STASINSKI AND VOLL’S HYPEROCTAHEDRAL GROUP CONJECTURE

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Abstract. In a recent paper, Stasinski and Voll introduced a length-like statistic on hyperoctahedral groups and conjectured a product formula for this statistic’s signed distribution over arbitrary descent classes. Stasinski and Voll proved this conjecture for a few special types of parabolic quotients. We prove this conjecture in full, showing it holds for all parabolic quotients. In the case that the descent class is a singleton, this formula gives the Poincaré polynomials for the varieties of symmetric matrices of a fixed rank.

1. Introduction

In this paper, we prove the conjecture of Stasinski and Voll [3, Conjecture 1] as written in Theorem 2.3.1 below. This result has relations to the Poincaré polynomials of the $n \times n$ symmetric matrices over $\mathbb{F}_q$ of a given rank, as noted in [3, p. 3]. The result also gives a formula for representation zeta functions associated to a particular group scheme, as described in [3, p. 3-4],[4, Definition 1.2, Theorem C]. Additionally, this result immediately implies the validity of the interesting identity given in [4, Proposition 5.5]. Finally, an conjectural analog of Theorem 2.3.1, which sums monomials over $S_n^T$ instead of $B_n^T$, can be found in [2, Conjecture C].

The main result states that a certain generating function for a length-like statistic on parabolic quotients of the hyperoctahedral group, $B_n$, always has a simple formula. The main idea of the proof will be to find an appropriate recursion for the product form of the generating function, and then to check combinatorially that the generating function as a summation satisfies this same recursion. The results described in the background section are mostly due to previous work, and the results in the following sections are original.

The outline of the paper is as follows. First, in Section 2, we present background material, including the relevant previous partial progress toward the conjecture made in [3]. Then, in Section 3, we give a recursion that the product form of the generating function satisfies. Next, in Section 4 we state Theorem 4.0.17, which is our main result, and we show why this theorem implies Stasinski and Voll’s conjecture. In Section 5 we describe some crucial involutions which will allow us to restrict which elements we sum the generating function over. In Sections 6, 7, and 8 we show why the first, second, and third parts of Theorem 4.0.17 hold. By far the most difficult is part 3, which will involve characterizing certain elements in $B_n$, which we call “initially uncanceled” and ”finally uncanceled,” and then describing a bijection between these two types of elements. Finally, in Section 9, we state a further conjecture, which involves a generalization of the generating function to two variables.

2. Background

2.1. Notation for $B_n$.

Notation 2.1.1. Throughout this paper, we will only use congruence conditions of the form $a \equiv b \pmod{2}$, and so we drop the \pmod{2}, and henceforth write $a \equiv b$ to simplify notation.

Notation 2.1.2. We let $[n] = \{1,\ldots,n\}, [n]_0 = \{0,\ldots,n\}, [\pm n]_0 = \{-n,\ldots,n\}, \mathbb{N} = \{n \in \mathbb{Z}, n > 0\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
Definition 2.1.3. Regarding \( S_{2n+1} \) as acting on the set \([\pm n]_0\), the hyperoctahedral group is by definition \( B_n = \{ w \in S_{2n+1} \mid w(-n) = -w(n) \} \).

Notation 2.1.4. A convenient way to notate \( w \in B_n \) will be \( w = [t_1, \ldots, t_n] \) which means that for \( t \in [n] \), we have \( w(t) = t_i, w(0) = 0 \), and \( w(-t) = -t_i \).

Definition 2.1.5. For \( w \in B_n \), define the signed permutation matrix which we also notate as \( w \in M_{n \times n} \). It is defined by

\[
 w_{i, j} = \begin{cases} 
 1, & \text{if } w(j) = i \\
 -1, & \text{if } w(j) = -i \\
 0 & \text{otherwise}
\end{cases}
\]

Remark 2.1.6. We often go back and forth between thinking about \( w \in B_n \) and \( w \in M_{n \times n} \). So, when encountering the notation \( w(k) \), think of \( w \in B_n \).

Lemma 2.1.7. [1, p. 243, Proposition 8.1.3] \( B_n \) is the Coxeter group generated by the involutions \( s_0, s_1, \ldots, s_{n-1} \), where \( s_0 = [-1, 2, 3, \ldots, n] \), and \( s_i = [1, 2, \ldots, i-1, i+1, i, \ldots, n] \) for \( i > 0 \), and satisfying the Coxeter relations \( s_i^2 = 1, (s_0 s_i)^2 = 1, (s_i s_j)^2 = e \) for \( i \in [n-2] \) and \( (s_i s_j)^2 = e \) if \( j \neq i \pm 1 \).

Notation 2.1.8. Let \( l : B_n \to \mathbb{N}_0 \) denote the usual Coxeter length function. In this paper, we only need the sign function, \( \text{sgn} : B_n \to \{ \pm 1 \} \), where \( \text{sgn}(w) = (-1)^l(w) \) by definition.

Remark 2.1.9. A useful equivalent characterization is that \( \text{sgn}(w) = \det(w) \), the determinant of the signed permutation matrix associated to \( w \). This will help us easily compute how \( \text{sgn}(w) \) changes as \( w \) changes.

Notation 2.1.10. For \( w \in B_n \), denote the descent set \( D(w) = \{ i \in [n-1]_0 \mid (w s_i) < l(w) \} \), which is the usual Coxeter right descent set. Also, say \( w \) has a descent at \( k \) if \( k \in D(w) \).

Lemma 2.1.11. [1, p. 248, Proposition 8.1.2] For all \( w \in B_n \), there is an equality \( D(w) = \{ i \in [n-1]_0 \mid w(i) > w(i+1) \} \).

Remark 2.1.12. We almost exclusively use this characterization of descent set in the remainder of the paper. The reason for the word “descent” is that if \( w(i) > w(i+1) \), the value of \( w(i) \) descends to the value of \( w(i+1) \).

We now recall some notation introduced in [3, p. 2]. Fix \( n \). Let \( I \subset [n-1]_0 \). Notate the quotient \( B_n^I = \{ w \in B_n \mid D(w) \subset I^c \} \), where by definition \( I^c = [n-1]_0 \setminus I \). For \( I \) a set, the notation \( I = \{ i_1, \ldots, i_t \} \) means \( i_j < i_2 < \cdots < i_t \).

The key statistic in this paper is the \( L \) statistic, defined by \( L : B_n \to \mathbb{N}_0 \), with

\[
 L(w) = \frac{1}{2} \left| \{(i, j) \in [\pm n]_0 \times [\pm n]_0 \mid i < j, w(i) > w(j), i \neq j \} \right| .
\]

For \( w \in B_n \), and \( k \) a column, denote by \( i_w(k) \) the unique row such that \( w s_i(k), k \neq 0 \). That is, \( i_w(k) \) is simply the row in which that \( \pm 1 \) in column \( k \) lies. Similarly, if \( k \) is a row, let \( j_w(k) \) be the unique column such that \( w s_i, k \neq 0 \). Beware; we often abuse notation by simply writing \( i_w(k) = i(k), j_w(k) = j(k) \), when it is clear what \( w \) is. Furthermore, we often have expressions in which the column involved is \( j \), which means its corresponding row is \( i(j) \).

Definition 2.1.13. For a fixed \( w \in B_n \), an entry \( w_{i, j} \) is (row) above of \( w_{k, l} \) if \( i < k \), i.e., if \( w_{i, j} \) lies in a row that is above the row \( w_{k, l} \) lies in. Also, \( w_{i, j} \) is (row) below \( w_{k, l} \) if \( i > k \). Similarly, \( w_{i, j} \) is (column) right of \( w_{k, l} \) if \( j > l \) and (column) left if \( j < l \). Finally, \( w_{s, t} \) is row between \( w_{i, j}, w_{k, l} \) if \( \max(i, k) \geq s \geq \min(i, k) \) and \( w_{s, t} \) is column between \( w_{i, j}, w_{k, l} \) if \( \max(j, l) \geq t \geq \min(j, l) \). When it is clear, we drop the parenthesized “column” and “row.”

Again, for a fixed \( w \in B_n \), a column is \( i \) weakly right of \( k \) if \( i \geq k \). There are analogous notions for weakly left, weakly above, and weakly below.
Notation 2.1.14. Fix $w ∈ B_n$. By abuse of notation, for $j,l$ columns, say $j$ is row above (respectively row below, column left, column right) $l$ if $w_{i(j),j}$ is row above (respectively row below, column left, column right) $w_{i(l),l}$, and for $s$ another column, say $s$ is row between (respectively column between) $j$ and $l$ when $w_{i(s),s}$ is row between (respectively column between) $w_{i(j),j}$ and $w_{i(l),l}$.

Similarly, when $i,k$ are rows, then $i$ is row above (respectively row below, column left, column right) $k$ if $w_{i,j(i)}$ is row above (respectively row below, column left, column right) $w_{k,j(k)}$, and for $t$ another column, then $t$ is row between (respectively column between) $i$ and $k$ when $w_{i,j(t)}$ is row between (respectively column between) $w_{i,j(i)}$ and $w_{k,j(k)}$.

We also sometimes say a row $i$ is above a column $j$, which just means $w_{i,j(i)}$ is above $w_{i(j),j}$. Take the analogous notions for a row being below, to the left of, or to the right of a column. Take the analogous definitions for a column $j$ being above a row $i$.

2.2. Notation for $f_{n,l}$. Let us briefly recall the notation for $f_{n,l}$ introduced in [3, p. 2].

For $n ∈ N_0$, define

$$
(n) = \begin{cases} 
1, & \text{if } n = 0 \\
1 - X^n & \text{if } n > 0
\end{cases}
$$

$$(n)! = (n) \cdot (n-1) \cdots (2) \cdot (1)
$$

$$(n)!! = \prod_{i \in [n], i \neq 0} (i)
$$

That is, $(n)!!$ is simply the product of $(i)$ with $i$ even such that $i < n$.

Fix $n$ and let $I = \{i_1, \ldots, i_l\} \subset [n-1]_0$. For $0 \leq t \leq l$, define

$$
\begin{array}{l}
\begin{align*}
   i_t &= i_{t+1} - i_t, & \text{if } 1 \leq t < l \\
   i_1 &= i_l & \text{if } t = 0 \\
   n - i_l &= i_l & \text{if } t = l.
\end{align*}
\end{array}
$$

That is, the $i_t$ are just consecutive differences of the $i_i$. If $i_1 \geq 0$, Define

$$f_{n,l}(X) = \frac{(n)!}{(i_1)! \cdot \prod_{k=1}^{l} (j_k)!!}
$$

In the case $i_1 < 0$, (or equivalently any element of $I$ is negative), define $f_{n,l}(X) = 0$.

Remark 2.2.1. There is an alternate, but equivalent way to write $f_{n,l}$. Namely,

$$f_{n,l}(X) = \binom{n}{I} X^{l(w)} \prod_{k=1}^{l} f_{[0], jk}(X)
$$

Where of course $f_{[0], jk} = \prod_{i \leq n, i \neq 1} (i)$. This formulation may lend some intuition to understanding Equation (2.1). The author conjectures that there may be a way of providing a noninductive proof of Theorem 2.3.1 by relating (2.2) to the well known equation

$$
\sum_{w \in S_n^l} X^{l(w)} = \binom{n}{I} X,
$$

perhaps by creating a map $F : B_n^l \to S_n^l$ similar to $H$ as defined in Algorithm 8.4.2, such that $L(w) = l(F(w)) + C$, for some constant $C$ depending only on $n, I$, but not on $w$.

2.3. The Main Result.

Theorem 2.3.1. For $n \in N, I = \{i_1, \ldots, i_l\} \subset [n-1]_0$, then

$$\sum_{w \in B_n^l} (-1)^{l(w)} X^{L(w)} = f_{n,l}(X).$$
Theorem 2.3.1 was shown in certain special cases in [3, Theorem 2]. Namely, it was shown when \( I = \{0\} \), when \( I = [n-1]_0 \) and when all elements of \( I \cup \{n\} \) are even. We spend the rest of the paper presenting a proof that holds for any \( I \) when \( I \subset [n-1]_0 \) The proof proceeds by induction on \( n \). We first demonstrate a recursive relation for the function \( f_{n,I}(X) \) by simple algebraic means, and then show that this same recursion applies to the expression \( \sum_{w \in B_n^I} (-1)^{l(w)} X^{L(w)} \).

2.4. Relevant Results from Stasinski and Voll. Before beginning the proof, we first recount the relevant progress toward Theorem 2.3.1 made in [3].

**Definition 2.4.1.**\(^\) Denote

\[ C_n = \{ w \in B_n | w_{i,j} \neq 0 \iff i + j \equiv 0 \} .\]

If \( w \in C_n \), say \( w \) is a chessboard element.

**Remark 2.4.2.** In the original paper [3], the chessboard elements that we are calling \( C_n \) are called \( C_{n,0} \), Stasinski and Voll also define a class called \( C_{n,1} \), which we do not need.

**Notation 2.4.3.** For \( I \subset [n-1]_0 \), Denote \( C^I_n = C_n \cap B^I_n \).

**Lemma 2.4.4.** (Chessboard Lemma)[3, p. 9, Lemma 8] For \( n \in \mathbb{N}, I \subset [n-1]_0 \),

\[ \sum_{w \in B^I_n} (-1)^{l(w)} X^{L(w)} = \sum_{w \in C^I_n} (-1)^{l(w)} X^{L(w)} \]

**Notation 2.4.5.** For \( I \subset [n-1]_0 \) define

\[ S_{n,I}(X) = \sum_{w \in C^I_n} (-1)^{l(w)} X^{L(w)} \]

**Corollary 2.4.6.** The main result Theorem 2.3.1 is equivalent to showing \( S_{n,I}(X) = f_{n,I}(X) \).

**Proof.** By the Chessboard Lemma 2.4.4,

\[ \sum_{w \in B^I_n} (-1)^{l(w)} X^{L(w)} = \sum_{w \in C^I_n} (-1)^{l(w)} X^{L(w)} = S_{n,I}(X) .\]

Hence, \( S_{n,I}(X) = f_{n,I}(X) \) it true if and only if \( \sum_{w \in B^I_n} (-1)^{l(w)} X^{L(w)} = f_{n,I}(X) \) is true. \( \square \)

**Definition 2.4.7.** In the following definition fix some \( n \) with \( w \in B_n \) and let \( 1 \leq j \leq n, 1 \leq j' \leq n \). Then, define the statistics

\[ a(w) = | \{ j \in [n] | w_{i(j),j} = -1, j \neq 0 \} | \]

\[ b_{j,j'}(w) = \begin{cases} 1, & \text{if } j < j', i(j) > i(j'), j \neq j' \\ 0, & \text{otherwise} \end{cases} \]

\[ c_{j,j'}(w) = \begin{cases} 1, & \text{if } j < j', w_{i(j'),j'} = -1, i(j) < i(j'), j \neq j' \\ 0, & \text{otherwise} \end{cases} \]

\[ b_{j'}(w) = \sum_{j=1}^{j'} b_{j,j'}(w), \]

\[ b_{j'}(w) = \sum_{j=1}^{j'} c_{j,j'}(w), \]

\[ b(w) = \sum_{j=1}^{n} b_j(w), \]

\[ c(w) = \sum_{j=1}^{n} c_j(w). \]
Remark 2.4.8. It is easy to see the definitions for the statistics $b, c$ given here agree with those in [3, Definition 4]. Namely,
\[ b(w) = |\{(j, j') \in [n] \times [n] | j < j', i(j) > i(j'), j \neq j'\}| \]
\[ c(w) = |\{(j, j') \in [n] \times [n] | w_{i(j), j'} = -1, j < j', i(j) < i(j'), j \neq j'\}| \]

Lemma 2.4.9. (abc Lemma) [3, p. 7, Lemma 6, Equation (11)] For $w \in B_n$, $L(w) = a(w) + b(w) + 2 \cdot c(w)$.

3. The $f_{n,t}$ Recursion

Now that we have the groundwork in place, we devote this section to describing the recursion which the polynomials $f_{n,t}(X)$ satisfy.

Notation 3.0.10. For the remainder of the paper, we shall fix $n$ and use $I$ for the set $I = \{i_1, i_2, \ldots, i_t\}$.\[ \]

Notation 3.0.11. Since the expression will come up a lot, for $k \in [l+1]$ define
\[ I^{(k)} = \begin{cases} I & \text{if } k = l + 1 \\ \{i_1, \ldots, i_{k-1}, i_k - 1, \ldots, i_l - 1\}, & \text{if } k \leq l \text{ and } i_k - 1 \neq i_{k-1} \\ \{i_1, \ldots, i_{k-2}, i_k - 1, \ldots, i_l - 1\}, & \text{otherwise} \end{cases} \]

Theorem 3.0.12. Let $I = \{i_1, \ldots, i_t\} \subset [n-1]_0$, and let $m$ be the biggest index such that $j_m$ is odd, if such an $m$ exists. Then, the $f_{n,t}$ satisfy the recurrence
\[ f_{n,t} = -X^n \cdot f_{n-1,t(m+1)}(X) + \sum_{t=m}^{l} X^{n-i_{t+1}} \cdot f_{n-1,t^{(t+1)}}(X) \]

If such an $m$ does not exist, then
\[ f_{n,t} = \sum_{t=0}^{l} X^{n-i_{t+1}} \cdot f_{n-1,t^{(t+1)}}(X) \]

Proof. We shall just prove (3.1) since the proof for (3.2) is analogous but with the first term dropped. Recall
\[ f_{n,t}(X) = \frac{(n)!}{(i_1)! \cdot \prod_{k=1}^{t} (j_k)!!} \]
\[ f_{n-1,t^{(t+1)}}(X) = \frac{(n-1)! \cdot (j_t)}{(i_1)! \cdot \prod_{k=1}^{t} (j_k)!!} \]

These equations both follow directly from the definition. To see why (3.3) holds, observe first that $t > m$, where $m$ is maximal such that $j_m$ is odd, and so this implies $j_t \equiv 0$. Now, we see that the consecutive differences in $I^{(k)}$ are all the same as the consecutive differences in $I$, except $j_t$ is reduced by 1 to $j_t - 1$. Then, since $j_t$ is even, the term $1/(j_t!!)$ in equation (2.1) turns into $1/(j_t-1!!) = (j_t)/(j_t!!)$, which explains the mysterious factor of $(j_t)$ in the numerator of (3.3).

So, substitute these values into (3.1). Substitution yields
\[ \frac{(n)!}{(i_1)! \cdot \prod_{k=1}^{t} (j_k)!!} = \sum_{t=m+1}^{l} X^{n-i_{t+1}} \cdot \frac{(n-1)! \cdot (j_t)}{(i_1)! \cdot \prod_{k=1}^{t} (j_k)!!} \]
\[ + X^{n-i_{t+1}} \cdot \frac{(n-1)!}{(i_1)! \cdot \prod_{k=1}^{t} (j_k)!!} - X^n \cdot \frac{(n-1)!}{(i_1)! \cdot \prod_{k=1}^{t} (j_k)!!} \]
Hence, dividing both sides of the equation by \( \frac{(n-1)!}{(l!) \prod_{i=1}^{l} (j_i)!} \) gives us that it is equivalent to show

\[
(n) = X^{n-i+1} - X^n + \sum_{t=m+1}^{l} X^{n-i+1} \cdot (j_t)
\]

Now, using the telescoping series,

\[
\sum_{t=m+1}^{l} X^{n-i+1} \cdot (j_t) = \sum_{t=m+1}^{l} X^{n-i+1} \cdot (1 - X^{i+1-i})
\]

\[
= \sum_{t=m+1}^{l} (X^{n-i+1} - X^{n-i})
\]

\[
= \sum_{t=m+2}^{l+1} X^{n-i} - \sum_{t=m+1}^{l} X^{n-i}
\]

\[
= X^{n-n} + \sum_{t=m+2}^{l} X^{n-i} - \sum_{t=m+2}^{l} X^{n-i} - X^{n-i_{m+1}}
\]

\[
= 1 - X^{n-i_{m+1}}
\]

Therefore,

\[
\sum_{t=m+1}^{l} X^{n-i+1} \cdot (j_t) + X^{n-i+1} - X^n = 1 - X^{n-i_{m+1}} + X^{n-i_{m+1}} - X^n = (n).
\]

Hence the induction step holds. Note that the base case of \( n = 1, I = \emptyset \) and \( n = 1, I = \{0\} \) are both trivial to compute, which means the base case holds as well. Thus, by induction, the Theorem is proved. \( \square \)

**Remark 3.0.13.** While the above proof is technically correct, it is important to note that the \( f_{n,I} \) were exactly defined to avoid an important boundary case. It may be that \( I = \{i_1, \ldots, i_l\} \) has has \( i_t = i_{t+1} - 1 \). Then, the definition of \( f_{n,I(t+1)}(X) \) says that we remove one instance of the repeated index \( i_t \), so that \( I(t) \) is a set and not a multiset. However, the telescoping sum in the proof ensures that it won’t matter whether we include \( i_t \) or not, and so we are justified in removing the repeated indices in Notation 3.0.11.

### 4. The Combinatorial Induction

We have just shown the \( f_{n,I}(X) \) satisfy a certain recursion, and so the rest of the paper will be devoted to showing that \( \sum_{w \in B_n^I} (-1)^{l(w)} X^{L(w)} \) satisfy this same recursion. It turns out that Theorem 2.3.1 actually follows from the stronger result Theorem 4.0.17. In this section, we will explain how Theorem 4.0.17 implies Theorem 2.3.1, and in the remaining sections we will go on to prove Theorem 4.0.17.

**Notation 4.0.14.** Recall \( B_n^I = \{ w \in B_n | D(w) \subseteq I^c \} \). Let \( (B_n^I)_k \) denote the set of \( w \in B_n^I \), such that \( w(k) = n \) or \( w(k+1) = n \). Let \( (C_n^I)_k = (B_n^I)_k \cap C_n^I \).

**Remark 4.0.15.** Note that in \( (C_n^I)_k \) only one of \( k \equiv n, k+1 \equiv n \) can be true, and so an equivalent way of writing it is

\[
(C_n^I)_k = \begin{cases} \{ w \in C_n^I | w(k) = n \}, & \text{if } k \equiv n \\ \{ w \in C_n^I | w(k+1) = n \}, & \text{if } k \not\equiv n. \end{cases}
\]

**Notation 4.0.16.** Let \( (S_n,I)_k(X) = \sum_{w \in (C_n^I)_k} (-1)^{l(w)} X^{L(w)} \).
Theorem 4.0.17. Suppose $I = \{i_1, \ldots, i_t\}$, and recall that $j_k$ are the consecutive differences of the $i_k$, that is, $j_k = i_{k+1} - i_k$. Let $i_m$ be the greatest element of $I$ such that $j_m$ is odd. Then,
\begin{enumerate}
\item $\sum_{i_k < i_m} (S_{n,I})_{i_k}(X) = 0$,
\item if $i_k = i_m$, then $(S_{n,I})_{i_k}(X) = -X^n \cdot f_{n-1,I(k+1)}(X)$,
\item let $i_{i+1} = n$. For $i_k \in I \cup \{i_{i+1}\}$ with $i_k > i_m$, then $(S_{n,I})_{i_k}(X) = X^{n-i_k} \cdot f_{n-1,I(i+1)}(X)$.
\end{enumerate}

In the rest of this section, we show why Theorem 4.0.17 implies Theorem 2.3.1. We will just be using the recursive formula for the $f_{n,I}(X)$ given by Theorem 3.0.12.

Lemma 4.0.18. We have the equality $S_{n,I}(X) = (S_{n,I})_n(X) + \sum_{k \in I} (S_{n,I})_k(X)$.

Proof. First, for each element $w$, there must be exactly one $i \in [n]$ such that $w(i) = \pm n$. Therefore, $S_{n,I}(X) = \sum_{k \in [n]} (S_{n,I})_k(X)$. However, we know that any element $w \in (B^w_n)_k$ which has $w(k) = n$ must have $k \in D(w)$ (except in the case $k = n$, and each element $w$ which has $w(k) = -n$ must have a descent at $k = n-1 \in D(w)$). Furthermore, for $w \in (C^w_n)_k$, a chessboard element, it can only satisfy $w(k) = \pm n$ for $n \equiv k$. This means that if $i \in I$, with $n - i$ even, then we may have $w(i) = n$, and if $i \in I$ and $n - i$ is odd, then we may have $w(i + 1) = -n$. However, there are columns $i$ for which $w(i) = \pm n$, except for $i = n$. Hence, $S_{n,I}(X) = \sum_{k \in [n]} (S_{n,I})_k(X) = (S_{n,I})_n(X) + \sum_{k \in I} (S_{n,I})_k(X)$.

Theorem 4.0.19. Assuming Theorem 4.0.17, for $n \in \mathbb{N}$ and $I = \{i_1, \ldots, i_t\} \subset [n-1]_0$,
\[ \sum_{w \in B^w_n} (-1)^{l(w)} X^{L(w)} = f_{n,I}(X). \]

Proof. (Proof of main result given Theorem 4.0.17) First, the base case of $n = 1$ is trivial to check, but for completeness, the case $n = 1, I = \emptyset$ was covered in Stanisiski and Voll [3], and the case $n = 1, I = \emptyset$, is easy because the only $w \in B^0_n$ is the identity, $e$, so $S_1,e(X) = 1$ and $f_{1,e}(X) = 1$ by definition.

So, it only remains to prove the induction step. First, by Corollary 2.4.6, it suffices to prove that $S_{n,I}(X)$ satisfies the same recursion as $f_{n,I}(X)$. Using Lemma 4.0.18, we have the equality $S_{n,I}(X) = (S_{n,I})_n(X) + \sum_{i_k \in I} (S_{n,I})_{i_k}(X)$.

We will cover that case that there exists an $m$ for which $j_m$ is odd. The case that there is no $m$ for which $j_m$ is odd is similar, except with the term corresponding to $i_m$ omitted. Using part 1 of Theorem 4.0.17, it follows that for $i_t \in I$ with $i_k < i_m$, we have $\sum_{i_k < i_m} (S_{n,I})_{i_k}(X) = 0$. Combining this result with Lemma 4.0.18 gives us that $S_{n,I}(X) = (S_{n,I})_n(X) + \sum_{i_k \in I} (S_{n,I})_{i_k}(X)$. Next, using part 2 of Theorem 4.0.17,
\[ (S_{n,I})_{i_k}(X) = -X^n \cdot f_{n-1,I(m+1)}(X). \]

And finally, for $i_k > i_m$, we use part 3 of Theorem 4.0.17 to obtain
\[ (S_{n,I})_{i_k}(X) = X^{n-i_k} \cdot f_{n-1,I(k+1)}(X), \]
\[ (S_{n,I})_n(X) = X^{n-n} \cdot f_{n-1,I(t+1)}(X). \]

Therefore,
\[ S_{n,I}(X) = -X^n \cdot f_{n-1,I(m+1)}(X) + \sum_{i_k \in I} f_{n-1,I(k+1)}(X). \]

However, we also know that
\[ f_{n,I}(X) = -X^n \cdot f_{n-1,I(m+1)}(X) + \sum_{i_k \in I} f_{n-1,I(k+1)}(X). \]

from our recurrence relation for $f_{n,I}$ given in Theorem 3.0.12. Therefore, the two polynomials satisfy the same recurrence, which tells us they are equal. Hence, $f_{n,I}(X) = S_{n,I}(X)$, proving the theorem. \qed
5. Swapping

5.1. Swapping Definitions. We now wish to prove Theorem 4.0.17. To do this, we will split up $C_n^I$ based on which $k \in [n]$ has $w(k) = \pm n$. However, before doing so, we will need a crucial lemma, which we call the swapping lemma. The point of the lemma is to create an involution on a certain set of “swappable” elements. Swaps are moves applicable to matrices which either interchange two rows (swap the rows) or change the sign of a row (which is heuristically a sort of degenerate interchanging of rows). This involution will preserve $L$ and change the sign of $w$. Hence, we can use this to cancel the contribution of all “swappable” elements to the sum $S_{n,I}(X)$, and restrict the sum to a much smaller set.

Definition 5.1.1. Let $k$ be a fixed column, and suppose $w \in C_n^I$. Then, a sign swap is a pair $(t, w)$ where $t \in [n]$ satisfies the following properties:

1. $t \equiv 0$
2. $t \leq k$
3. For all rows $s$ above $t$, we have $j(s)$ is right of $k$.

Define $\text{Swap}^k_{\text{sign}}(w) = \{(t, w) | (t, w) \text{ is a sign swap}\}$. When it is clear which element $w$ we are referring to, we will simply refer to the sign swap $(t, w)$ as $(t)$. Furthermore, define the matrix

$$\text{swap}_{(t, w)}(w)_{y, z} = \begin{cases} -w_{y(t), t} & \text{if } y = i(t), z = t \\ w_{y, z} & \text{otherwise} \end{cases}$$

I.e., the difference between $w$ and $\text{swap}_{(t, w)}(w)$ is that we switch the sign of the element in $w$ that lies in column $t$.

Remark 5.1.2. Note that even though $w$ does not explicitly appear in condition (3) above, condition (3) depends on $w$ because the statement “$j(s)$ is right of $k$” really means “$j_w(s) \geq k$.”

Example 5.1.3. Below is an example of a $(2) \in \text{Swap}^3_{\text{sign}}(w)$ sign swap. Note that $(2)$ refers to the column, not the row. It is a sign swap because $2 \equiv 0$ and there are no entries to the left of column 3 which lie above row $i(2) = 2$.

$$w = \begin{pmatrix} t & k \\ -1 & 1 \\ 1 & \end{pmatrix} \quad \text{swap}(2) = \begin{pmatrix} t & k \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Definition 5.1.4. A two column swap is a triple $(b, t, w)$ such that $w \in C_n^I$ and $b < t \in [n]$ are two columns such that

1. $b \equiv t$,
2. For all rows $s$ row between $i(b), i(t)$, we have $j(s)$ is right of $k$.

Define $\text{Swap}_{\text{two}}(w) = \{(b, t, w) | (b, t, w) \text{ is a two swap}\}$. When it is clear which element $w$ we are referring to, we simply refer to the two column swap as the pair $(b, t)$. Now, for any two swap $(b, t, w)$ define the matrix $\text{swap}_{(b, t, w)}(w)$ by

$$\text{swap}_{(b, t, w)}(w)_{y, z} = \begin{cases} w_{i(b), b} & \text{if } y = i(t), z = b \\ w_{i(t), t} & \text{if } y = i(b), z = t \\ 0 & \text{if } y = i(b), z = b \\ 0 & \text{if } y = i(t), z = t \\ w_{y, z} & \text{otherwise} \end{cases}$$
I.e., the difference between $w$ and swap$_{(b,t)}(w)$ is that we switch the rows in which $b,t$ lie.

**Definition 5.1.5.** Let $k \in [n]$ be a column. A **left swap** is a triple $(b,t,w)$, such that $(b,t,w) \in \text{Swap}_\text{two}(w)$ if $t$ is weakly left of $k$ Define $\text{Swap}^k_{\text{left}}(w) = \{(b,t,w)\mid (b,t,w) \text{ is a left swap}\}$.

**Example 5.1.6.** Here is an example of a $(1,3) \in \text{Swap}^4_{\text{left}}(w)$ left swap. Note that $3 - 1 \equiv 0$ and the only row between $i(1) = 3, i(3) = 5$ is 4 and $j(4) = 6$ is to the right of column 4.

$$w = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{swap}_{(1,3)}(w) = \begin{pmatrix} 1 & -1 & 1 \\ \bullet & 1 & \bullet \\ 1 & -1 & 1 \end{pmatrix}$$

**Definition 5.1.7.** Let $k \in [n]$ be a column. A **double-minus swap** is a triple $(b,t,w)$ with $(b,t,w) \in \text{Swap}_\text{two}(w)$ satisfying the following properties:

1. $w_{i(b),b} = w_{i(t),t} = -1$.
2. $k < b < t$
3. For all rows $s$, such that $s$ is row between $i(b), i(t)$, we have $w_{s,j(s)} = 1$.

We define $\text{Swap}^k_{\text{dm}}(w) = \{(b,t,w)\mid (b,t,w) \text{ is a double-minus swap}\}$.

**Example 5.1.8.** Here is an example of a $(2,6) \in \text{Swap}^7_{\text{dm}}(w)$ double-minus swap. Note that $6 - 2 \equiv 0$ and the only rows between $i(2) = 2, i(6) = 6$ are 3, 4, 5 all of which lie to the right of column 1.

$$w = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{swap}_{(2,6)}(w) = \begin{pmatrix} 1 & -1 & 1 \\ \bullet & 1 & \bullet \\ -1 & 1 & \bullet \end{pmatrix}$$

**Definition 5.1.9.** For $k \in [n]$ a column. A **single-minus swap** is a triple $(b,t,w)$ with $(b,t,w) \in \text{Swap}_\text{two}(w)$ if additionally:

1. $w_{i(t),t} = -1$.
2. $b \leq k < t$
3. For all rows $s$ row between $i(b), i(t)$, we have $w_{s,j(s)} = 1$.

We define $\text{Swap}^k_{\text{sm}}(w) = \{(b,t,w)\mid (b,t,w) \text{ is a single-minus swap}\}$. 
Example 5.1.10. Here is an example of a \((1, 3) \in \text{Swap}_{sm}^2(w)\) single-minus swap. Note that \(3 - 1 \equiv 0\) and the rows between \(i(1) = 3, i(3) = 1\) is row 2 which has a 1.

\[
\begin{pmatrix}
1 & -1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

Swap_{(2,6)}(w) =
\[
\begin{pmatrix}
1 & \bullet & 1 \\
\bullet & -1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

Note that if \(w\) had a \(-1\) in the first column, instead of a 1, we would still have \((1, 3) \in \text{Swap}_{sm}^2(w)\).

Definition 5.1.11. For \(k \in [n]\) a column. A double-plus swap is a triple \((b, t, w)\) with \((b, t, w) \in \text{Swap}_{\text{two}}(w)\) such that:

1. \(w_{i(b), b} = w_{i(t), t} = 1\).
2. \(k < b < t\).
3. For all rows \(s\) row between \(i(b), i(t)\), we have \(w_{s,j(s)} = -1\).

We define \(\text{Swap}_{dp}^k(w) = \{(b, t, w) | (b, t, w) \text{ is a double-plus swap}\}\).

Definition 5.1.12. For \(k \in [n]\) a column. A single-plus swap is a triple \((b, t, w)\) such that \((b, t, w) \in \text{Swap}_{\text{two}}(w)\) satisfying the following conditions:

1. \(w_{i(t), t} = 1\).
2. \(b \leq k < t\).
3. For all rows \(s\) row between \(i(b), i(t)\), we have \(w_{s,j(s)} = -1\).

We define \(\text{Swap}_{sp}^k(w) = \{(b, t, w) | (b, t, w) \text{ is a single-plus swap}\}\).

Remark 5.1.13. It is important to note that left swaps, double-plus swaps, double-minus swaps, single-plus swaps, and single-minus swaps will very often be notated as pairs \((b, t)\) instead of triples \((b, t, w)\), as in accordance with Definition 5.1.4.

Remark 5.1.14. As defined above, double plus and double-minus swaps are somewhat reminiscent of a type 1 “odd sandwich in \(w\)” as described in [3, Definition 15], however, we place additional restrictions on the parity of the columns involved, and the relation of the columns to \(k\). Also, sign swaps are reminiscent of a type 2 “odd sandwich” as defined in [3, Definition 15], but again with additional restrictions. In Subsections 5.2 and 5.3, we develop results for swaps, similar to [3, Lemma 19] for odd sandwiches.

Example 5.1.15. Let \(w = [9, 2, 5, 10, 3, 4, 7, -6, 1, -8]\), and when \(w\) is written in permutation matrix notation, we have

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & 1 \\
-1 & 1 & -1
\end{pmatrix}
\]
Observe that $w \in C_{10}$, because $\forall i \in [10], i \equiv w(i)$. Also observe that $D(w) = \{0, 4, 7, 9\}$. We see

- $(2) = \text{Swap}_s^4(w)$,
- $(1, 3) = \text{Swap}_{l}^4(w)$
- $(8, 10) = \text{Swap}_{l}^4(w)$
- $(1, 7, 3, 7) = \text{Swap}_{l}^4(w)$
- $\emptyset = \text{Swap}_{s}^4(w) = \text{Swap}_{s}^4(w)$

As some nonexamples, the following are not swaps:

- $(2, 3) \notin \text{Swap}_{s}^4(w)$, because $2 \neq 3$.
- $(3, 5) \notin \text{Swap}_{s}^4(w)$, because $w_{4,6} = +1$, but you can only have single-plus swaps when the rows between $i(3) = 5, i(5) = 3$ only have $-1$'s.
- $(5, 7) \notin \text{Swap}_{s}^4(w)$ because row 4, is row between $i(5) = 3, i(7) = 5$, but $w_{4,j(4)} = 1$.

**Definition 5.1.16.** Define

$$\text{Swap}_{s}^k(w) = \text{Swap}_{s}^k(w) \cup \text{Swap}_{l}^k(w) \cup \text{Swap}_{s}^k(w) \cup \text{Swap}_{l}^k(w) \cup \text{Swap}_{s}^k(w) \cup \text{Swap}_{l}^k(w).$$

For the rest of this definition we shall use $x$ to stand for an element of $\text{Swap}_{s}^k(w)$. That is, either $x = (t, w)$ or $x = (b, t, w)$. Call any $x \in \text{Swap}_{s}^k(w)$ a swap. If $\text{Swap}_{s}^k(w) \neq \emptyset$, call $w$ a general-plus $k$-swappable. Define

$$\text{Swap}_{s}^k(w) = \text{Swap}_{s}^k(w) \cup \text{Swap}_{l}^k(w).$$

Say $x \in \text{Swap}_{s}^k(w)$ is a general-left $k$-swap. If $\text{Swap}_{s}^k(w) \neq \emptyset$, call $w$ a general-left $k$-swappable.

Define

$$\text{Swap}_{s}^k(w) = \text{Swap}_{s}^k(w) \setminus \text{Swap}_{s}^k(w).$$

Say $x \in \text{Swap}_{s}^k(w)$ is a general-minus swap. If $x \in \text{Swap}_{s}^k(w)$, call $x$ a general-minus $k$-swap. If $\text{Swap}_{s}^k(w) \neq \emptyset$, call $w$ a general-minus $k$-swappable.

Define

$$\text{Swap}_{s}^k(w) = \text{Swap}_{s}^k(w) \setminus \text{Swap}_{s}^k(w).$$

Say $x \in \text{Swap}_{s}^k(w)$ is a general-plus $k$-swap. If $\text{Swap}_{s}^k(w) \neq \emptyset$, call $w$ a general-plus $k$-swappable.

**5.2. The Swapping Lemma.** Now we have finally managed to define all the types of swaps we need. The next steps are to show that swaps preserve descent sets, that swaps preserve $L$, and that swaps change the sign of $w$. We can then use these properties to cancel out certain swappable elements from $S_{n,t}(X)$.

**Lemma 5.2.1.** For any $x \in \text{Swap}_{s}^k(w)$, we have $D(w) \setminus \{k\} = D(\text{swap}_{s}(w)) \setminus \{k\}$. That is, swapping preserves the descent set, except for possibly $k$.

**Proof. The case of a sign swap.**

If $x = (t, t)$, it is clear

$$D(w) \setminus \{t - 1, t\} = D(\text{swap}_{s}(w)) \setminus \{t - 1, t\},$$

because only column $t$ is altered. Since $|w(t-1)| = i(t-1) > i(t) = |w(t)|$, by definition of sign swap,

$$w(t-1) > w(t) \iff (\text{swap}_{s}(w))(t-1) > (\text{swap}_{s}(w))(t).$$

If $t \neq k$, by the same argument,

$$w(t) > w(t+1) \iff (\text{swap}_{s}(w))(t) > (\text{swap}_{s}(w))(t+1),$$

which implies that in this case $D(w) \setminus k = D(\text{swap}_{s}(w)) \setminus k$.

The case of a left swap.
Suppose \((b, t)\) is a left swap. We immediately know that
\[
D(w) \setminus \{b - 1, b, t - 1, t\} = D(\text{swap}_{(b, t)}(w)) \setminus \{b - 1, b, t - 1, t\},
\]
because only columns \(b, t\) are altered. In fact, this statement is obviously true for all two
swaps, and so we will use it in the remaining cases as well. Since \((b, t)\) is a left swap, we
know that for all rows \(r\) which are row between \(i(b), i(t)\), \(r\) lies to the right of \(k\). This means
that none of \(i(t - 1), i(t + 1), i(b - 1), i(b + 1)\) are row between \(i(b), i(t)\) except possibly when
\(t = k\). Therefore,
\[
w(s) > w(s + 1) \iff (\text{swap}_{(b, t)}(w))(s) > (\text{swap}_{(t)}(w))(s + 1),
\]
for \(s = b - 1, b, t - 1,\) and also for \(s = t\) if \(t \neq k\). Therefore, \(D(w) \setminus k = D(\text{swap}_{(b, t)}(w)) \setminus k\).

The case of a single-minus or single-plus swap.

Suppose \((b, t)\) is a single-minus swap. The case of a single-plus swap is analogous with
signs reversed. Once again, we only have to show that the descents at \(b - 1, b, t - 1, t\) are
preserved, except possibly when \(b = k\). By the same argument as in the case of a left swap,
we know this property holds for \(b, b - 1\). In order to see the descent at \(t, t - 1\) is preserved,
note that if \(i(t - 1)\) is row between \(i(t), i(b)\) then \(w_{t - 1, j(t - 1)} = 1\), which means there is an
ascent from \(t - 1\) to \(t\) in \(w\) if and only if there is an ascent from \(t - 1\) to \(t\) in \(\text{swap}_{(b, t)}(w)\).
Similarly, if \(i(t + 1)\) is row between \(i(t), i(b)\), there is a descent from \(t\) to \(t + 1\) in both \(w\) and
\(\text{swap}_{(b, t)}(w)\). Hence, the descent set is preserved if \(i(t + 1), i(t - 1)\) are row between
\(i(t), i(b)\). However, if either of \(i(t + 1), i(t - 1)\) is not row between \(i(t), i(b)\), then the descent
set is preserved, as can be seen using the same argument as in the case of the left swap.
Therefore, \(D(w) \setminus k = D(\text{swap}_{(b, t)}(w)) \setminus k\).

The case of a double-minus or double-plus swap.

The case of a double-minus swap follows the same argument as for column \(t\) of a single
minus swap \((b, t)\). The argument for showing a double-plus swap preserves descent sets is
analogous to that for the double-minus swap, with 1’s and −1’s reversed. □

Lemma 5.2.2. If \(x\) is a swap for \(w\), then \(x\) is still a swap for \(\text{swap}_x(w)\).

Proof. For all types of swaps, it is obvious that the same conditions on the one or two
columns involved in the swap will still be satisfied after the swap is performed. For example,
in the case of a left swap \((b, t, w)\) if all rows which are row between \(i_w(b), i_w(t)\) are right of \(k\),
then \((b, t, \text{swap}_{(b, t, w)}(w))\) will also be a left swap. That is, since we are swapping two rows,
the rows which are row between \(i_w(b), i_w(t)\) before the swap will be the same as the rows
which are row between \(i_{\text{swap}_{(b, t, w)}(w)}(b), i_{\text{swap}_{(b, t, w)}(w)}(t)\) after the swap, and hence they will all
be to the right of \(j\). Completing this proof for the other types of swaps is similar. □

Lemma 5.2.3. (Swapping Lemma) If \(w \in C_n^L\) and \(x \in \text{Swap}^k(w)\), then
\[
\begin{align*}
L(w) &= L(\text{swap}_x(w)) \\
\text{sgn}(w) &= -\text{sgn}(\text{swap}_x(w)).
\end{align*}
\]

Proof. First, it is quite apparent that \(\text{sgn}(w) = -\text{sgn}(\text{swap}_x(w))\). This is because swapping
two rows changes the determinant of the signed permutation matrix representing \(w\) by a
factor of \(-1\). Also, multiplying a row by \(-1\) changes the sign of the determinant by a
factor of \(-1\). Since \(\text{sgn}(w)\) is the sign of the determinant of the signed permutation matrix
therefore, \(\text{sgn}(w) = -\text{sgn}(\text{swap}_x(w))\).

So, we only have to check that \(L\) is invariant. While it is possible to do this using the
definition of \(L\) directly, it’s a little easier to see it using the abc Lemma 2.4.9. In order to
see how \(L\) changes, it suffices to check how \(a, b, c\) changes.

There are now two cases to verify, depending on what type of swap \(x\) is. First, there is
the case of a sign swap, second the case of a left swap, double-minus, single-minus, double-plus,
or single-plus swap.

The case of a sign swap.
Let the sign swap be \((t)\). In this case, \(a(w)\) changes by 0, because we are assuming that \(t\) is an even column, and \(a\) only counts the number of \(-1\)'s in odd columns. Next, \(b\) does not change at all, because the rows less than \(t\) do not change. And finally, \(c\) does not change because by assumption there are no columns \(j \in [n] \) \(j < t\), \(i(j) < i(t), j \equiv t\). Therefore,

\[
a(w) = a(\text{swap}_{(t)}(w)), b(w) = b(\text{swap}_{(t)}(w)), c(w) = c(\text{swap}_{(t)}(w))
\]

are all preserved, it follows that \(L(w) = L(\text{swap}_{(t)}(w))\). Hence, we have completed the case of a sign swap.

### The case of a left, double-minus, single-minus, double-plus, or single-plus swap.

Let the swap be \((\beta, \tau)\).\(^1\) We will show a slightly stronger statement. Namely, suppose \((\beta, \tau)\) is a two swap, such that for all rows \(s\) which are row between \(i(\beta), i(\tau)\) and column between \(\beta, \tau\), we have \(w_{s,j(i(s))} \neq w_{rt(j(\tau))}\). For such two swaps, \(L(w) = L(\text{swap}_{(\beta, \tau)}(w))\). By definition, all five types of two swaps satisfy this property. For instance, left swaps have no rows which are row between \(i(\beta), i(\tau)\) and column between \(\beta, \tau\).

Of course \(a(w) = a(\text{swap}_{(\beta, \tau)}(w))\) because we do not change the signs of any of the columns. Using the abc Lemma, to complete the proof, it suffices to show \(b(w) + 2c(w) = b(\text{swap}_{(\beta, \tau)}(w)) + 2c(\text{swap}_{(\beta, \tau)}(w))\).

In order to prove this, first observe that if \(\{j, j'\} \cap \{\tau, \beta\} = \emptyset\) then

\[
b_{j,j'}(w) = b_{j,j'}(\text{swap}_{(\beta, \tau)}(w)), c_{j,j'}(w) = c_{j,j'}(\text{swap}_{(\beta, \tau)}(w)).
\]

The above equations are true because both \(i(j), i(j')\) are unchanged by the swap.

Additionally, for any column \(j\), let

\[
L_{\beta, \tau, j}(w) = b_{j,\beta}(w) + b_{j,\tau}(w) + b_{\beta,j}(w) + b_{\tau,j}(w) + 2c_{j,\beta}(w) + 2c_{j,\tau}(w) + 2c_{\beta,j}(w) + 2c_{\tau,j}(w).
\]

Then, we claim

\[
L_{\beta, \tau, j}(w) = L_{\beta, \tau, j}(\text{swap}_{(\beta, \tau)}(w)) \tag{5.1}
\]

Let us now see why Equation (5.1) completes the lemma, and then we shall prove Equation (5.1) holds. Assuming Equation (5.1),

\[
b(w) + 2c(w) = \sum_{j=1}^{n} \sum_{j'=1}^{n} (b_{j,j'}(w) + 2c_{j,j'}(w))
\]

\[= \sum_{\{j,j'\} \cap \{\tau, \beta\} = \emptyset} (b_{j,j'}(w) + 2c_{j,j'}(w)) + \sum_{j=1}^{n} L_{\beta, \tau, j}(w)
\]

\[= \sum_{\{j,j'\} \cap \{\tau, \beta\} = \emptyset} (b_{j,j'}(\text{swap}_{(\beta, \tau)}(w)) + 2c_{j,j'}(\text{swap}_{(\beta, \tau)}(w)))
\]

\[+ \sum_{j=1}^{n} L_{\beta, \tau, j}(\text{swap}_{(\beta, \tau)}(w))
\]

\[= \sum_{j=1}^{n} \sum_{j'=1}^{n} (b_{j,j'}(\text{swap}_{(\beta, \tau)}(w)) + 2c_{j,j'}(\text{swap}_{(\beta, \tau)}(w)))
\]

\[= b(\text{swap}_{(\beta, \tau)}(w)) + 2c(\text{swap}_{(\beta, \tau)}(w)).
\]

So, to complete the proof, we only need show (5.1) holds. There are now three cases, depending on where \(i(\beta)\) lies relative \(i(\beta), i(\tau)\).

First, the case that \(i(\beta)\) is below both \(i(\beta), i(\tau)\). Then, in fact all terms in \(L_{\beta, \tau, j}(w)\) are preserved after swapping. That is, \(b_{j,\beta}(w) = b_{j,\beta}(\text{swap}_{(\beta, \tau)}(w))\), and similarly for the other seven terms. Hence, Equation (5.1) holds in this case.

The case that \(i(\beta)\) is above both \(i(\beta), i(\tau)\) is similar to the case that it is below both.

---

\(^1\)For the rest of this proof only, use \((\beta, \tau)\) instead of \((b, t)\), because we wish to reserve \(b\) for the function \(b\) as defined in Definition 2.4.7 and we don’t want to use the confusing subscript \(b_b\).
So, we only need consider the remaining case that $i(j)$ is row between $i(\beta), i(\tau)$. There are now three further subcases of this, depending on where $j$ is in relation to $\beta, \tau$.

If $j$ is right of $\beta$ and right of $\tau$, then we see

$$L_{\beta, \tau, j}(w) = b_{\beta, j}(w) + b_{\tau, j}(w) + 2c_{\beta, j}(w) + 2c_{\tau, j}(w),$$

as the other four terms are 0. Now, for definiteness, assume $i(\beta) < i(\tau)$, as the other case is analogous. Then,

$$b_{\beta, j}(w) = 0, b_{\tau, j}(w) = 1, b_{\beta, j}(\text{swap}_{\beta, \tau}(w)) = 1, b_{\tau, j}(\text{swap}_{\beta, \tau}(w)) = 0.$$

Therefore,

$$b_{\beta, j}(w) + b_{\tau, j}(w) = b_{\beta, j}(\text{swap}_{\beta, \tau}(w)) + b_{\tau, j}(\text{swap}_{\beta, \tau}(w)).$$

Analogously,

$$c_{\beta, j}(w) + c_{\tau, j}(w) = c_{\beta, j}(\text{swap}_{\beta, \tau}(w)) + c_{\tau, j}(\text{swap}_{\beta, \tau}(w)),$$

and so $L_{\beta, \tau, j}(w) = L_{\beta, \tau, j}(\text{swap}_{\beta, \tau}(w)).$

Next, the subcase in which $j$ is left of both $\beta$ and $\tau$ is analogous to the the subcase that $j$ is right of both $\beta$ and $\tau$.

Hence the final subcase remaining is when $j$ is column between $\beta, \tau$ and $i(j)$ is row between $i(\beta), i(\tau)$. In this subcase,

$$L_{\beta, \tau, j}(w) = b_{\beta, j}(w) + b_{\beta, j}(w) + 2c_{\beta, j}(w) + 2c_{\beta, j}(w),$$

and one can easily compute that

$$L_{\beta, \tau, j}(w) = 2 = L_{\beta, \tau, j}(\text{swap}_{\beta, \tau}(w)),$$

using the fact that $w_{i(j), j} \neq w_{i(\tau), j}$.

Hence, in all cases, $L_{\beta, \tau, j}(w) = L_{\beta, \tau, j}(\text{swap}_{\beta, \tau}(w))$, completing the proof. \hfill \Box

5.3. The Swapping Involution. We have now shown that swaps interact nicely with $L$ and $l$. These properties will allow us to use swaps to cancel out many terms in $S_{n, l}(X)$. However, in order to do so, we will carefully define an involution on the set of $k$-swappable elements.

**Definition 5.3.1.** Let $w \in C^k_n$ and define a total ordering, which we call the **swapping ordering** on $\text{Swap}^k_m(w)$, $\text{Swap}^k_m(w)$, and $\text{Swap}^k_m(w)$ associated with $w$ as follows:

- If $(p), (t)$ are both sign swaps, then $(p) < (t)$ if $i(p) < i(t)$.
- If $(p)$ is a sign swap, and $(b, t)$ is a two swap, then $(p) < (b, t)$.
- If $(p, q), (b, t)$ are both two swaps, then $(p, q) < (b, t)$ if either
  - $\min(i(p), i(q)) < \min(i(b), i(t))$
  - $\min(i(p), i(t)) = \min(i(b), i(t))$ and $\max(i(p), i(q)) < \max(i(b), i(t))$.

**Lemma 5.3.2.** There exists an involution $\text{swap}^l : C^k_n \to C^k_n$, (respectively $\text{swap}^r : C^k_n \to C^k_n$, and $\text{swap}^m : C^k_n \to C^k_n$) given by $w \mapsto \text{swap}_x(w)$, where $x$ is the lowest possible general-left $k$-swap (respectively general-plus $k$-swap, general-minus $k$-swap) that can be made according to the swapping ordering defined in Definition 5.3.1. If there are no swaps, then $\text{swap}^l$ (respectively $\text{swap}^m, \text{swap}^r$) acts as the identity. This involution satisfies

$$L(\text{swap}^l(w)) = L(w),$$

$$\text{sgn}(w) = -\text{sgn}(\text{swap}^l(w))$$

(respectively

$$L(\text{swap}^r(w)) = L(w),$$

$$\text{sgn}(w) = -\text{sgn}(\text{swap}^r(w)),$$

$$L(\text{swap}^m(w)) = L(w),$$

$$\text{sgn}(w) = -\text{sgn}(\text{swap}^m(w)).$$
Proof. Let us just prove the statement for the case of Swap\(^k_m\)(w) since Swap\(^k_x\)(w) is exactly analogous, and Swap\(^k_y\)(w) simply follows from restricting the case of Swap\(^k_m\)(w).

The facts that \(L(\text{swap}^m(w)) = L(w)\) and \(\text{sgn}(w) = -\text{sgn}(\text{swap}^m(w))\) are immediate from the Swapping Lemma. So, it suffices to show that \(\text{swap}^m\) as defined is indeed an involution. That is, we need to show that if \(\text{swap}^m(w) = \text{swap}_x(w)\), then \(\text{swap}^m(\text{swap}_x(w)) = \text{swap}_x(\text{swap}_x(w)) = w\). Of course, we know that once we apply a swap, we are able to apply it again, so we just have to show that making a swap does not allow us to make any lesser swaps. We now show that if \(x\) is the least swap of \(w\), and \(y\) is the least swap of \(\text{swap}_x(w)\), then \(x = y\). We do this by ruling out the three cases listed below.

The case \(x = (t)\) is a sign swap, and \(y = (p)\) is a sign swap with \((p) < (t)\).

By definition of sign swap, the only sign swap that can be made to a matrix \(w\) is \((t)\) where \(t\) to the left of \(k\) and \(i(t)\) is minimal. It is clear that \(\text{swap}_x((t))\) will still have \(t\) as the column to the left of \(k\) with \(i(t)\) minimal, and so no lesser sign swap can be made.

The case \(x = (b, t)\) is a two swap, and \(y = (p)\) is a sign swap with.

If \((b, t)\) is a double-minus or double-plus swap, \(\text{swap}_{(b, t)}(w)\) cannot have any sign swaps \((p)\), since \(w\) had no sign swaps \((p)\), and interchanging two rows to the right of \(k\) cannot create the existence of a sign swap.

Suppose instead that \((b, t)\) is a left, single-minus or single-plus swap. First, we must have either \((p) = (b)\) or \((p) = (t)\), since there are no columns \(s\) such that \(i(s)\) is row between \(i(t)\) and weakly left of \(k\). If \(i_w(b) > i_w(t)\), we must have \((p) = (t)\), but this would then imply that \((b) \in \text{Swap}_x^{k_m}(w)\), contradicting the assumption that \((b, t)\) was the least swap on \(w\).

So, the only remaining case is when \(i_w(b) < i_w(t)\), and \((p) = (b)\). Observe that there are no rows \(s\) with \(s < \min(i(b), i(t))\) so that \(s\) lies to the left of \(k\). Ergo, if \((b)\) is a sign swap on \(\text{swap}_{(b, t)}(w)\), then \(b = 0\) and there must be no column \(s < b\) with \(s \equiv 1\), and \(i(s) < \min(i(b), i(t))\). But these two observations combined imply that \((t) \in \text{Swap}_x^{k_m}(w)\), contradicting the assumption that \((b, t)\) was minimal in \(\text{Swap}_x^{k_m}(w)\).

The case \(x = (b, t)\) is a two swap, and \(y = (p, q)\) is a two swap with \((p, q) < (b, t)\).

If \((b, t)\) is a double-minus swap, \(\text{swap}_{(b, t)}(w)\) is a left swap, then such a situation is clearly impossible, since the set of rows and signs of the rows to the left of \(k\) and to the right of \(k\) in \(\text{swap}_{(b, t)}(w)\) and \(w\) are the same, and hence we can do the same set of swaps for both \(\text{swap}_{(b, t)}(w)\) and \(w\).

So, the only remaining case is when \((b, t)\) is a single-minus swap, or single-plus swap. The two are similar, so we’ll just prove it when \((b, t)\) is a single-minus swap. If \(i(b) < i(t)\), then if there is a higher swap \((p, q)\), on \(\text{swap}_{(b, t)}(w)\), the only possibility is that one of \(p, q\) is equal to \(t\), and either \((p, q)\) is a single-minus swap or a double-minus swap. For concreteness, let us say \(q = t\), the other case is the same with \(p, q\) reversed. If \((p, q)\) is a single-minus swap, then one of \((p, b)\) or \((b, p)\) would be a left swap on \(w\), which is less than \((b, t)\). If instead, \((p, t)\) is a double-minus swap, then one of \((b, p)\) or \((p, b)\) would have been a single-minus swap on \(w\). Both of these cases contradict the assumption that \((b, t)\) was the least swap for \(w\). The same argument goes through in the case \(i(b) > i(t)\), with \(b, t\) reversed. \(\square\)

6. Part 1 of the Theorem

Now that the theory of swapping is complete, we are able to easily prove part 1 of 4.0.17. We can use the involution from Lemma 5.3.2 to cancel off all the elements in part 1 of Theorem 4.0.17.

Notation 6.0.3. For Section 6 and Section 7, reserve the use of column \(i_m\) for the column such that \(j_m\) is odd, but \(j_s\) is even for all \(s > m\), assuming such an \(i_m\) exists.

Definition 6.0.4. For \(j\) a fixed column, and for rows \(r, s \in [n]\) with \(r < s\) call the set of rows

\[B^j_{r,s} = \{t | r \leq t \leq s\}\]
a block if it satisfies the following conditions:
- For all \( t \in B_{r,s} \), we have that \( j(t) > j \).
- If \( s + 1 \neq n \), then \( j(s + 1) \leq j \).
- If \( r - 1 \neq 0 \), then \( j(r - 1) \leq j \).

If \( B_{r,s} \) is a block with \( r \neq s \), call \( B_{r,s} \) an even block, since it contains an even number of rows. If instead \( r \equiv s \), call \( B_{r,s} \) an odd block.

Remark 6.0.5. When it is clear which column \( j \) we are using, we often write \( B_{r,s} \) for \( B_{j,r,s}^{j} \). In particular, in Section 8, we have a fixed column \( j \) with \( j \in I, j > i_m \) throughout the section, and all blocks will be taken with \( B_{r,s} = B_{j,r,s}^{j} \), for the particular fixed \( j \) in that section. In this section, we normally take \( j = i_m \), but still use the notation \( B_{r,s}^{i_m} \) for clarity.

Lemma 6.0.6. For all \( w \in (S_{n,t})_{n}(X) \) such that \( i_b < i_m \), then there exists an odd block \( B_{r,s}^{i_m} \).

Proof. By assumption, \( n - i_m + 1 \) is odd. Therefore, let \( B_{l_1,u_1}^{i_m}, B_{l_2,u_2}^{i_m}, \ldots, B_{l_k,u_k}^{i_m} \) be the collection of all blocks with respect to the column \( i_m \). Notice that we cannot have any \( k \) such that \( l_k = n \) ever because \( i(i_u) = n \). Since the size of block \( p \) is \( u_p - l_p + 1 \), then the total size of all the blocks is \( \sum_{p=1}^{k} (u_p - l_p + 1) = n - i_m \). Furthermore, since \( n - i_m \) is odd, there exists a \( p \) for which \( u_p - l_p + 1 \) is odd, and so there is some odd block.

Lemma 6.0.7. If \( w \in (C_n)_{n} \) has an odd block \( B_{r,s}^{i_m} \), with \( k < i_m \), then \( w \) is general-left \( k \)-swappable with \( x \) in \( \text{Swap}_{l}^{i_m}(w) \), and \( \text{swap}_x(w) \in (C_n)_{h} \) for \( h < i_m \).

Proof. Let \( B_{r,s}^{i_m} \) be the odd block. If \( r \neq 1 \), we can apply a left swap \((r - 1, s + 1)\), and if \( r = 1 \), we can apply the sign swap \((s + 1)\). Therefore, \( w \) is left \( k \)-swappable. Clearly \( \text{swap}_x(w) \in (C_n)_{h} \) with \( h < i_m \) since the left swaps here only involve elements to the left of \( i_m \).

Lemma 6.0.8. For \( w \in B_{n}^{i_m} \) there are no odd blocks \( B_{r,s}^{i_m} \) if and only if \( \text{Swap}_{r}^{k}(B_{n}^{i_m}) = \emptyset \).

Proof. Assume all blocks are even blocks. Let \( a, b \in [n] \) be two rows such that \( j(a), j(b) < k \) and there are no rows to the left of \( k \), which are row between \( a, b \). Then, we must have \( a - b \equiv 1 \), since they are either adjacent, or separated by an even number of rows. This implies \( w \) has no left swaps. Furthermore, the topmost row to the left of \( k \) must be odd, which means we cannot have a sign swap. Hence, there are no general-left \( k \)-swaps. Conversely, if \( w \) has an odd block, it is general-left \( k \)-swappable by Lemma 6.0.7.

Corollary 6.0.9. Part 1 of Theorem 4.0.17 holds.

Proof. Using Lemma 6.0.6, for any \( i_b < i_m, w \in (S_{n,t})_{i_b} \) then \( w \) has some odd block \( B_{r,s}^{i_m} \). But then, by Lemma 6.0.8, all such elements have some general-left \( k \)-swap, and so by Lemma 5.3.2 for the case of \( \text{Swap}_{l}^{i_m}(w) \), there exists a sign reversing involution on all such elements. Observe that this involution from Lemma 5.3.2 preserves the set \( \{ w \in (C_n)_{i_b} \mid w \in (S_{n,t})_{i_b} \text{ with } i_b < i_m \} \) by the last part of Lemma 6.0.7. Therefore, we can safely cancel all elements with \( w(i_b) = \pm n \), where \( i_b \) is left of \( i_m \), and hence part 1 of Theorem 4.0.17 holds.

7. Part 3 Implies Part 2

In this section, we present a fairly straightforward bijection, given by flipping the sign of the entry in the bottom row. This will show part 2 of Theorem 4.0.17 follows from part 3 of Theorem 4.0.17. After this section, we work on proving part 3.

Lemma 7.0.10. If \( w \in (C_n)_{i_b} \) with \( i_m - n \) odd, then the element \( u \) given by

\[
u(i) = \begin{cases} -w(i), & \text{if } i = i_m + 1 \\ w(i), & \text{otherwise} \end{cases}
\]

satisfies \( L(u) + i_m + 1 = L(w) \).
Proof. Note that since $i_m - n$ is odd, we necessarily have $w(i_m + 1) = -n$. So, we use the abc lemma to compute $L$. Since $w(i_m + 1) = -n$, $i_w(i_m + 1) = i_u(i_m + 1) = n$, and so $b_{i_m+1}(w) = b_{i_m+1}(u)$.

Now we have two cases: either $n$ is even or $n$ is odd. We just check the even case, since the odd case is similar. If $n$ is even, then $i_m + 1$ is even, which means there $i_m/2$ odd columns less than $i_m + 1$. This tells us that $c_{i_m+1}(w) = \frac{i_m+1}{2}$, $c_{i_m+1}(u) = 0$. Also, since $i_m + 1$ is even, $a_{i_m+1}(w) = a_{i_m+1}(u)$. Additionally, none of the $a_j, b_j, c_j$ statistics change when $j \neq i_m + 1$. This tells us that

$$L(w) - L(u) = a(w) - a(u) + b(w) - b(u) + 2c(w) - 2c(u)$$
$$= a_{i_m+1}(w) - a_{i_m+1}(u) + b_{i_m+1}(w) - b_{i_m+1}(u) + 2c_{i_m+1}(w) - 2c_{i_m+1}(u)$$
$$= 0 + 0 + 2 \frac{i_m+1}{2}$$
$$= i_m + 1.$$

Hence, $L(u) + i_m + 1 = L(w)$. \hfill \Box

Lemma 7.0.11. Assuming part 3 of the Theorem 4.0.17, part 2 holds.

Proof. Let $I = \{i_1, \ldots, i_l\}$, and let $J = \{i_1, \ldots, i_{m-1}, i_m + 1, i_{m+1}, \ldots, i_l\}$. That is, shift the index $m$ up to $m + 1$. We show there is a bijection between elements of $(C_n^{i_l'})_{i_m}$ and $(C_n^{j_l'})_{i_m+1}$ given by switching the sign of $w(i_m + 1)$. For $w \in (C_n^{i_l'})_{i_m}$, since $w$ is assumed to be a chessboard element, and $n - i_m$ is odd, we must have a descent from column $i_m$ to $i_m + 1$. So we must make $w(i_m + 1) = -n$. In the case of $w \in (C_n^{j_l'})_{i_m+1}$, by analogous reasoning, we must have $w(i_m + 1) = n$. Therefore, the maps $f : (C_n^{i_l'})_{i_m} \to (C_n^{j_l'})_{i_m+1}$, $g : (C_n^{j_l'})_{i_m+1} \to (C_n^{i_l'})_{i_m}$, both defined by switching the sign of $w(i_m + 1)$ define mutual inverses as maps of sets.

Now, since $l$ reverses sign under this involution. We also saw from Lemma 7.0.10 that $L$ decreases by $i_m + 1$. Thus,

$$\sum_{w \in (C_n^{i_l'})_{i_m}} (-1)^{(w)} X^L(w) = \sum_{w \in (C_n^{j_l'})_{i_m}} (-1)^{(g(w))} X^L(g(w)) + i_m + 1$$
$$= \sum_{u \in (C_n^{j_l'})_{i_m+1}} (-1)^{(u)} X^L(u) + i_m + 1$$
$$= -X^{i_m+1} \sum_{u \in (C_n^{j_l'})_{i_m+1}} (-1)^{(u)} X^L(u)$$
$$= -X^{i_m+1} X^{n-(i_m-1)} f_{n,J(m)}$$
$$= -X^n f_{n,J(m)}$$
$$= -X^n f_{n,J(m+1)}$$

where the equality from the third line to the fourth line crucially uses part 3 of Theorem 4.0.17. The equality of the penultimate line to the ultimate line uses the fact that there is an equality of sets $J(m) = I^{(m+1)}$. \hfill \Box

8. Part 3 of the Theorem

We now begin the arduous task of proving part 3 of Theorem 4.0.17. The loose outline is as follows. We will define a set of elements called the initially uncanceled elements, which satisfy 7 mysterious, seemingly complicated properties. However, it will turn out that a matrix $w$ satisfies these seven properties if and only if $w$ has no general-minus swaps. We will analogously define a set of elements called finally uncanceled elements, which again satisfy seven properties. This time, these seven properties will be equivalent to $w$ having no general-plus swaps. We will then give an algorithm which takes initially uncanceled elements $w \in B_n$ bijectively to finally uncanceled elements $w \in B_{n-1}$. We will check that this algorithm preserves $\text{sgn}(w)$ and alters $L(w)$ in exactly the way we want so that part 3
of Theorem 4.0.17 holds. We now turn to the proof, the first step of which is defining the initially uncancelled elements.

**Notation 8.0.12.** For the rest of this paper, we shall assume that all elements \( w \in C_n^j \) lie in \( \{ C_n^j \} \), where \( j = t_r \in I = \{ i_1, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_l \} \) is an element so that \( j_k \equiv 0 \) for \( k \geq r \). Observe that we necessarily have \( w_{i(j), j} = 1 \). If, instead, \( w_{i(j), j} = -1 \), there would be a descent at \( j + 1 \), contradicting the assumption that \( i_{r+1} - j = i_{r+1} - i_r = j_r = \equiv 0 \).

### 8.1. Initially Uncanceled Elements

Before defining initially uncancelled elements, we first need some terminology.

**Definition 8.1.1.** For any of the four possible values \((\pm 1, \pm 1)\), call the elements in columns \( t, t + 1 \) a \((\pm 1, \pm 1)\) \( k \) pair (with respect to \( k \)) if:
- \((\pm 1, \pm 1) = (w_{i(t), t}, w_{i(t+1), t+1})\)
- \( t \neq n \)
- \( t > k \)

**Notation 8.1.2.** We shall often simply write \((\pm 1, \pm 1)\) when it is understood to be a \((\pm 1, \pm 1)\) \( k \) pair with respect to \( j \), where \( j \) is the fixed column from Notation 8.0.12.

**Remark 8.1.3.** Note that we cannot have any \((1, -1)\) \( t \) pairs, as there are no descents at columns \( t \) to the right of \( j \) with \( t \neq n \). Hence, the only possibilities are \((1, 1)\), \((-1, -1)\) and \((-1, 1)\) \( t \) pairs.

**Definition 8.1.4.** For rows \( s, r \in [n] \) with \( r < s \), call the set of rows \( P_{r,s}^k = \{ t \mid r \leq t \leq s \} \) a block of ones if for all \( t \in P_{r,s}^k \), we have that \( j(t) \) lies to the right of \( k \) and \( w_{i(t), j(t)} = 1 \). Furthermore, if \( s + 1 \neq n \), then either \( j(t+1) \) is weakly left of \( k \) or \( w_{s+1, j(s+1)} = -1 \). Also, if \( r - 1 \neq 0 \), then either \( j(r-1) \) is weakly left of \( k \), or \( w_{r-1, j(r-1)} = -1 \). In the case \( r \neq s \), \( P_{r,s}^k \) is called an even block of ones, and if \( r \equiv s \), say \( P_{r,s}^k \) is an odd block of ones.

**Definition 8.1.5.** For rows \( s, r \in [n] \) with \( r < s \), call the set of rows \( N_{r,s}^k = \{ t \mid r \leq t \leq s \} \) a block of minus ones if for all \( t \in N_{r,s}^k \), we have that \( j(t) \) lies to the right of \( k \) and \( w_{i(t), j(t)} = -1 \). Furthermore, if \( s + 1 \neq n \), then either \( j(t+1) \) is weakly left of \( k \) or \( w_{s+1, j(s+1)} = 1 \). Also, if \( r - 1 \neq 0 \), then either \( j(r-1) \) is weakly left of \( k \), or \( w_{r-1, j(r-1)} = 1 \). Call a block of minus ones an even block of minus ones if \( r \neq s \). Call a block of minus ones an odd block of minus ones if \( r \equiv s \).

**Remark 8.1.6.** In the above notation, \( P \) is for positive and \( N \) is for negative. Blocks of ones are just subsets of blocks which have are maximal sets of adjacent rows with all 1’s, while blocks of minus ones are subsets of blocks which are maximal sets of adjacent rows with all \(-1\)’s.

**Remark 8.1.7.** For the rest of this paper, we shall omit the superscript and write \( N_{r,s}^k, P_{r,s}, B_{r,s} \) when we mean \( N_{r,s}, P_{r,s}, B_{r,s} \). That is, when we omit the superscript, we are assuming that blocks, blocks of minus ones, and blocks of plus ones are taken with respect to the column \( j \), where \( j \) is the fixed column as defined in Notation 8.0.12.

**Definition 8.1.8.** An element \( w \in (C_n^j) \) is initially uncancelled (with respect to \( j \)) if its signed permutation matrix satisfies the following properties:

1. Either \( j(n-1) \) is left of \( j \) or \( w_{n-1, j(n-1)} = -1 \).
2. \( w \) has no odd blocks.
3. Whenever \( t \) is a row for which \( w_{i(t), j(t)} = 1 \) and \( t \) lies to the right of \( j \), then there exists an even block of ones \( P_{r,s} \) with \( r \leq t \leq s \), except possibly if for all \( k < t \), \( k \) lies to the right of \( j \) and \( w_{k,j(k)} = 1 \).
4. If \( n \) is even, no \((-1, 1)_t \) pairs exist. If \( n \) is odd, there can be at most one \((-1, 1)_t \) pair, and if such a pair exists, then \( i(t+1) = 1 \). For all rows \( s \) above \( i(t) \), we have \( w_{s,j(s)} = 1 \) and \( j(s) \) is right of \( j \).
5. For any \((-1, -1)_t \) pair, any row \( r \) which is row between \( i(t), i(t+1) \), must satisfy \( w_{r,j(r)} = 1 \), and \( j(r) \) is to the right of \( j \).
(6) For any block of minus ones \( N_{p,q} \subset B_{r,s}, N_{p,q} \neq B_{r,s} \), then \( N_{p,q} \) is an odd block of minus ones if and only if \( p = r \) or \( q = s \).

(7) For \( B_{r,s} \) an even block, and \( P_{r,q} \) (respectively \( P_{p,s} \)) a block of ones, we must have \( q = s \) (respectively \( p = r \)).

We notate \( w \in (U^r_n)_j \) if \( w \) is initially uncanceled with respect to \( j \).

Example 8.1.9. Take

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
1 & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\]

Here, \( w \) is an example of an initially uncanceled matrix in \((U_{10}^{(2,6)})^c_2\). Note \( D(w) = \{2, 6\}\).

Property 1 holds because \( w_{0,3} = -1 \). Property 2 holds because the blocks \( B_{1,4} \) and \( B_{8,9} \) are both even blocks. Property 3 holds because rows 7, 8 determine an even block of ones \( P_{5,8} \), and rows 1, 2, 3, 4 also determine an even block of ones \( P_{1,4} \). Property 4 is satisfied because there are no \((-1, 1)_i\) pairs. Here, \( j = 2 \). Property 5 holds because columns 3, 4 are to the right of 2, and the only rows between \( i(3) = 9 \) and \( i(4) = 6 \) are rows 7, 8 which both contain \(+1's\) to the right of 2. Property 6 holds because the only times \( N_{p,q} \subset B_{r,s}, N_{p,q} \neq B_{r,s} \) are when \((r,s) = (6,9)\) and either \( p = q = 6 \) or \( p = q = 9 \), but in both cases \( N_{p,q} \) is an odd block of minus ones. Finally, property 7 is satisfied because in \( B_{1,4} \) there are no rows with a \(-1\), and in \( B_{8,9} \), the top and bottom rows are \(-1's\).

8.2. Equivalence of Initially Uncanceled and no General-Minus \( j \)-Swaps. In this subsection, we will show that a matrix satisfies the above seven properties if and only if it has no general-minus \( j \)-swaps. The if direction is much harder, and we will proceed to verify each of the seven properties, one at a time, assuming \( w \) has no general-minus \( j \)-swaps. To this aim, from this point, until Subsubsection 8.2.27, we shall assume that \( w \in (C^r_n)_j \) has no general-minus swaps.

8.2.1. Property 1.

Lemma 8.2.2. We can never have both that column \( j(n-1) \) lies to the right of \( j \) and \( w_{n-1,j(n-1)} = 1 \).

Proof. Suppose it did. Since \( j(n-1) \neq n \), there cannot be a descent from \( j(n-1) \) to \( j(n-1) + 1 \). Hence, the only possibility is that there is a 1 in column \( j(n-1) + 1 \). But, this is impossible because \( i(j) = n \) by assumption, and \( j < j(n-1) \) by assumption. \( \square \)

8.2.3. Property 2.

Lemma 8.2.4. Any \( w \) with no general-minus \( j \)-swaps cannot have any odd blocks. Furthermore, the least swap of \( w \) under the swapping order Definition 5.3.1 cannot be a left swap of the form \((b, j)\) or sign swap of the form \((j)\).

Proof. By Lemma 6.0.7, \( w \) cannot have any odd blocks \( B_{r,s} \). Since there is necessarily an even block above \( j \), it follows from Lemma 6.0.8 that \( w \) cannot have any left swaps of the form \((b, j)\) or sign swaps of the form \((j)\). \( \square \)

Lemma 8.2.5. For \( w \) a general-minus \( j \)-swappable element, the involution defined in Lemma 5.3.2 \( \text{swap}^m : C^r_n \to C^r_n \) restricts to a map \( \text{swap}^m : (C^r_n)_j \to (C^r_n)_j \).
Proposition 8.2.11. If such an odd block of minus ones existed, we could perform a double-plus swap \(8.2.10\) \(8.2.9\).

Proof. Suppose there is an odd block of ones \(8.2.7\). Of course, \(s \neq n - 1\), by Lemma 8.2.2.

Remark 8.2.6. For the remainder of this section, we implicitly use the above Lemma 8.2.5 through most of this section, noting that there is a restricted involution on the general left \(j\)-swappable elements such that \(w(j) = (\text{swap}^m(w))(j) = n\).

8.2.7. Property 3.

Lemma 8.2.8. Any matrix \(w\) with no general-minus \(j\)-swaps has no odd blocks of ones \(P_{r,s}\), except possibly if \(r = 1\), that is, if the block of ones goes through the top row.

Proof. If \(s \neq n - 1\), by Lemma 8.2.2.

8.2.9. Some Important Intermediate Lemmas.

Lemma 8.2.10. If \(B_{r,s}\) is an even block, then there cannot be an odd block of minus ones \(N_{p,q}\) with \(r < p < q < s\).

Proof. If such an odd block of minus ones existed, we could perform a double-plus swap \((p - 1, q + 1)\).

Notation 8.2.11. Let

\[
s_e = \{k \in [n]|k > j, k \equiv 0, w_{i(k),k} = -1\},
\]

\[
s_o = \{k \in [n]|k > j, k \equiv 1, w_{i(k),k} = -1\}.
\]

Also, for each \(k \in [n]\) with \(k > j, w_{j(k),k} = -1\), let

\[
up_k = |\{t \in [j - 1]|i(t) < i(k)\}|.
\]

Then define

\[
u_e = \{k \in [n]|k > j, w_{i(k),k} = -1, |up_k| \equiv 0\},
\]

\[
u_o = \{k \in [n]|k > j, w_{i(k),k} = -1, |up_k| \equiv 1\}.
\]

For \(a, b \in \{o, e\}\) let \(u_{a} s_{b} = u_{a} \cap s_{b}\).

Note, \(e\) stands for even and \(o\) stands for odd. These statistics are crucial for seeing how \(L\) changes when apply Algorithm 8.4.2.

Notation 8.2.12. For \(B_{r,s}\) an even block, define \(\text{minus}_{B_{r,s}} = \{t \in B_{r,s}|w_{t,j(t)} = -1\}\).

Lemma 8.2.13. For any even block \(B_{r,s}\) with \(r \neq 1\), then \(|\text{minus}_{B_{r,s}} \cap s_e| = |\text{minus}_{B_{r,s}} \cap s_o|\).

If \(r = 1\), then

\[
||\text{minus}_{B_{r,s}} \cap s_e| - |\text{minus}_{B_{r,s}} \cap s_o|| = \begin{cases} 1, & \text{if } |\text{minus}_{B_{r,s}}| \equiv 1 \\ 0, & \text{if } |\text{minus}_{B_{r,s}}| \equiv 0 \end{cases}.
\]

Proof. First, consider the case that \(B_{r,s}\) is an even block with \(r \neq 1\). Then, whenever some \(t \in B_{r,s}\) satisfies \(w_{t,j(t)} = 1\), we know by Lemma 8.2.8 that there is some even block of ones, \(P_{a,b}\) with \(a \leq t \leq b\). Write \(\text{minus}_{B_{r,s}} = \{k_1, \ldots, k_p\}\), so that it is increasing. First observe that by Lemma 8.2.4, \(p\) must be even, since \(|B_{r,s}| \equiv 0\), and \(|\{t \in B_{r,s}|w_{t,j(t)} = 1\}| \equiv 0\), since each even block of ones has even length by definition. So \(|B_{r,s}| - |\{t \in B_{r,s}|w_{t,j(t)} = 1\}| = |\text{minus}_{B_{r,s}}|\) is also even. Next, note that for \(1 \leq i \leq p - 1\), either \(k_i + 1 = k_{i+1}\), or
else there is an even block of ones \( P_{k_l+1,k_{l+1}-1} \). This tells us that \( k_l \neq k_{l+1} \). Therefore, for any even block \( B_{r,s} \), such that \( r \neq 1 \), we must have \( |\text{minus} B_{r,s} \cap s| = |\text{minus} B_{r,s} \cap s_o| \).

In the case that \( r = 1 \), the argument from the previous paragraph shows that the parities of the rows in \( \text{minus} B_{r,s} \) must alternate. However, if \( r = 1 \) we do not know the parity of \( |\text{minus} B_{r,s} \cap s| \). Hence, \( |\text{minus} B_{r,s} \cap s| - |\text{minus} B_{r,s} \cap s_o| \) will either be 0 or 1 depending on whether \( |\text{minus} B_{r,s} \cap s| \) is even or odd, which is exactly what we wanted to show. \( \square \)

**Lemma 8.2.14.** Any initially uncanceled \( w \) can have at most one \((-1,1)\) pair. Furthermore \( w \) has a \((-1,1)\) pair if and only if there exists an odd block of ones \( P_{p,q} \), with \( p = 1 \).

**Proof.** For definiteness, let us take the case that \( n \equiv 0 \). The case \( n \equiv 1 \) is exactly analogous, simply by reversing the roles of even and odd.

This proof is simply a parity counting argument, where we count parity using both rows and columns. First, if \( n \) is even, the only way there could be a \((-1,1)_m \) pair is if \( m \) is odd, because by assumption \( m \neq n \). So, if there were any \((-1,1)_m \) pairs, we must have \( u_o > u_e \).

Now let us count by rows. By Lemma 8.2.13 we know that for any even block \( B_{r,s} \), with \( r \neq 1 \), \( |\text{minus} B_{r,s} \cap s| = |\text{minus} B_{r,s} \cap s_o| \). The second statement of Lemma 8.2.13 tells us that for \( r = 1 \), we must have \( |\text{minus} B_{r,s} \cap s| - |\text{minus} B_{r,s} \cap s_o| \leq 1 \). Since the number of \((-1,1)\) pairs is exactly the difference \( \sum_{t} (|\text{minus} B_{t,s} \cap s| - |\text{minus} B_{t,s} \cap s_o|) \), summed over all blocks \( B_{r,s} \). By Lemma 8.2.13, we know that \( |\text{minus} B_{r,s} \cap s| - |\text{minus} B_{r,s} \cap s_o| = 0 \), if \( r \neq 1 \). Thus, the number of \((-1,1)\) pairs is exactly equal to \( |\text{minus} B_{r,s} \cap s_o| - |\text{minus} B_{r,s} \cap s| \). By Lemma 8.2.13 this is equal to the parity of \( |\text{minus} B_{r,s} \cap s| \). However, \( |\text{minus} B_{r,s} \cap s| \equiv 1 \) if and only if there is an odd block of ones. By Lemma 8.2.8 the only way \( |\text{minus} B_{r,s} \cap s| \) can contain an odd block of ones \( P_{p,q} \) is if \( p = 1 \). Hence \( w \) has a single \((-1,1)\) pair if and only if \( w \) has some odd block of ones \( P_{p,q} \), with \( p = 1 \). \( \square \)

**Lemma 8.2.15.** If \( B_{r,s} \) is a block and \( t \in B_{r,s}, t \neq n, w_{t,j(t)} = 1 \), then \( w_{t+1,j(t)+1} = 1 \).

**Proof.**

\[
\begin{pmatrix}
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  1 & & & & & 1 \\
  & 1 & & & & \\
  & & 1 & & & \\
  t & & & & & 1 \\
  t_2 & & & & & \\
  c & & & & & \\
\end{pmatrix}
\]

We know that column \( j(t) + 1 \) must have a 1 as there is no descent at \( t \) because \( t \neq n \). So, to complete the proof, we only need to show that \( j(t+1) = j(t) + 1 \). Suppose instead \( j(t+1) \neq j(t) + 1 \). Now, we will show we get stuck in an infinite descent. Let \( c = i(j(t) + 1) \). Then, by Lemma 8.2.8, \( w_{c,j(c)} = 1 \), and, \( c \) must be part of a block of ones, \( P_{b_1,t_1} \), with \( b_1 \neq t_1 \). So, either \( w_{c-1,j(c-1)} = 1 \), and lies to the right of \( j \) or else row \( c \) is the top row of an even block of ones \( P_{c,d} \).

Suppose we are in the former case, that \( w_{c-1,j(c-1)} = 1 \), and lies to the right of \( j \). Define \( t_2 = c - 1 \). We know \( j(t_2) \) is not equal to \( c - 1 \), since \( j(t) = c - 1 \), and \( t \) by assumption \( j(t) + 1 \neq j(t+1) \). Therefore, since \( t_2 \neq n \), we are again in the same situation as before, except \( t \) is replaced by \( t_2 \).

The latter case that \( c \in P_{c,d} \) is handled similarly, by applying the above argument to \( w_{c,j(c)} \) instead of \( w_{c-1,j(c-1)} \).

Thus, we will be forced to move down the matrix indefinitely. This implies \( n \) is unbounded, which is a contradiction. \( \square \)
8.2.16. Property 4.

Lemma 8.2.17. If w is initially uncanceled, and n is even, then w has no $(-1,1)_t$ pairs.

Proof. Assume w has some $(-1,1)_t$ pair. Let $B_{1,s}$ be the topmost block. We saw in Lemma 8.2.14 that there is an odd block of ones $P_{1,q}$ with $q \geq s$. However, we must then have that $w_{t+1,j(q+1)} = -1$, and that $q + 1$ is to the right of j. This is because if the element in row $q + 1$ were to the left of j, we would have $B_{1,s} = P_{1,q}$ which would imply $B_{1,s}$ is an odd block, which we cannot have by Lemma 8.2.4.

Therefore, $w_{q+1,j(q+1)} = -1$. We know $q \not\equiv n \equiv 0$, so $q \equiv 1$. But then, by Lemma 8.2.15, if follows that $w_{q+1,i(q)+1} = 1$, contradicting $w_{q+1,j(q+1)} = -1$, since a signed permutation matrix can only have one nonzero entry in each row. Therefore, w has no $(-1,1)_t$ pairs. □

Lemma 8.2.18. If n is odd, there is at most one t for which there is a $(-1,1)_t$ pair, and if such a pair exists, then $i(t+1) = 1$. If we let $i(t) = y + 1$, then for all $k, 1 \leq k \leq y$, we have that k lies to the right of j, and $w_{k,j(k)} = 1$.

Proof. Nearly everything was shown in Lemma 8.2.14. All that remains to check is that $i(t+1) = 1$, and for all $k, 1 \leq k \leq y$, we have that k lies to the right of j and $w_{k,j(k)} = 1$. First, will show that $i(t+1) = 1$. By Lemma 8.2.14, we know $w_{1,j(1)} = 1$. Note that $j(1)$ lies on an odd column to the right of j, which means there cannot be a descent at $j(1) - 1$. Hence it must be that $w_{j(1)-1,i(j(1)-1)} = -1$, which implies that we have a $(+1, -1)_{j(1)-1}$ pair. Since there is at most one such pair by Lemma 8.2.14, we must have $t = j(1) - 1$ and so $i(t+1) = i(j(1)) = 1$.

The last claim to show is that if we do have a unique $(-1,1)_t$ pair, then all elements row above t are $1$’s to the right of j. We already know they are to the right of j because there exists some s such that $i(t) \in B_{1,s}$. We know by Lemma 8.2.14 that there must be an odd block of ones $P_{1,p}$, for $p < s$. However, since $B_{1,s}$ is an even block, we must have $p + 1 \in B_{1,s}$ and so $w_{p+1,i(p+1)} = -1$. Since $p + 1 \not\equiv n$, and all the rows above $p + 1$ have 1’s, we must have a $(-1,1)_{j(p+1)}$ pair. So, since there is only a single $(-1,1)_t$ pair, we must have $t = j(p + 1)$, as claimed. □

Corollary 8.2.19. For any initially uncanceled $w \in (C_n^t)^t$, we have $|u_e s_o| = |u_o s_o| = |u_o s_e| = 0$ and

$$|u_e s_e| \begin{cases} \leq 1, & \text{if } n \equiv 1 \\ = 0 & \text{otherwise}. \end{cases}$$

Proof. By definition, the only way we can have any of $|u_a s_b|$ > 0 is if there is a some $(-1,1)_t$ pair. By Lemma 8.2.18, Lemma 8.2.17 the only way we can have a $(-1,1)_t$ pair is if n is odd, t is even, and there are no rows s with s to the left of j such that s < t. Therefore, if we do have a $(-1,1)_t$ pair, then we must have $t \in u_e$, (in particular there are zero entries which lie above it and to the left of j). Finally, by Lemma 8.2.18 $i(t) \equiv 0$, and so $t \in s_e$. □

8.2.20. Property 5.

Lemma 8.2.21. When two adjacent columns t, t + 1 have $w_{i(t),t} = -1$ and $t \not\equiv n$ then for any row r which is row between $i(t), i(t + 1)$, we have $w_{r,j(r)} = 1$ and j(r) is to the right of j.

Proof. First, we will show that i(t), i(t + 1) must lie in the same block, and second we will show there are only +1’s row between $i(t), i(t + 1)$. The proofs of both of these facts involve an infinite descent.

Suppose that i(t), i(t + 1) lie in different blocks. Say $B_{r,s}$ is the even block with $t \in B_{r,s}$. By Lemma 8.2.13, there are the same number of $-1$’s in odd columns of $B_{r,s}$ as $-1$’s in even columns. Using this, since $i(t + 1) \not\in B_{r,s}$, there must be some other $(-1,1)_{t_2}$ pair with $i(t_2 + 1) \in B_{r,s}$ but $i(t_2) \not\in B_{r,s}$. This tells us there is a block $B_{r_2,s_2}$ beneath $B_{r,s}$. However, replacing t by $t_2$ and $B_{r,s}$ by $B_{r_2,s_2}$, we are in the same situation as we started in, but in a lower block, and therefore we must always have a lower block. Of course, this is a contradiction, as the matrix is of finite size.
So, we only have to show there are only 1's row between \(i(t), i(t+1)\). We have already seen in Lemma 8.2.18 that this statement holds for a \((−1,1)\) pair. So, we only have to verify the statement for \((−1,−1)\) pairs. Suppose there is such a \((−1,−1)\) pair. Since each even block has an even number of −1's, define \(s_1\) to be the column such that \(s_1 < t, w_{i(s_1), s_1} = −1\), and such that for all \(k\) which are row between \(i(s_1), i(t)\), we have \(w_{k,j(k)} = 1\). Note we must have \(s_1 \neq t\), since they are either adjacent or separated by an even block of ones. Further, note that \(s_1 \neq t + 1\), by assumption. Since \(s_1 \equiv n\), there cannot be an ascent at column \(s_1 - 1\), but we also know \(i(s_1) < i(t)\) because \(s_1 \neq t + 1\). Hence, we are in the same situation as the Theorem, if we replace \((t, t + 1)\) by \((s_1 - 1, s_1)\), but with \(i(s_1 - 1) < i(t)\). So, we have an infinite descent, yielding a contradiction. □

8.2.22. Property 6.

Lemma 8.2.23. If \(B_{r,s}\) is an even block with \(r \neq 1\), then there cannot be an even block of minus ones \(N_{p,q} \subset B_{r,s}, N_{p,q} \neq B_{r,s}\) and \(p \equiv r\).

Proof. Suppose there is such an \(N_{p,q}\), with \(p \equiv r\). Then, the block \(B_{r,s}\) is an even block, all blocks of ones \(P_{l,m} \subset B_{r,s}\) are even blocks of ones, and all blocks of minus ones \(N_{p,q}\) with \(p \neq r, q \neq s\) are even blocks of minus ones. These facts, together with the fact that \(p \equiv r\) imply that all blocks of minus ones \(N_{p,q} \subset B_{r,s}\) must be even blocks of minus ones. Now, fix \(p, q\) such that \(N_{p,q}\) is the block of minus ones in \(B_{r,s}\) which is below all other blocks of minus ones in \(B_{r,s}\). There are now two cases depending on whether \(p \equiv n\).

Case 1: \(p \equiv n\)

Since \(N_{p,q} \neq B_{r,s}\), either \(p - 1 \in B_{r,s}\), or else \(P_{p+1,s}\) is an even block of ones. However, we know that \(p - 1 \not\equiv n\), and \(s \not\equiv n\). If \(w_{p-1,j(p-1)} = 1\), since \(w_{p,j(p)} = −1\), we have a contradiction to Lemma 8.2.15. If instead \(w_{s,j(s)} = 1\), then \(j(s+1)\) is to the left of \(j\), which again contradicts Lemma 8.2.21. Hence, this situation is impossible.

Case 2: \(p \not\equiv n\)

In this case, note that \(q \equiv n\), and hence \(j(q) - 1 \not\equiv n\), and \(i(j(q) - 1) > q\). But this means that \(i(j(q) - 1) \not\in B_{r,s}\), as \(N_{p,q}\) was the lowest block of minus ones in \(B_{r,s}\). Therefore, there is a \((−1,−1)_{j(q)−1}\) pair, which is not contained in a single even block, contradicting Lemma 8.2.21.

Corollary 8.2.24. If \(N_{p,q} \subset B_{r,s}, N_{p,q} \neq B_{r,s}\), then \(N_{p,q}\) is an odd block of minus ones if and only if \(p = r\) or \(q = s\).

Proof. By Lemma 8.2.23, if \(p = r\) or \(q = s\), we must have that \(N_{p,q}\) is an odd block of minus ones. However, by Lemma 8.2.10 if \(p \neq q, r \neq s\) then \(N_{p,q}\) is an even block of minus ones.

8.2.25. Property 7.

Lemma 8.2.26. For \(B_{r,s}\) an even block, and \(P_{r,q}\) (respectively \(P_{p,s}\)) a block of ones, we must have \(q = s\) (respectively \(p = r\).

Proof. Suppose there exists \(P_{r,q} \subset B_{r,s}\) with \(P_{r,q}\) an even block of ones, with \(q \neq s\). Then, \(P_{r,q}\) must be an even block of ones, which implies there is some even block of minus ones \(N_{q+1,t}\) immediately beneath it. However, this means \(q + 1 \equiv r\), with \(q + 1\) the top of an even block of minus ones, contradicting Lemma 8.2.23.

An analogous argument goes through in the case of \(P_{p,s}\). Namely, if \(P_{p,s}\) is an even block of ones, the block of minus ones immediately above it, \(N_{l,s−1}\) has \(t \equiv r\), contradicting Lemma 8.2.23.

8.2.27. Equivalence of the Seven Properties and no General-Minus \(j\)-Swaps.

Theorem 8.2.28. An element \(w \in (B_n^r)_j\) is initially uncanceled if and only if it has no general-minus \(j\)-swaps.
Proof. First, let us check that if \( w \) has no general-minus \( j \)-swaps, then it is initially uncanceled. The first property holds by Lemma 8.2.2. Property 2 holds by Lemma 8.2.4. Property 3 holds by Lemma 8.2.8. Property 4 is satisfied by Lemma 8.2.18 and Lemma 8.2.17. Property 5 is proven in Lemma 8.2.21. By Lemma 8.2.24, property 6 holds. Finally, by Lemma 8.2.26, property 7 holds.

The converse also follows in a straightforward fashion. We have already seen using Lemma 6.0.8 that if all blocks are even, then there are no sign swaps or left swaps. So, we just need to show there are no double-plus swaps, double-minus swaps, or single-minus swaps. If we had any double-minus swap or double plus swap, then there must be some block \( B_{r,s} \) with either an odd block of ones \( P_{p,q} \) or an odd block of minus ones \( N_{p,q} \) with \( p \neq r \) and \( q \neq s \). However, by property 3, the former cannot happen, and by property 6 the latter cannot happen.

Finally, we just need to check there are no single-minus swaps \((b,t), \) with \( t \in B_{r,s} \). To see this, if \( B_{r,s} \) has any rows \( k \) with \( w_{k,j(k)} = -1 \), then by properties 6 and 7, we must have both \( w_{r,j(r)} = w_{s,j(s)} = -1 \). Therefore, we cannot possibly have any minus swaps involving a column \( t \) with \( i(t) \in B_{r,s} \) as to perform a single-minus swap we would either need \( w_{r,j(r)} = 1 \) or \( w_{s,j(s)} = 1 \).

\[ \square \]

Corollary 8.2.29. Recall \( U_n^c \) is the set of initially uncanceled elements. It follows that

\[ \sum_{w \in (C_n^h)_j} (-1)^{l(w)} X_L(w) = \sum_{w \in (U_n^c)_j} (-1)^{l(w)} X_L(w) \]

Proof. To prove this, we only need to show there is an involution on the set \((C_n^h)_j \setminus (U_n^c)_j \). However, we have shown \( w \in (C_n^h)_j \setminus (U_n^c)_j \) if and only if \( w \) is not initially uncanceled in Theorem 8.2.28. Therefore, using Lemma 8.2.5 there is indeed a sign reversing, \( L \) preserving involution on \((C_n^h)_j \setminus (U_n^c)_j \), completing the proof.

\[ \square \]

8.3 Finally Uncanceled Elements.

Definition 8.3.1. Fix \( k \) and let \( I = \{i_1, \ldots, i_l\} \) with \( k \in I \cup \{n\} \) and \( k \equiv i \) for all \( i \in I \cup \{n\} \) such that \( i > k \). An element \( w \in C_n^h \), is finally uncanceled (with respect to \( k \)) if its signed permutation matrix satisfies the following properties.

(1) Either \( j(n) \) is left of \( k \) or \( w_{n,j(n)} = 1 \).
(2) \( w \) has no odd blocks with respect to \( k \).
(3) Whenever \( t \) is a row for which \( w_{t,j(t)} = -1 \) and \( t \) lies to the right of \( k \), then there exists an even block of minus ones \( N_{r,s}^k \) with \( r \leq t \leq s \), except possibly if for all \( p < t \), it is the case that \( j(p) \) lies to the right of \( k \) and \( w_{p,j(p)} = -1 \).
(4) If \( n \) is even, there can be at most one \( t \) for which \( t, t+1 \) is a \((-1,1)_t^k \) pair, and if such a pair exists, then \( i(t) = 1 \), and if \( s \) is row above \( i(t+1) \), then \( w_{s,j(s)} = -1 \) and \( j(s) \) is to the right of \( k \). If \( n \) is odd, no \((-1,1)_t^k \) pairs exist.
(5) For any \((1,1)_t^k \) pair, any row \( r \) which is row between \( i(t), i(t+1) \), must satisfy \( w_{r,j(r)} = -1 \) and \( j(r) \) is to the right of \( k \).
(6) If \( P_{p,q}^k \subset B_{r,s}^k \), \( P_{p,q}^k \neq B_{r,s}^k \), then \( P_{p,q}^k \) is an odd block of ones if and only if \( p = r \) or \( q = s \).
(7) For \( B_{r,s}^k \) an even block, and \( N_{r,s}^k \) (respectively \( N_{p,q}^k \)) a block of minus ones, we must have \( q = s \) (respectively \( p = r \)).

Notate \( w \in (V_n^c)_k \) if \( w \) is finally uncanceled with respect to \( k \).

Remark 8.3.2. It is important to note that for \( w \in (U_n^c)_j \), we must have \( w(j) = n \). However, for \( w \in (V_n^c)_k \), we make no such restriction. It is for this reason that the left hand side of the next Theorem 8.3.3 is a sum over all of \( C_n^h \) as opposed to just \((C_n^h)_k \).

Theorem 8.3.3. Let \( I, k \) be as in Definition 8.3.1. We have the equality

\[ \sum_{w \in C_n^h} (-1)^{l(w)} X_L(w) = \sum_{w \in (V_n^c)_k} (-1)^{l(w)} X_L(w) \]
Proof. The proof of this result is essentially the same as that of Corollary 8.2.29, but with the role of −1’s and +1’s reversed. One key difference is that here we are using the set Swap_k^p(C_n^r), whereas in Corollary 8.2.29 we used Swap_k^p(C_n^t), i.e., we used general-minus j-swaps before, and now we are using general-plus k-swaps. A second crucial difference here is that the maximal element of Swap_k^p(C_n^r), does not necessarily preserve the kth column. The analogs of Lemma 8.2.2, Lemma 8.2.4, Lemma 8.2.8, Lemma 8.2.18, Lemma 8.2.17, Lemma 8.2.21, Lemma 8.2.24, Lemma 8.2.26 all easily go through similarly with 1’s and −1’s interchanged. Then, the analog of Theorem 8.2.28 tells us w has no general-plus k-swaps, if and only if w is finally uncanceled. Finally, we can use the general-plus k-swapping invocation to cancel off all elements of C_n^r \setminus (V_n^r)^k, and obtain this theorem. □

8.4. The Lowering Map.

Remark 8.4.1. Fix I ⊂ [n − 1]_0, n ∈ I and let j = i_r so that
\[ I = \{i_1, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_t\} \]
\[ J = I^c = \{i_1, \ldots, i_{r-1}, j-1, i_{r+1}-1, \ldots, i_l-1\}. \]

With I, j as in Notation 8.0.12. For the rest of this section, retain this notation for I and J. Now, in order to complete the proof of our Theorem 4.0.17, all that remains is to show there is a bijection between elements of (U_n^r)_j and (V_n^r)_j, which decreases L by n − j and preserves the sign of w. We next define this map, and then prove it satisfies the desired properties.

Algorithm 8.4.2. The lowering map \( H : (U_n^r)_j \to B_{n-1} \), taking an uncanceled element \( w \in (U_n^r)_j \), to an element \( H(w) \), where \( H(w) \) is computed by the following algorithm:

Step 1
Make the transpositions \( w \mapsto w_1 = ws_j \cdots s_{n-2}s_{n-1} \). Now there is a 1 in the lower right-hand corner, so henceforth, treat \( w_1 \) as in \( B_{n-1} \), which is naturally embedded in \( B_n \) as the set of all elements fixing \( n \).

Step 2
Whenever in \( w_1 \) there is a block of ones \( P_{r,t}^{j-1} \), define the function \( p_{r,t} : B_{n-1} \to B_{n-1}, x \mapsto s_r s_{r+1} \cdots s_t x \). Let \( w_2 \) be the result of the composite of all maps \( p_{r,t} \), applied to \( w_1 \), as one runs through all even blocks of ones \( P_{r,t}^{j-1} \).

Step 3
If in \( w_2 \) there is an odd block of minus ones, \( N_{r,t}^{j-1} \), let \( w_3 = s_0 w_2 \).

Step 4
For each block of minus ones \( N_{r,t}^{j-1} \) in \( w_3 \) define the function \( n_{r,t} : B_{n-1} \to B_{n-1}, x \mapsto s_{r-1} \cdots s_{t-1}x \). Let \( w_4 \) be the result of the composite of all maps \( n_{r,t} \), applied to \( w_3 \), as one runs through all blocks of minus ones \( N_{r,t}^{j-1} \). Then, declare that \( H(w) = w_4 \).

Remark 8.4.3. Note that in steps 2 and 4 that \( w_2 \) and \( w_4 \) are independent of the order in which we apply \( p_{r,t}, n_{r,t} \) since nonadjacent transpositions commute.

Remark 8.4.4. An alternative characterization of the above algorithm is as follows. First cross out column \( j \) and row \( n \), to get a matrix in \( B_{n-1} \). Then, start at the bottom of the matrix, and move upward, and whenever there is a 1 in row \( t \) to the right of \( j \), switch rows \( t, t + 1 \). Once we hit the top of the matrix, we then proceed downward, and whenever there is a −1 in row \( t \) to the right of \( j \), interchange rows \( t, t - 1 \). Here, interchanging rows \( 1, 0 \) means we change the sign of row \( 1 \).

We tend to use this point of view of interchanging two adjacent rows in the ensuing proofs, although of course it is equivalent to applying transpositions to the matrix.

Example 8.4.5. Let us now give an example of how the algorithm would proceed. Let
Here, since \( w(3) = 9 \), we set \( j = 3 \), and so blocks in the algorithm will be taken with respect to column \( j - 1 = 2 \). In step 1, we move \( j(9) = 3 \) to the far right, resulting in

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
\end{pmatrix}
\]

Now, viewing this as an element of \( B_8 \), we have

\[
w_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
\end{pmatrix}
\]
Next, we perform step 2, which moves the \((1,1)^2\) pair down by 1, and pushes the \(-1\) in column 3 up by 2.

\[
\begin{pmatrix}
1 \\
-1 \\
1 \\
-1 \\
-1 \\
\end{pmatrix}
\]

However, we have not completed step 2, because there is another block of ones, namely \(P_{1,1}^2\) and so we transpose the top two rows. The resulting matrix is \(w_2\).

\[
w_2 = \\
\begin{pmatrix}
-1 & \bullet & \downarrow & \downarrow & 1 \\
1 \\
-1 \\
-1 \\
1 \\
\end{pmatrix}
\]

Next, we apply step 3 to the odd block of minus ones \(N_{1,1}^2\), by changing the sign of the top row. This results in the matrix \(w_3\).

\[
w_3 = \\
\begin{pmatrix}
\mathbf{1} \\
1 \\
1 \\
-1 \\
-1 \\
1 \\
\end{pmatrix}
\]
Finally, we perform step 4, moving the $-1$’s in rows 5 and 6 up by one row, and moving the 1 in the 4th row down two rows.

$$H(w) = \begin{pmatrix} 1 & & & 1 \\ & -1 & & \\ & & -1 & \\ 1 & & & \end{pmatrix}$$

This is our final result. Lo and behold, $H(w)$ is a chessboard element with a very similar descent set to our original matrix, and furthermore, it is actually a finally uncanceled matrix. We can also see by direct computation that $\text{sgn}(w) = \text{sgn}(H(w)) = -1$, and $L(w) = 15, L(H(w)) = 9$. Since $n = 9, j = 3$, we have that $L(w) - L(H(w)) = 15 - 9 = 6 = n - j$.

We will soon see that this is no coincidence, and that in fact this lowering map defines a bijection between the initially uncanceled elements and the finally uncanceled elements of one lower dimension, in a way that preserves $\text{sgn}(w)$ and changes $L(w)$ by $n - j$.

**Lemma 8.4.6.** For any $w \in (U_n^I)_j$, the matrix $w_3$, after step 3 of Algorithm 8.4.2, has no odd blocks of minus ones $N_{1,3}^{j-1}$. Furthermore, all $(-1, -1)_t$ pairs in $w_3$ satisfy $i(t+1) = i(t) - 1$, that is, they lie in adjacent rows.

**Proof.** If some block $B_{p,q}$ in $w$ is not an even block of minus ones or an even block of ones, and if $p > 1$, we know by property 6 and property 7 of initially uncanceled matrices, that the pattern of $+1$’s and $-1$’s in a given $B_{p,q}$, is as follows: First, there is an odd block of minus ones, then there are alternating even blocks of plus ones and even blocks of minus ones, until finally there is an odd block of minus ones. Hence, moving all the even blocks of plus ones down by a row in step 2 forces the top row of each block of minus ones to move to a higher block of minus ones. This ensures that all the resulting blocks of minus ones will be even blocks of minus ones. (For an example, see the movement between $w_2$ and $w_3$ in Example 8.4.5.)

The fact that $i_{w_3}(t + 1) = i_{w_3}(t) - 1$, follows directly from property 5. That is, since we move the 1’s between each $(-1, -1)_t$ pair in $w$ down, the $(-1, -1)_t$ pair in $w$ becomes a $(-1, -1)_{t-1}$ pair in $w_3$, which is moved together so that $i_{w_3}(t + 1) = i_{w_3}(t) - 1$. \hfill $\square$

**Lemma 8.4.7.** The image of $H$ is contained in $C_{n-1}^{J^c}$.

**Proof.** We must show that the descent set at the end of the algorithm is $J$, and that the final result is a chessboard element. First, it is clear that after step 1 of Algorithm 8.4.2 we have $D(w_1) \subset J$, because $D(w) \subset I$, and we simply decrease the descents to the right of $j$ by 1 column. In steps 2,3,4, the descent set also remains unchanged: Any row to the left of $j$ can never change places with another row to the left of $j$, which shows that the descent set is preserved to the left of $j$. And for rows to the right of $j$, we are only moving rows which contain $+1$’s up, and rows which contain $-1$’s down, and we are always preserving the relative order of the rows with $-1$’s and preserving the relative order of the rows with $+1$’s. So, the final descent set is contained $J$ and the image of $H$ is contained in $B_{n-1}^{J^c}$.

Next, we just have to check that the image is chessboard. After step 1, all the columns $t$ to the left of $j$ are on squares with $t \equiv i_{w_1}(t)$. For this argument, we will call squares $(i(t),t)$ with $t \equiv i(t)$ chessboard squares. After step 1, all the columns to the right of $j$ are not on chessboard squares.
In step 2, there are two cases. First, the case that the topmost block of ones is not an odd block of ones. Then, each +1 to the right of \( j \) is moved down by a single row, so that it will then be on a chessboard square. Also, the row \( s \) below the block of ones is moved up by an even number of rows, because the block of ones is an even block of ones. So, in \( w_2 \), we have that \((s, j_{w_2}(s))\) is a chessboard square if and only if it was on a chessboard square before applying the transpositions.

If instead, the topmost block of ones is an odd block of ones, then the \(-1\) in the row beneath this topmost block is moved up to the top row, and so the parity of the row of the \(-1\) is changed, and the parities of all the rows in this odd block of ones are also changed.

In step 3, only a sign is changed.

Finally, in step 4, if the even block of minus ones contains the top row of \( w \), then the \(-1\) in the top row is changed to a 1 and moved down by an odd number of rows, and the other \(-1\)’s are moved up by a single row. Otherwise, all the \(-1\)’s are moved up by a single row, so that they too will lie on chessboard squares. Finally, thanks to Lemma 8.4.6, all blocks of minus ones are even blocks of minus ones, and so we will always preserve the parity of the row immediately above the even block of minus ones.

Hence, from steps 2, 3, and 4, we have changed the parity of the rows to the right of \( j \) and preserved the parity of the rows weakly to the left of \( j \), resulting in a chessboard element. \( \square \)

**Lemma 8.4.8.** The image \( H((U^I_n)_{j}) \) is contained in \((V^F_n)_{j-1}\).

**Proof.** We now have to check that the seven properties of finally uncanceled matrices hold.

**Property 1**

Property 1 is obvious because if before step 4 property 1 does not hold, then the \(-1\) in row \( n-1 \) is moved up by a row, and either a 1 or an element to the left of \( j \) must occupy row \( n-1 \).

**Property 2**

Property 2 is clear because the parities of the columns to the left of \( j \) are preserved by this process as shown in Lemma 8.4.7.

**Property 3**

Property 3 holds by Lemma 8.4.6, since step 4 of the algorithm does not separate any blocks of minus ones.

**Property 4**

In order to show property 4 of finally uncanceled matrices holds after the algorithm, we break up the argument into five cases depending on what the topmost block of \( w \) is. That is, we can check it in the cases that the topmost row lies to the left of \( j \), the topmost row is part of an odd block of minus ones, the topmost row is part of an even block of minus ones, the topmost row is part of an odd block of ones, or the topmost row is part of an even block of ones. Below, we’ll just write out the details for the case the topmost row lies in an even block of minus ones. The other cases are similar.

Suppose there is an even block of minus ones containing the first row. Since the topmost row is not part of an odd block of minus ones, there are no \((-1, 1)_{t} \) pairs. So, the topmost row must be part of a \((-1, -1)_{(j(1)-1)} \) pair, meaning the topmost row is of the same parity as \( n \). Then, after applying the algorithm, the topmost row will have its sign changed, and be moved to the bottom of the topmost block of minus ones. So, \( H(w) \) will have an odd block of minus ones at the top, with a \((-1, 1)_{(j(1)-1)^{-1}} \) pair. Additionally, \( i_{H(w)}(t) = 1 \), and \( i_{H(w)}(t+1) \) is immediately below the topmost odd block of minus ones, as is allowed in property 4 of finally uncanceled matrices.

**Property 5**

Property 5 holds because if we start with \( w \in B^I_n \) which has a \((+1, 1)_{t} \) pair, then by Lemma 8.2.15 we initially have \( i_{w}(t+1) = i_{w}(t)+1 \). Therefore, these 1’s will stay in adjacent
rows until step 4, and then in step 4, they will either remain together, or be separated by a single even block of minus ones.

**Properties 6 and 7**

Note, that it is easy to check these conditions are satisfied when we are dealing with the block $B_{1,s}$ in $w$, and so for the remainder of this verification, we assume that we are dealing with $B_{r,s}$ for $r > 1$.

In order to deduce these properties, we can use Lemma 8.4.6, which tells us that $(-1, -1)$ pairs lie in adjacent rows. Similarly, whenever we have a $(1, 1)$ pair, we must have that the only rows between $i_w(t), i_w(t+1)$ are to the right of $j$ and have $-1$'s. Furthermore, since the parity of the columns of these pairs change, we automatically obtain that any even block which contains both plus ones and minus ones must be alternating in the following sense:

For $B_{r,s}^{-1}$ a block in the matrix $H(w)$ after the algorithm, which contains both ones and minus ones,

$$B_{r,s}^{-1} = P_{r,s_1}^{-1} \cup N_{s_1+1,s_2}^{-1} \cup P_{s_2+1,s_3}^{-1} \cdots \cup N_{s_{k-2}+1,s_{k-1}}^{-1} \cup P_{s_{k-1}+1,s}^{-1},$$

where $P_{r,s_1}^{-1}$ and $P_{s_{k-1}+1,s}^{-1}$ are odd blocks of ones in $H(w)$, but all the other blocks of ones are even blocks of ones, and all the blocks of minus ones are even blocks of minus ones. This is exactly what properties 6 and 7 of finally uncanceled matrices say.

8.5. **The Lowering Map's Effect on L.** In this subsection, we check that $H$ preserves $\text{sgn}(w)$ and changes $L(w)$ by $n - j$.

**Lemma 8.5.1.** For a fixed $w \in \langle C_n^- \rangle_j$, let $y = s_{n-1} s_{n-2} \cdots s_j w$. Note that $y(n) = n$. Then, $L(y) = L(w) - |u_s s_e| - |u_e s_o| + |u_e s_e| + |u_e s_o| - \frac{n-j}{2}$.

**Proof.** By performing these swaps, it is fairly straightforward to calculate

$$a(y) - a(w) = |s_o| + |s_e|,
$$

$$b(y) - b(w) = \frac{n-j}{2},
$$

$$c(y) - c(w) = -|u_s s_e| + |u_o s_o|.$$

One can check these computations in similar fashion to those in the swapping Lemma 5.2.3. Then, using the abc Lemma 2.4.9, $L(w) = a(w) + b(w) + 2c(w)$, so

$$L(y) - L(w) = a(y) + b(y) + 2c(y) - a(w) - b(w) - 2c(w)
$$

$$= a(y) - a(w) + b(y) - b(w) + 2(c(y) - c(w))
$$

$$= -|s_o| + |s_e| - \frac{n-j}{2} - 2|u_s s_e| + 2|u_o s_o|
$$

$$= -|u_s s_e| - |u_e s_o| + |u_e s_e| + |u_o s_o| - \frac{n-j}{2}.
$$

**Lemma 8.5.2.** The lowering map preserves the sign of $w$. i.e. $\text{sgn}(w) = \text{sgn}(H(w))$.

**Proof.** In all four steps, we make a multiply $w$ by a certain number of Coexeter generators. It suffices to check that the we multiply $w$ by an even number of Coexeter generators to obtain $H(w)$. In step 1, there are $n - j \equiv 0$ Coexeter generators by which $w$ is multiplied to obtain $w_1$. In steps 2, 3, 4, $w$ is also multiplied by $n - j$ Coexeter generators. Hence, $w$ is multiplied by a total of $2(n - j)$ Coexeter generators to obtain $H(w)$, and hence $\text{sgn}(w) = \text{sgn}(H(w))$.

**Lemma 8.5.3.** We have

\begin{align*}
(8.1) & \quad L\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} - L\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \\
(8.2) & \quad L\begin{pmatrix} 0 & -1 \\ \pm 1 & 0 \end{pmatrix} - L\begin{pmatrix} \pm 1 & 0 \\ 0 & -1 \end{pmatrix} = -1
\end{align*}
we will have one move as in (8.4). Since this decreases

There is no

through 4, we change

where if we choose + for the ± sign on the first matrix, we must also choose + for the ± sign on the second matrix.

Proof. The proof is given by calculating $L$ in the four situations. □

Lemma 8.5.4. Let $t, t+1, t+2$ be three adjacent rows with $t \equiv n$ and suppose $w_{t+1,j(t+1)} = w_{t+2,j(t+2)} = 1$, but with $t$ not in the block of ones containing $t+1, t+2$, (respectively $w_{t,j(t)} = w_{t+1,j(t+1)} = -1$, but with $t+2$ not in the block of minus ones containing $t, t+1$).

Then, if we first exchange rows $t, t+1$ and then exchange rows $t+1, t+2$ (respectively first exchange rows $t+1, t+2$ and then exchange rows $t, t+1$) the total value $L(w)$ decreases by 1.

Proof. Let us prove it for the case $w_{t+1,j(t+1)} = w_{t+2,j(t+2)} = 1$ using Lemma 8.2.15. The other case for $w_{t,j(t)} = w_{t+1,j(t+1)} = -1$ is analogous, by reversing the role of 1’s and −1’s.

Since $t+2 \equiv n$, it follows that $t+1, t+2$ is a $(1, 1)$ pair, and hence they lie in adjacent rows and columns. Define $w_t$ to be the 3×3 submatrix whose entries are the intersection of rows $t, t+1, t+2$ and of columns $j(t), j(t+1), j(t+2)$.

First, since rows $t, t+1, t+2$ are adjacent, the change in $L(w)$ due to these transpositions is the same as the change in $L(w_t)$. One can prove this using methods analogous to those in the Swapping Lemma, by noting that the values of $a_j, b_j, c_j$ are unaltered by performing these swaps, as long as $j \notin \{j(t), j(t+1), j(t+2)\}$.

Now, there are two cases, either $j(t+2) < j(t)$ or else $j(t+2) > j(t)$. For simplicity, let us assume $j(t+2) > j(t)$; the other case is analogous. So, by the above, we have $L(v) - L(w) = L(v_t) - L(w_t)$. Now, we may note that clearly $a(v_t) = a(w_t)$, and also that

\[
b_{2,3}(w_t) = b_{2,3}(v_t),
\]

\[
c_{2,3}(w_t) = c_{2,3}(v_t).
\]

So, the only difference in $L(v_t)$ and $L(w_t)$ comes from columns 1,2, which is precisely calculated by equations (8.1), (8.2) from Lemma 8.5.3. Thus, $L(v_t) - L(w_t) = -1$, which implies $L(v) - L(w) = -1$. □

Lemma 8.5.5. For a fixed $w \in (C_n^r)$, $L(H(w)) = L(w) - n + j$.

Proof. By Lemma 8.5.1, if $w_1$ the result of $w$ after step 1 of Algorithm 8.4.2, then $L(w_1) = L(w) - |u_o s_c| - |u_c s_o| + |u_o s_o| + |u_c s_c| - \frac{n-2}{2}$. So, we only need to show that in steps 2 through 4, we change $L$ by $|u_o s_c| + |u_c s_o| - |u_o s_o| - |u_c s_c| - \frac{n-2}{2}$.

There are now three cases depending on whether or not there is a $(-1,1)$ pair, and if there is not, what the sign of the first row is.

There is no $(-1,1)_t$ pair and $w_{1,j(1)} = 1$.

Since there is no $(-1,1)_t$ pair, $|u_o s_c| + |u_c s_o| - |u_o s_o| - |u_c s_c| = 0$, in both this case and the next. In this case, we will make exactly $\frac{n-2}{2}$ moves of the type as in Lemma 8.5.4. By Lemma 8.5.4, each such move decreases $L$ by 1. Hence, we change $L$ by a total of

$-\frac{n-2}{2} = |u_o s_c| + |u_c s_o| - |u_o s_o| - |u_c s_c| - \frac{n-2}{2}$, because $|u_o s_c| + |u_c s_o| - |u_o s_o| - |u_c s_c| = 0$.

There is no $(-1,1)_t$ pair and $w_{1,j(1)} = -1$.

In this case, we will only have $\frac{n-2}{2} - 1$ moves as in Lemma 8.5.4. However, during step 2, we will have one move as in (8.4). Since this decreases $L$ by one, steps 2 through 4 change $L$ by a total of

$-\frac{n-2}{2} + 1 - 1 = -\frac{n-2}{2} = |u_o s_c| + |u_c s_o| - |u_o s_o| - |u_c s_c| - \frac{n-2}{2}$, as desired.

There is a $(-1,1)_t$ pair
8.6. The Inverse to the Lowering Map. We have essentially shown so far that there is a one way map from \((U_n^I)^{-1})_{j-1}\) to \((V_n^J)^{-1})_{j-1}\), which alters \(L(w)\) by \(n-j\), and preserves \(sgn(w)\). In order to complete the Theorem, we just need to show this is a bijection. In what follows, we produce the inverse map.

**Algorithm 8.6.1.** Define the raising map \(R: (V_n^J)_{j-1}\) → \(B_n\), which takes a finally uncanceled element \(w \in (V_n^J)_{j-1}\), to an element \(R(w)\), where \(R(w)\) is computed by the following algorithm.

**Step 1**
For each block of minus ones \(N_{r,t}^j\) in \(w\) define \(rn_{r,t}: B_{n-1} \to B_{n-1}, x \mapsto s_{r} \cdots s_{t}x\). Then, let \(w_1\) be the result of the composite of all maps \(rn_{r,t}\) applied to \(w\), as one runs through all even blocks of minus ones \(N_{r,t}^j\).

**Step 2**
If \(P_{r,t}^j\) is an odd block of ones in \(w_1\), let \(w_2 = s_0w_1\).

**Step 3**
Whenever in \(w_2\) there is a block of ones \(P_{r,t}^j\), define \(np_{r,t} : B_{n-1} \to B_{n-1}, x \mapsto s_{t-1} \cdots s_{r+1}x\). Let \(w_3\) be the result of the composite of all maps \(np_{r,t}\) applied to \(w_2\), as one runs through all blocks of ones \(P_{r,t}^j\).

**Step 4**
Let \(w_4\) be the element of \(B_n\) defined by

\[
w_4(i) = \begin{cases} w_3(i), & \text{if } i \neq \pm n \\ i, & \text{otherwise} \end{cases}
\]

Here, we are just naturally embedding \(B_{n-1}\) into \(B_n\).

**Step 5**
Define \(w_5 = ws_{n-1} \cdots s_{j+1}s_j\). Then, define \(L(w) = w_5\).

**Lemma 8.6.2.** The image \(R((V_n^J)_{j-1})\) is contained in \((U_n^I)^{-1}\)

**Proof.** The proof is exactly analogous to that of Lemma 8.4.8. \(\square\)

**Lemma 8.6.3.** The maps \(H: (U_n^I)^{-1})_{j} \to (V_n^J)_{j-1}\), \(R: (V_n^J)_{j-1} \to (U_n^I)^{-1})_{j}\) are mutual inverses.

**Proof.** By Lemma 8.4.8 and Lemma 8.6.2, \(H\) is indeed a map \(H: (U_n^I)^{-1})_{j} \to (V_n^J)_{j-1}\) and \(R\) is indeed a map \(R: (V_n^J)_{j-1} \to (U_n^I)^{-1})_{j}\). Furthermore, the algorithms were precisely constructed so that \(H \circ R = id, R \circ H = id\), since in \(H\) we are multiplying by transpositions \(s_i\) and in \(R\) we are multiplying by the same transpositions in the opposite order. Here we are crucially using that \(s_i^2 = 1\), as \(B_n\) is a Coxeter group. \(\square\)

8.7. Completion of Part 3.

**Corollary 8.7.1.** Part 3 of Theorem 4.0.17 holds. That is, for \(I \subset [n-1], j = i_k \in I \cup \{n\}, \) then \((S_{n, j})_{i_k}(X) = f_{n,j}(X), \) where \(J\) is as defined in Notation 8.4.1.

**Proof.** In Lemma 8.6.3, we showed that \(H\) induces a bijection \(H: (U_n^I)^{-1})_{j} \to (V_n^J)_{j-1}\), and by Lemma 8.5.5, we see \(L(H(w)) = L(w) - n + j\) and \(l(H(w)) = l(w)\).
This tells us
\[
\sum_{w \in (U^I_{n,j})} (-1)^{l(w)} X^L(w) = \sum_{w \in (U^I_{n,j})} (-1)^{l(H(w))} X^{L(H(w)) + n - j}
= X^{n-j} \sum_{w \in (V^J_{n-1,j-1})} (-1)^{l(w)} X^L(w)
\]

Therefore, using Corollary 8.2.29, the above observation, and Theorem 8.3.3
\[
(S_{n,I})_j(X) = \sum_{w \in (C^I_{n,j})} (-1)^{l(w)} X^L(w)
= \sum_{w \in (U^I_{n,j})} (-1)^{l(w)} X^L(w)
= X^{n-j} \sum_{w \in (V^J_{n-1,j-1})} (-1)^{l(w)} X^L(w)
= X^{n-j} f_{n-1,J}(X),
\]

The equality between the penultimate line and ultimate line above holds by an inductive assumption. That is, we are inductively assuming Theorem 4.0.17 holds for \(n - 1\) and proving that it holds for \(n\).

9. A Further Conjecture

In the above sections, we found a nice way to factor \(\sum_{w \in B^I_{n,j}} (-1)^{l(w)} X^L(w)\). It is natural to ask if there is any nice way to factor the expression \(\sum_{w \in B^I_{n,j}} t^{l(w)} X^L(w)\), where we replace \(-1\) by a general variable \(t\). It seems that in general, there is not a nice factorization, but in the case that \(0 \in I\), there may be a nice single factor.

**Conjecture 1.** The two variable polynomial \(xt + 1\) divides \(\sum_{w \in B^I_{n,j}} t^{l(w)} X^L(w)\) if and only if \(0 \in I\).

**Remark 9.0.2.** This conjecture is true for all subsets \(I \subset [n-1]_0\) with \(n \leq 6\), as was verified by the computer. However, we do not see a way to generalize the techniques used in this paper to prove Conjecture 1. Note also that when we plug in \(t = -1\), we obtain the formula in Theorem 2.3.1, but \(-x + 1\) almost always divides \(f_{n,I}\), and so Theorem 2.3.1 does not appear to be very helpful in proving Conjecture 1.

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**References**


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