to an upper face of $C(n,d')$. Let T' be the subdivision of $C(n+1,d')$ obtained adding the triangle $\{1,n,n+1\}$. Then T' satisfies the above hypotheses with n replaced by $n+1$ and*

(i) *If the π-coherence of T depends upon the choice of parameters, the same is true for T'.*
(ii) *If T is π-incoherent for every choice of parameters, then so is T'.*

Proof. Lower faces of $C(n,d')$ are lower faces of $C(n+1,d')$ as well and $\{1,n,n+1\}$ is a contour face of $C(n+1,d')$ for $d' \geq 4$. Thus, T' satisfies the hypotheses. Parts (i) and (ii) follow trivially from Lemma 5.4. Part (ii) of the lemma applies to our case because the fact that T has a lower face of $C(n,d')$ implies that *any* functional $\sum_{i=1}^{d'} w_i x_i$ which exhibits the π-coherence of T has $w_{d'} < 0$. $\square$

We can now complete Step 2. It is easy to check that the subdivision T of $C(n,2)$ produced in Lemma 5.3 satisfies the hypotheses of the previous corollary with $d' = n-2$, if $n \geq 6$: the polygon $P_2 = \{1,2,5,6,7,\dots,n-1,n\}$ corresponds to a lower facet of $C(n,n-2)$ and the polygon $P_1 = \{2,3,4,5\}$ corresponds to a lower facet of $C(6,4)$ and to a contour face of $C(n,n-2)$ for $n \geq 7$. Therefore by iterating part (i) of the corollary we obtain for each $d' \geq 4, n-d' \geq 2$ a subdivision T of $C(n,2)$ which is π-induced for $\pi : C(n,d') \to C(n,2)$, but whose π-coherence depends upon the choice of parameters.

STEP 3. Here we make use of Lemma 2.7. As was said in Section 2, the natural projections $\pi : C(n,d') \to C(n,d)$ and $\hat{\pi} : C(n+1,d'+1) \to C(n+1,d+1)$ are compatible with the single element liftings $C(n+1,d'+1), C(n+1,d+1)$ of $C(n,d'), C(n,d)$, respectively. Recall from that section that this required us to choose the parameters $t_1 < \cdots < t_n < t_{n+1}$ so that $t_{n+1} = 0$. Again this is not a problem, as an affine transformation $t_i \mapsto at_i + b$ leads to compatible affine transformations of $C(n,d), C(n+1,d+1), C(n,d'), C(n+1,d'+1)$ and π-coherence of subdivisions is easily seen to be preserved by such transformations. Since this situation makes $C(n,d)$ the *vertex figure* [31, §2.1] of $C(n+1,d+1)$ for the vertex $n+1$, any polytopal subdivision T' of $C(n+1,d+1)$ gives rise to a polytopal subdivision T of $C(n,d)$ by taking the *link* of $n+1$ in T'. In this situation we say that T' *extends* T.

Proposition 5.6. *Let T be a polytopal subdivision of $C(n,d)$ which is π-induced for the projection $\pi : C(n,d') \to C(n,d)$ and some choice of parameters $t_1 < \cdots < t_n$.*

- (a) *If T is π-coherent then for every choice of the parameter t_{n+1}, T extends to a subdivision T' of $C(n+1,d+1)$ which is $\hat{\pi}$-coherent.*
- (b) *If T is not π-coherent then it does not extend to any subdivision T' of $C(n+1,d+1)$ which is $\hat{\pi}$-coherent.*

Proof. If T is π-coherent then by Theorem 2.5 there is a functional $f \in im(\phi^*_{P,Q}) \subset ker(\phi_Q)^*$ which induces it. Under the identification given by the isomorphism in Lemma 2.7, this same functional f will induce some $\hat{\pi}$-coherent subdivision T' that extends T. Conversely, if T' were $\hat{\pi}$-coherent and extended T, then the vector $f' \in im(\phi^*_{\hat{P},\hat{Q}}) \subset ker(\phi_{\hat{Q}})^*$ which induces T' would map under the reverse of this isomorphism to a vector that induces T and would demonstrate its π-coherence. □

This finally gives us the result needed to complete Step 3.

Corollary 5.7. *If $\pi : C(n,d') \to C(n,d)$ has a π-induced polytopal subdivision of $C(n,d)$ whose π-coherence depends upon the choice of parameters then so does $\hat{\pi} : C(n+1,d'+1) \to C(n+1,d+1)$.*

Proof. Given such a subdivision T of $C(n,d)$ and a choice of parameters which makes it π-coherent, part (a) of the previous corollary produces a subdivision T' of $C(n+1,d+1)$ which is $\hat{\pi}$-coherent for some choices of the parameters. But if one chooses the first n parameters $t_1 < \cdots < t_n$ so that T is π- incoherent then T' must also be $\hat{\pi}$-incoherent by part (b), regardless of how t_{n+1} is chosen. □

6. An instance of the Baues problem

The discussion in Step 1 of Section 5 shows that the case of the projections $\pi : C(n,n-2) \to C(n,2)$ plays a special role in the family of projections $\pi : C(n,d') \to C(n,d)$, in that they provide examples of non-canonical behavior (π-induced subdivisions whose π-coherence depends upon the choice of parameters) with d and $n-d'$ both minimal. This prompted our study of the Generalized Baues Problem (or GBP) in this case, leading to Theorem 1.2. Let $\bar{\omega}(P \xrightarrow{\pi} Q)$ denote the proper part of the Baues poset $\omega(P \xrightarrow{\pi} Q)$, i.e. $\omega(P \xrightarrow{\pi} Q)$ with its maximal element removed. Recall that the GBP asks whether $\bar{\omega}(P \xrightarrow{\pi} Q)$ has the homotopy type of a $(\dim(P)-\dim(Q)-1)$-sphere. When we refer to the topology of a poset $\mathcal{L}$ we always mean the topology of the *geometric realization* $|\Delta(\mathcal{L})|$ of its *order complex* $\Delta(\mathcal{L})$, so that $\Delta(\mathcal{L})$ is the simplicial complex of chains in $\mathcal{L}$ (see [9]). We now restate and prove Theorem 1.2.

Theorem 1.2. *Let $\pi : P \to Q$ be a linear surjection of polytopes with the following two properties:*

- *P has $\dim(P)+2$ vertices and*
- *the point configuration $\mathcal{A}$ which is the image of the set of vertices of P under π has only coherent subdivisions.*

Then the GBP has a positive answer for $\pi : P \to Q$, i.e. the poset $\bar{\omega}(P \xrightarrow{\pi} Q)$ of all proper π-induced subdivisions of Q has the homotopy type of a $(\dim(P)-\dim(Q)-1)$-sphere.

Proof. Consider the tower of projections

$$\Delta^{n-1} \to P \xrightarrow{\pi} Q,$$

where Δ^{n-1} is the standard $(n-1)$-simplex in $\mathbb{R}^n$, as in Section 2. There is an obvious inclusion

$$\omega(P \xrightarrow{\pi} Q) \hookrightarrow \omega(\Delta^{n-1} \to Q)$$

which simply identifies every π-induced subdivision of Q with a subdivision of Q which uses only the points of $\mathcal{A}$ as vertices. Since all subdivisions of the point configuration $\mathcal{A}$ are coherent, $\bar{\omega}(\Delta_{n-1} \to Q)$ can be described as follows. Let $Q^* = \{q_1^*, \dots, q_n^*\}$ be the Gale transform of Q. Every subdivision of Q corresponds uniquely to a face of the *chamber complex* of Q^*, i.e. to a non-zero cone in the common refinement of all the simplicial fans generated by vectors of Q^*. Thus, the poset $\bar{\omega}(\Delta_{n-1} \to Q)$ is the opposite or dual poset $\mathcal{L}^{opp}$ to the poset $\mathcal{L}$ of faces in the chamber complex of Q^*. Equivalently, $\bar{\omega}(\Delta_{n-1} \to Q)$ is isomorphic to the poset of proper faces of the *secondary polytope* of Q (see [17, Chapter 7] or [3] for details).

Only certain of the cones in the chamber complex of Q^* correspond to π-induced subdivisions of Q, that is, to elements of the subposet $\mathcal{L}' := \bar{\omega}(P \xrightarrow{\pi} Q)^{opp} \subseteq \mathcal{L}$. We know that $\mathcal{L}$, considered as a topological space, is homeomorphic to a $(\dim(P) - \dim(Q))$-sphere and wish to show that the subspace $\mathcal{L}'$ is homotopy equivalent to a $(\dim(P) - \dim(Q) - 1)$-sphere. We will show that the cones corresponding to the elements of $\mathcal{L} - \mathcal{L}'$ form two disjoint, convex (but not necessarily closed) unions of cones U^+, U^- and that there is a linear functional f which separates them, in the sense that $f(x) > 0$ for all $x \in U^+$ and $f(x) < 0$ for all $x \in U^-$. The result then follows immediately from the technical Lemma 6.1.

Since P has $dim(P) + 2$ vertices, its vertices contain a unique (up to a scaling factor) affine dependence

$$\sum_{i \in F^+} c_i p_i = \sum_{j \in F^-} c_j p_j \tag{4}$$

with $c_i, c_j > 0$ for all i, j. Let $F^0 = \{1, \dots, n\} \setminus (F^+ \cup F^-)$. Observe that the affine dependence projects to an affine dependence in Q and induces a functional f in the Gale transform Q^* such that $f(q_i^*)$ is zero, positive or negative if i is in F^0, F^+ or F^-, respectively. In oriented matroid terms, the affine dependence is a *vector* in Q and, thus, a *covector* in Q^*. This f will be the linear functional of the statement of Lemma 6.1.

A subset F of indices represents a face of P if and only if it either contains $F^+ \cup F^-$ or contains neither F^+ nor F^-. Thus, F represents a non-face if it contains F^+ or F^-, but not both. The complements of these non-faces are the “forbidden cones” in the Gale transform, according to Lemma 2.3. Hence, the forbidden cones are of the following two types:

(+) $A^+ \cup A^0$ where $\emptyset \neq A^+ \subseteq F^+$, $A^0 \subseteq F^0$, or
(−) $A^- \cup A^0$ where $\emptyset \neq A^- \subseteq F^-$, $A^0 \subseteq F^0$.

We claim that the unions of the forbidden open cones of the types (+) and (−) are:

$$U^+ = pos(F^0 \cup F^+) \cap \{f(x) > 0\} \quad \text{and} \quad U^- = pos(F^0 \cup F^-) \cap \{f(x) < 0\},$$

respectively. Indeed, let v be a vector in the relative interior of a cone of type (+). Clearly, v lies in $pos(F^0 \cup F^+)$ and $f(v) > 0$ since every cone of type (+) contains among its generators a vector on which f is positive and no vector on which f is negative. Conversely, if $v \in U^+ \subset pos(F^+ \cup F^0)$ then v lies in a certain cone C generated by a subset of $F^0 \cup F^+$ and this subset must contain an element of F^+ or otherwise $f(v) = 0$. A similar argument applies to U^-.

The sets U^+ and U^-, as defined above, are clearly convex and separated by the functional f, so the proof is complete. ∎

The following technical lemma was needed in the preceding proof.

Lemma 6.1. *Let $\mathcal{F}$ be a complete polyhedral fan in $\mathbb{R}^d$, i.e. a collection $\{C\}$ of relatively open polyhedral pointed cones covering $\mathbb{R}^d$, each having its vertex at the origin, such that $C \cap C'$ is a boundary face of C, C' for each pair of cones.*

Let $\mathcal{L}$ be the poset of non-zero cones in $\mathcal{F}$ and $\mathcal{L}'$ the subposet corresponding to the elements in two non-empty convex (but not necessarily closed) unions of cones U^+, U^-. Assume that the cones U^+, U^- are separated by some functional $f \in \mathbb{R}^d$, in the sense that $f(x) > 0$ for $x \in U^+$ and $f(x) < 0$ for $x \in U^-$.

Then $\mathcal{L} - \mathcal{L}'$ is homotopy equivalent to a $(d-2)$-sphere.

Proof. Note that $\mathcal{L}$ is the face poset of the regular (actually polyhedral) cell complex obtained by intersecting the cones in the fan $\mathcal{F}$ with the unit sphere $\mathbb{S}^{d-1}$. The subposet $\mathcal{L} - \mathcal{L}'$ then indexes the cells in the cones which cover the complement $X := \mathbb{S}^{d-1} - (U^+ \cup U^-)$. If we show that X is homotopy equivalent to $\mathbb{S}^{d-2}$ then the proof follows immediately by applying Lemma 6.2 below with $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$.

Let $\mathbb{S}^{d-2}$ be the equatorial sphere $\mathbb{S}^{d-1} \cap \{x : f(x) = 0\}$ defined by the function f and H^+ be the "upper" hemisphere $H^+ := \mathbb{S}^{d-1} \cap \{x : f(x) \geq 0\}$ (similarly define H^-). It suffices to show that $\mathbb{S}^{d-2}$ is a deformation retract of $H^+ - U^+$ and similarly for $H^- - U^-$, since then we can retract X onto $\mathbb{S}^{d-2}$ by first retracting $H^+ - U^+$ onto $\mathbb{S}^{d-2}$, keeping H^+ fixed, and then retracting $H^- - U^-$ onto $\mathbb{S}^{d-2}$. To retract $H^+ - U^+$ onto $\mathbb{S}^{d-2}$, note that the pair (H^+, U^+) is homeomorphic to the pair $(\mathbb{B}^{d-1}, C)$ by forgetting the last coordinate and then scaling, where C is some convex subset inside the unit disk $\mathbb{B}^{d-1}$. We then need to retract $\mathbb{B}^{d-1} - C$ onto the boundary $\partial\mathbb{B}^{d-1} = \mathbb{S}^{d-2}$, which can be done as follows. Pick any point $p \in C$. For $x \in \mathbb{B}^{d-1} - C$, let $s(x)$ be the unique point on the boundary $\partial\mathbb{B}^{d-1}$ that lies on the ray which emanates from p and passes through x. Define the deforming homotopy $f : (\mathbb{B}^{d-1} - C) \times [0,1] \to \mathbb{B}^{d-1} - C$ by $f(x,t) = (1-t)\,x + t\,s(x)$. Convexity of C shows that f is well-defined, i.e. that its image lies in $\mathbb{B}^{d-1} - C$. Also f is continuous because s is the restriction to $\mathbb{B}^{d-1} - C$ of a continuous map $\mathbb{B}^{d-1} - \{p\} \to \partial\mathbb{B}^{d-1}$. ☐

The following lemma is probably well-known but we do not know of a proof in the literature, so we include one here.

Lemma 6.2. *Let K be a regular cell complex with face poset $\mathcal{L}$ and let σ_l be the cell of K indexed by $l \in \mathcal{L}$. Then for any subposet $\mathcal{L}'' \subseteq \mathcal{L}$, the subspace $K'' := \cup_{l \in \mathcal{L}''} \sigma_l$ is homotopy equivalent to $\mathcal{L}''$.*

Proof. Since K is a regular cell complex and $\mathcal{L}$ is its face poset, the order complex $\Delta(\mathcal{L})$ is the first barycentric subdivision $Sd(K)$ and the order complex $\Delta(\mathcal{L}'')$ is a subcomplex of $Sd(K)$. The subspace K'' of K may be identified with a subspace of $Sd(K)$. We will describe a deformation retraction of K'' inside $Sd(K)$ onto the subcomplex $\Delta(\mathcal{L}'')$. The retraction will be defined piecewise on each simplex of $Sd(K)$. A simplex σ of $Sd(K)$ is represented by a chain $l_1 < \cdots < l_r$ in $\mathcal{L}$ and has vertices labelled $l_1, \ldots, l_r$. Let σ_1 be the subface of this simplex spanned by the l_i's which lie in the subposet $\mathcal{L}''$ and σ_2 be the opposite subface, i.e. the one spanned by the rest of the l_i. Then $|\sigma|$ is the *topological join*

$$|\sigma| = |\sigma_1| * |\sigma_2| := |\sigma_1| \times |\sigma_2| \times [0,1]/((x,y,0) \sim x, (x,y,1) \sim y).$$

One can check from the definition that $|\sigma \cap K''| \subseteq |\sigma| - |\sigma_2|$ and $|\sigma \cap \Delta(\mathcal{L}'')| = |\sigma_1|$. Since for any topological join $X * Y$ one can retract $X * Y - Y$ onto X, we can retract $|\sigma \cap K''|$ onto $|\sigma \cap \Delta(\mathcal{L}'')|$ for each simplex σ. It is easy to see that all of these retractions can be done coherently, giving a retraction of K'' onto $|\Delta(\mathcal{L}'')|$, as desired. ☐

As was mentioned earlier, Theorem 1.2 has the following corollary.

Corollary 6.3. *The GBP has a positive answer for* $\pi : C(n, n-2) \to C(n, 2)$.

It is perhaps worthwhile to look more closely at the first interesting example, i.e. the projection $\pi : C(6,4) \to C(6,2)$. From the results of Section 5, this is the minimal example where the fiber polytope depends upon the choice of parameters. We first compute the Baues poset $\omega(C(6,4) \to C(6,2))$ (which does not depend on the choice of parameters) by using the technique in the proof of Theorem 1.2. The secondary polytope of $C(6,2)$ is the well-known 3-dimensional associahedron, which is a simple 3-polytope with six pentagons and three quadrilaterals as facets. A picture of it can be found in [17, page 239]. There are two special vertices, each incident to three pentagons, which correspond to the triangulations $\{135, 123, 345, 561\}$ and $\{246, 234, 456, 126\}$. In the chamber complex of the Gale transform, these triangulations correspond to chambers that are triangular cones generated by 246 and 135, respectively. The other twelve vertices of the associahedron are incident to two pentagons and a quadrilateral each.

The minimal non-faces of $C(6,4)$ are $F^+ = 135$ and $F^- = 246$. Since in this case F^0 is empty (this is always the case in $C(n, n-2)$), the regions U^+ and U^- in the proof of Theorem 1.2 to be removed from the chamber complex of $C(6,2)$ are the closed triangular cones generated by 135 and 246. In other words, the Baues poset $\omega(C(6,4) \to C(6,2))$ is isomorphic to the poset of proper faces of the associahedron not incident to the two special vertices mentioned above. This leaves us with twelve vertices of the associahedron, representing twelve π-induced triangulations, 15 edges, representing 15 π-induced subdivisions of height one in the poset and 3 quadrilaterals, representing 3 π-induced subdivisions of height 2. The cell complex of these faces is depicted in Figure 1, where we have drawn the subdivision corresponding to each face. The twelve subdivisions in thick are the triangulations, which we have numbered from **1** to **12**. In the following discussion we will refer to a π-induced subdivision by the π-induced triangulations which refine it. Thus, **(1,2,3,4)**, **(5,6,7,8)** and **(9,10,11,12)** represent the three subdivisions of height two (the quadrilaterals in Figure 1).

We now want to study the fiber polytope associated to the projection π and its dependence with the choice of parameters. We first recall that a π-coherent subdivision cannot have both lower and upper faces of $C(6,4)$. The upper and lower facets of $C(6,4)$ are $\{1234, 1245, 1256, 2345, 2356, 3456\}$ and $\{1236, 1346, 1456\}$, respectively. Thus, the faces 136 and 146 are upper, while 234 and 345 are lower. This implies that the following subdivisions are not π-coherent:

$$(\mathbf{8}) = \{124, 234, 146, 456\}, \qquad (\mathbf{9}) = \{123, 136, 345, 356\},$$
$$(\mathbf{7},\mathbf{8}) = \{1234, 146, 456\}, \quad (\mathbf{5},\mathbf{8}) = \{124, 234, 1456\}, \quad (\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}) = \{1234, 1456\},$$
$$(\mathbf{9},\mathbf{10}) = \{1236, 345, 356\}, \quad (\mathbf{9},\mathbf{12}) = \{123, 136, 3456\}, \quad (\mathbf{9},\mathbf{10},\mathbf{11},\mathbf{12}) = \{1236, 3456\}.$$

Incidentally, the same argument shows that for the projection $\pi : C(2d'-2, d') \to C(2d'-2, 2)$, $d' \geq 4$, there are π-induced subdivisions which are π-incoherent in every choice of parameters. Namely, the subdivision with only two cells $\{1, \ldots, d'\}$ and $\{1, d', d'+1, \ldots, 2d'-2\}$. Other cases can be obtained from this by applying Proposition 5.6. We do not know whether $n-d', d'-d, d \geq 2$ always implies there exists a subdivision of $C(n,d)$ which is π-induced for $\pi : C(n,d') \to C(n,d)$ but π-incoherent for every choice of parameters.

After removing those non-coherent subdivisions, the poset is already almost the face poset of a polygon, except for the quadrilateral **(1,2,3,4)**. In particular, the triangulations **2**, **5**, **6**, **7**, **10**, **11**, **12** and **4** are always π-coherent as well as the height one subdivisions (bistellar flips) joining them. The possible subdivisions whose π-coherence may depend upon the choice of parameters are listed in the following rows:

$$(\mathbf{1},\mathbf{2}) = \{125, 156, 2345\}, \quad (\mathbf{1}) = \{125, 156, 235, 345\}, \quad (\mathbf{1},\mathbf{4}) = \{1256, 235, 345\},$$
$$(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}) = \{1256, 2345\},$$

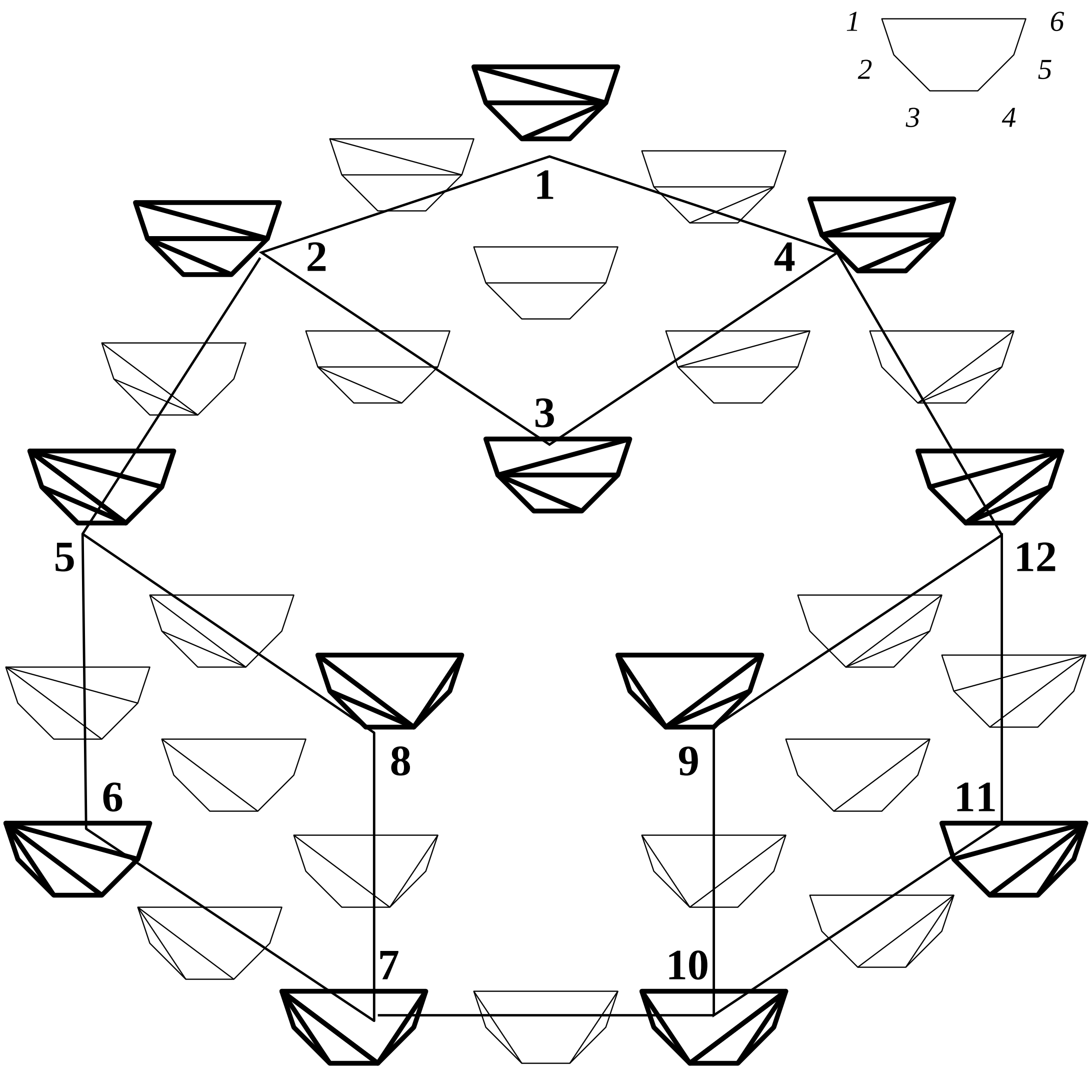

FIGURE 1. The structure of π-induced subdivisions.

$$(\mathbf{2},\mathbf{3}) = \{1256, 234, 245\}, \qquad (\mathbf{3}) = \{126, 256, 234, 245\}, \qquad (\mathbf{3},\mathbf{4}) = \{126, 256, 2345\}.$$

The π-coherence of the subdivision $\{1256, 2345\}$ is precisely what was studied in Step 1 of Section 5 and, actually, the three cases of $|c_3|$ being less, equal or greater than $|c_4|$ which appeared there produce the three cases for the fiber polytope. In the two extreme cases the fiber polytope is a 9-gon and in the middle one it is an octagon.

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