Minnesota REU Day 1 June 1, 2015

Pasha Pylyavskyy

Pizza & bowling party
Wed 5:30-7:30pm
Goldy's Game Room - Coffman Union basement

		Vincent Hall
TAs:	Elise DelMas	556
	Al Garver	550
	Thomas McConville	526
	Becky Patrias	552

Commuting elements of plactic monoid

Free monoid
has an alphabet $\{1, \ldots, n\}$
with no relations (but associative)
and multiplication by concatenation

$$123 \cdot 34 = 12334$$

Q: When do two words commute?

$$34 \cdot 123 = 34123 \neq 12334$$

THM: If and only if both are powers of the same word.

EX: $1212 \cdot 12 = 12 \cdot 1212$

One direction is easy.

EXERCISE 1: Prove this theorem.

The plactic monoid

Elements are still words in the *ordered* alphabet $1<2<\dots<n$.

But now there are *equivalence relations* imposed: (locally)

$$\dots xzy \dots \equiv \dots zxy \dots \text{ if } x \le y < z$$

$$\dots yxz \dots \equiv \dots yzx \text{ if } x < y \le z$$

EXAMPLE: $23\underline{412} \equiv \underline{231}42 \equiv 21\underline{342}$

$\equiv \underline{213}24$

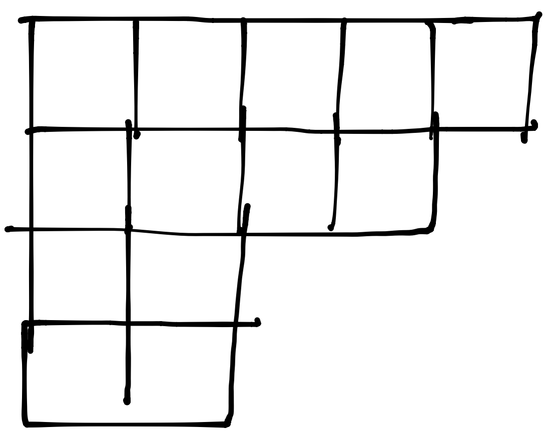
$\equiv 23124$

But 1234 has no other equivalents.

Multiplication is still given by concatenation.

Q: What are the sizes of the classes? When are 2 words equivalent?

Tableaux (semistandard Young tableaux)



$\leftrightarrow$ the number partition $\lambda = (5,4,2,2)$

A filling with $1, 2, \ldots$ so that numbers increase $\leq$ across rows

$<$ down columns

is called a semistandard Young tableau of the shape λ

e.g. $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 6 \\ \hline 3 & 3 & 3 & 5 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$

The tableau has a reading <u>word</u> by reading it in a certain order

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array} \rightsquigarrow 2311$

<u>THM</u>: There exists a unique tableau (= reading word of a tableau) in each plactic class.

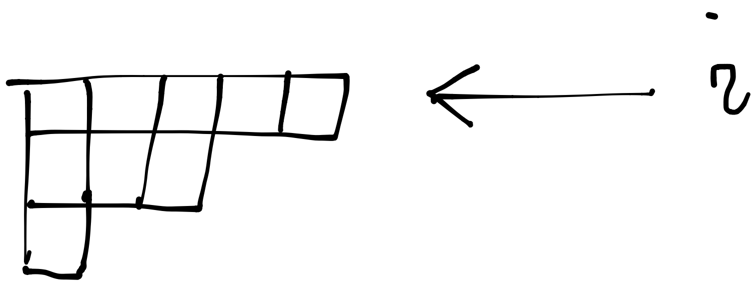
e.g. $23124 \leftsquigarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$

in the previous class

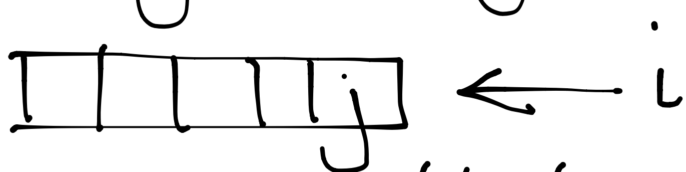
Q: How to find the tableau for each plactic class?

The RSK insertion algorithm

We insert letters into tableaux one at a time



The rule goes row-by row



1. If $j \le i$, attach i on the right.

2. If not, i bumps out smaller entry $j > i$.

3. Insert bumped out thing into the next row.

EXAMPLE:

21342

The tableau starts out empty.

$\emptyset \leftarrow 2$

$\boxed{2}$

$\boxed{2} \leftarrow 1$ produces $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow 3 \text{ produces } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \leftarrow 4 \text{ produces } \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \leftarrow 2 \text{ produces } \underbrace{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}}_{\text{the insertion tableau for } 21342}$$

EXERCISE 2: Prove that this insertion tableau (or its reading word) is plactic-equivalent to the word inserted.

REU PROBLEM 1:

When do two elements of the plactic monoid commute?

EXAMPLE:

$$21 \cdot 2 \equiv 2 \cdot 21$$
$$(y x z \quad y z x, \quad x < y \leq z)$$

Assume $n=2$ so the alphabet is $\{1<2\}$.

Bracketing: match/pair 2's left of 1's, remove and repeat

1 2 2 1 2 1 1 2 2

THM: Two words in alphabet $\{1<2\}$ commute if and only if AB and BA have the same # of bracketing pairs.

EXAMPLE: $\overset{\frown}{21}2 \equiv 2\overset{\frown}{21}$

both have 1 pair.

EXERCISE 3: Prove this THM.

This bracketing is very important in Kashiwara's theory of crystal operators.

It's not hard to see that if A, B commute, then restricting them to their subwords of 1,2's they commute,
of 2,3's they commute,
etc.

So we get a sequence of <u>necessary</u> conditions for A, B to commute.

Sadly, they are <u>not sufficient</u>:

$$21 \cdot 32 \not\equiv 32 \cdot 21$$

even though

$$21 \cdot 2 \equiv 2 \cdot 21$$

$$2 \cdot 32 \equiv 32 \cdot 2.$$

REU SubPROBLEM 1a:

Find the answer for $n=3$, i.e. $\{1<2<3\}$.

Note that asking which <u>words</u> commute is equivalent to asking which <u>tableaux</u> commute.

<u>EXAMPLE:</u> $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \cdot \boxed{2} = \boxed{2} \cdot \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$

We can <u>encode</u> tableaux differently...

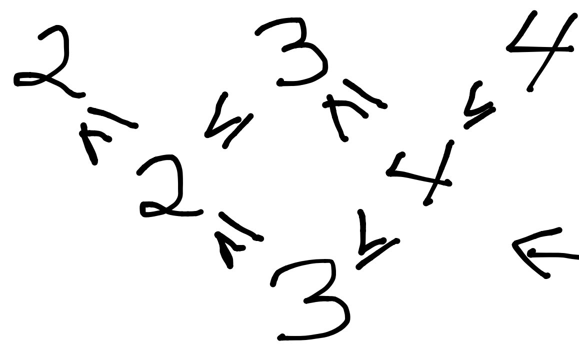
<u>Gelfand-Tsetlin patterns</u>

$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&2&3\\ \cline{1-3} 3&3\\ \cline{1-2}\end{array} \xrightarrow{1\text{'s}} \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline\end{array}$ shape 3

$\xrightarrow{1,2\text{'s}} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&2\\ \cline{1-2}\end{array}$ shape 42

shape 432

Record the sequence of shapes:

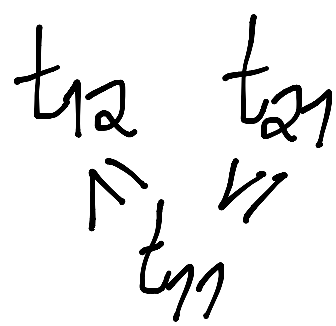


certain inequalities are forced

For only 2 letters,

$$\begin{array}{|l}
\overbrace{1\,1\cdots1}^{t_{11}}\,2\,2\cdots\cdots2 \\
\underbrace{2\,2\cdots2}_{t_{22}}
\end{array}$$

$$\underbrace{\hspace{6cm}}_{t_{21}}$$



What inequality characterizes commutation of

$$T = \begin{array}{cc} t_{12} & t_{21} \\ \multicolumn{2}{c}{t_{11}} \end{array} \qquad T' = \begin{array}{cc} t'_{12} & t'_{21} \\ \multicolumn{2}{c}{t'_{11}} \end{array} \; ?$$

$$\min(t_{21} - t_{11},\, t'_{11} - t'_{22})$$
$$= \min(t'_{21} - t'_{11},\, t_{11} - t_{22})$$

$$22\cdots 211\cdots 122\cdots 2 \bullet 22\cdot 211\cdot 122 \cdot 2$$

Variants - Google their definitions!

i.e. other monoids

- shifted plactic monoid
- hypoplactic monoid
- Sylvester monoid
- Hecke monoid

REU Sub PROBLEM 1b.

EXERCISE 4:

- Google for the shifted plactic monoid (paper by L. Serrano)
- Find the answer for how to characterize commutation when $n=2$ i.e. $\{1<2\}$ in the shifted plactic monoid.