REU Day 3

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Rogers-Ramanujan identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})}$$

(1st R-R identity)

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})}$$

(2nd R-R identity)

History

Ramanujan's letter to Hardy 1913.

P.A. Macmahon gave a combinatorial interpretation.

Rogers in 1894 had already given a proof.

Rogers & Ramanujan wrote a paper with another proof.

Schur in 1917 independently found them, with two proofs!

They appear lots of places - number theory, representation theory, etc.

Macmahon (1st R–R)

THM:

# of partitions of n into distinct parts, with differences at least 2

= # of partitions of n into parts congruent to 1 or 4 mod 5

EXAMPLE: $n=9$

$$\left.\begin{array}{ll} 9 & 9 \\ 81 & 6111 = 61^3 \\ 72 & 4^2 1 \\ 63 & 41^5 \\ 531 & 1^9 \end{array}\right\}\ \text{5 of each!}$$

Why is Macmahon's Thm same as 1^{st} R-R?

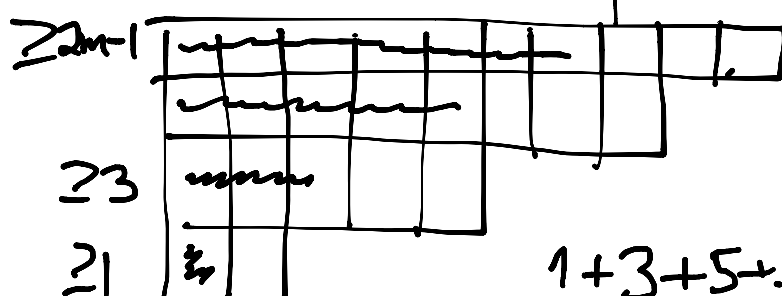
RHS (of 1^{st} R-R)

$$= \frac{1}{(1-q^1)(1-q^4)(1-q^6)\cdots\cdots}$$

$$= \left(\sum_{j=0}^{\infty}(q^1)^j\right)\left(\sum_{k=0}^{\infty}(q^4)^k\right)\cdots\cdots$$

$$= \sum_{n\geq 0} q^n \cdot (\text{Macmahon's RHS for } n)$$

What about partitions on LHS of MacMahon? If m parts total



$$1+3+5+\ldots+2m-1 = m^2$$

REU Problem 3(a):

Find a proof of 1^{st} R-R that at each step has only positive terms.

Use your proof to give a bijection for Macmahon's Thm.

None known, as of today (June 3, 2015)

A proof of 1st R-R due to Bressoud
(but with negative terms, and cancellation...)

TWO FACTS:

1. EXERCISE 8:

(Finite q-binomial theorem)

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k = (1+x)(1+xq)\cdots(1+xq^{N-1})$$

for $N \geq 0$ integer

q-binomial coefficient

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{N!_q}{k!_q (N-k)!_q} \quad \text{where } N!_q := 1(1+q)(1+q+q^2)\cdots(1+q+\dots+q^{N-1})$$

$\in \mathbb{N}[q]$

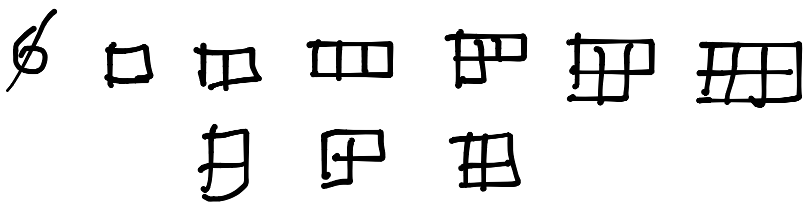
$=$ generating function for $\lambda \subset {}_{k}\boxed{}^{N-k}$

EXAMPLE: $k=2 \quad N=5$

$$\begin{bmatrix}5\\2\end{bmatrix}_q = \frac{5!_q}{2!_q\,3!_q} = \frac{[5]_q[4]_q}{[2]_q[1]_q}$$

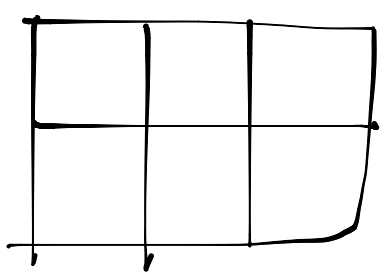
$$= \frac{(1+q+q^2+q^3+q^4)(1+q+q^2+q^3)}{(1+q)(1)}$$

$$= 1+q+2q^2+2q^3+2q^4+q^5+q^6$$



$N-k=3$

$k=2$



2. EXERCISE 9:

(Jacobi Triple Product Identity)

$$\sum_{n=-\infty}^{+\infty} q^{n^2} x^n = (q^2;q^2)_\infty (-qx;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty$$

where

$$(A;q)_\infty := \prod_{m=0}^{\infty} (1-Aq^m)$$

$$= (1-A)(1-Aq)(1-Aq^2)\cdots$$

[Also define the useful notation

$$(A;q)_N := \prod_{m=0}^{N-1} (1-Aq^m)$$
]

e.g. $(-x;q)_N = (1+x)(1+xq)\cdots(1+xq^{N-1})$

Bressoud's proof:

$$P_n(z;a) := \sum_{m=-n}^{n} \begin{bmatrix} 2n \\ n-m \end{bmatrix}_q q^{am^2} z^m$$

a Laurent polynomial in z.

EXERCISE 10: Prove this

PROP: $$\frac{P_n(z;a)}{(q;q)_{2n}} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}} \frac{P_s(z;a-1)}{(q;q)_{2s}}$$

How would this help?

Let $a=\frac{1}{2}$, and replace

$$z \mapsto -zq^{1/2}$$

Then

$$p_n(z;a) = \sum_{m=-n}^{n} \begin{bmatrix} 2n \\ n-m \end{bmatrix}_q q^{\frac{m^2+m}{2}} (-z)^m$$

q binomial thm

$$= \frac{1}{(-z)^n} (1-qz)(1-q^2z)\cdots(1-q^nz) \times (1-z)(q-z)\cdots(q^{n-1}-z)$$

$$\xrightarrow{z=1} 0$$

$$\frac{p_n(z;a)}{(q;q)_{2n}} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}} \sum_{s_1=0}^{s} \frac{q^{s_1^2}}{(q;q)_{s-s_1}} \frac{p_{s_1}(z;a-2)}{(q;q)_{2s_1}}$$

apply EXER 10 twice

$a = 5/2 \quad z = -q^{1/2}$

$$\frac{p_n(z;a)}{(q;q)_{2n}} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}(q;q)_s}$$

Let $n \to \infty$

$$p_n(z;a) = \sum_{m=-n}^{n} \begin{bmatrix} 2n \\ n-m \end{bmatrix}_q q^{\frac{5}{2}m^2} \left(-q^{1/2}\right)^m$$

$$\begin{bmatrix} 2n \\ n-m \end{bmatrix}_q \xrightarrow{n\to\infty} \frac{1}{(q;q)_\infty}$$

Jacobi Triple Product

$$\longrightarrow \frac{1}{(q;q)_\infty} (q^5;q^5)_\infty (q^3;q^5)_\infty (q^2;q^5)_\infty$$

$$= \frac{1}{(q^1;q^5)_\infty (q^4;q^5)_\infty}$$

Meanwhile

$$\frac{p_n(z;a)}{(q;q)_{\cancelto{\infty}{2n}}} = \sum_{s=0}^{\cancelto{\infty}{n}} \frac{q^{s^2}}{(q;q)_{\cancelto{\infty}{n-s}}(q;q)_s}$$

$$\xrightarrow{n\to\infty} P_n(z;a) = \sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q)_s} \quad \blacksquare$$

Here is what Schur proved:

$$\sum_{j=0}^{\frac{n+1}{2}} \begin{bmatrix} n+1-j \\ j \end{bmatrix}_q q^{j^2} =$$

$$\sum_{\text{all } j} (-1)^j q^{\frac{j(5j+1)}{2}} \begin{bmatrix} n+1 \\ \left\lfloor \frac{n+1-5j}{2} \right\rfloor \end{bmatrix}_q$$

$\Big\downarrow n \to \infty$ $\qquad\qquad\qquad$ $\Big\downarrow n \to \infty$

LHS of (1^{st} R-R) $\qquad$ RHS of (1^{st} R-R)

$\blacksquare$

Involutions

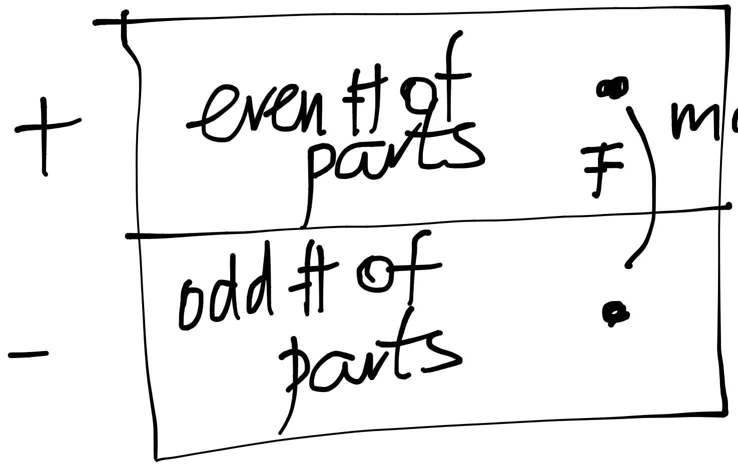
Euler's pentagonal number theorem:
(EPNT)

$$\prod_{n=1}^{\infty}(1-q^n) = (q;q)_\infty = 1-(q+q^2)+(q^5+q^7)-\dots$$
$$= \sum_{k=-\infty}^{+\infty} q^{\frac{k(3k-1)}{2}}(-1)^k$$

F. Franklin's involution proves this...

Consider all partitions with distinct parts counted with + if it has an even # of parts

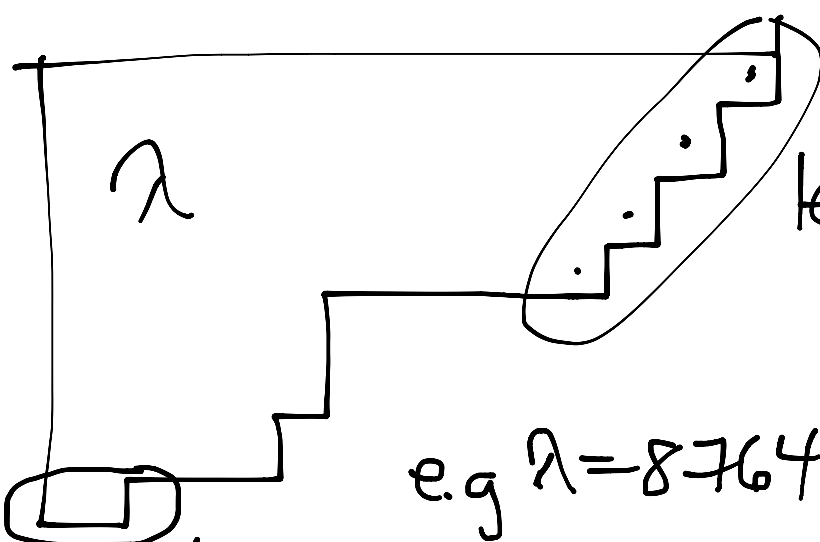
− if it has an odd # of parts



match them up, via Franklin's involution F

F will change # parts by <u>one</u>.

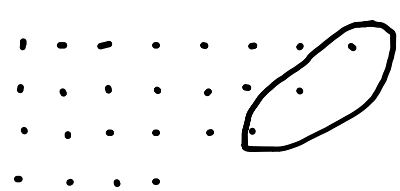
Picture for F:



leading run of λ

smallest part of λ

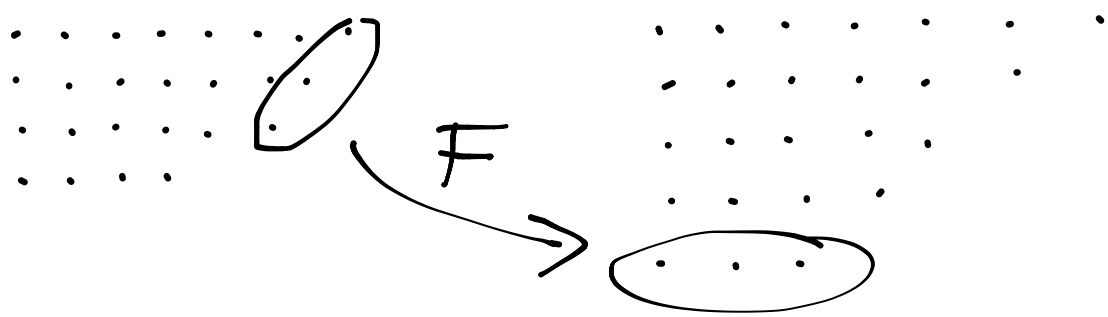
e.g. $\lambda = 8764$



If the leading run is < smallest part, move it to make it a new smallest part.

If the leading run is $\geq$ smallest part, move the smallest part to make a new leading run.

$F(8764) = 76543$



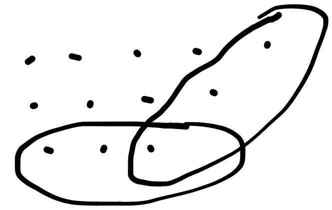
When does this fail?

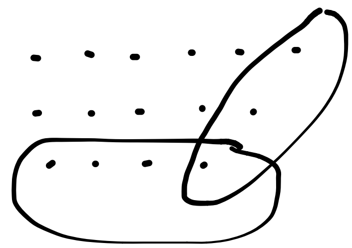
$$\lambda = (2k-1)(2k-2)\cdots k$$

or

$$\lambda = 2k(2k-1)\cdots k+1$$

$k=3$





We want something like this for R-R.

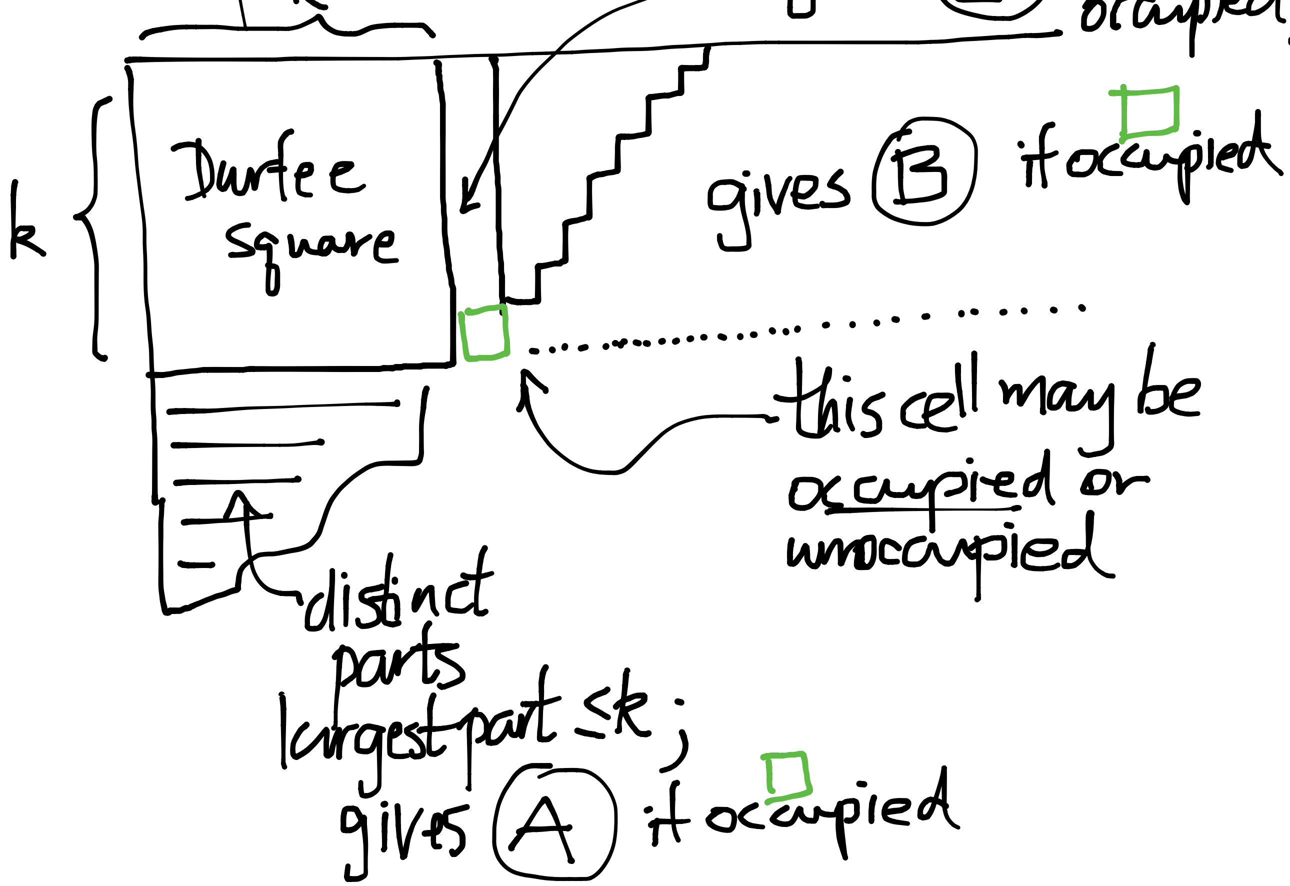
First, an identity of Sylvester.

$$(-aq;q)_\infty = 1+\sum_{k=1}^{\infty} q^{k^2} q^{\binom{k}{2}} a^k \left(\underset{E}{q^k} \frac{\overset{A}{(-aq;q)_k}}{\underset{B}{(q;q)_k}} + \frac{\overset{C}{(-aq;q)_{k-1}}}{\underset{D}{(q;q)_{k-1}}} \right)$$

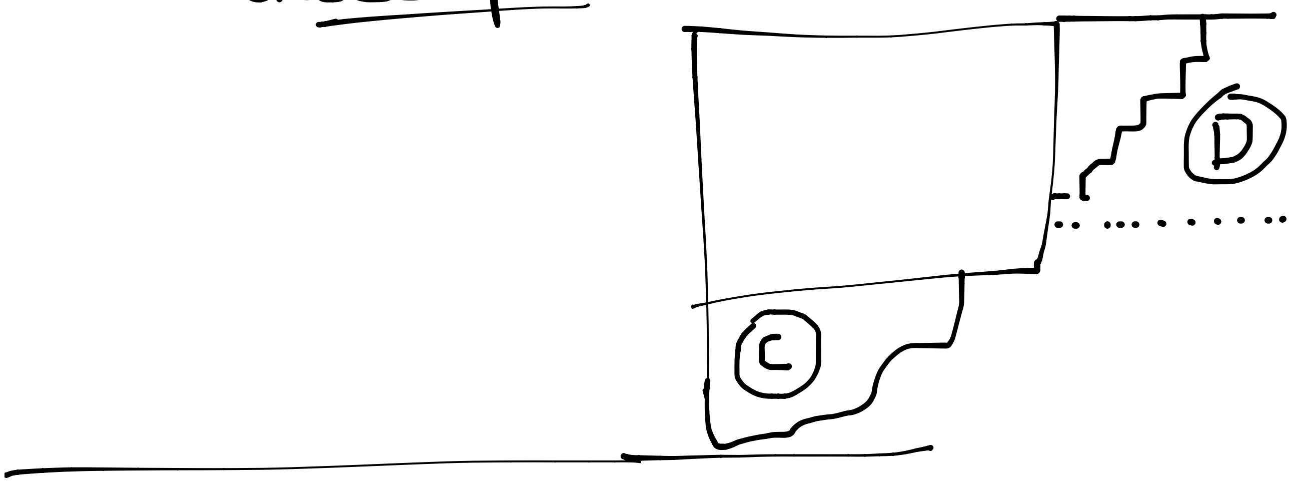
proof:

$$\text{LHS} = \sum_{\substack{\theta \text{ a partition} \\ \text{with distinct} \\ \text{parts}}} q^{\|\theta\|} a^{\#\text{parts of } \theta} = \prod_{k=1}^{\infty} (1 + a q^k)$$

For the RHS, consider the Durfee square of the partition θ:



The (C), (D) come similarly from the case where that cell □ is <u>unoccupied</u>:



EXERCISE 11: Prove via Durfee squares:

$$(-aq;q)_N = 1 + \sum_{k=1}^{N} q^{k(3k-1)/2} a^k \times \left(q^k (-aq;q)_k \begin{bmatrix} N-k \\ k \end{bmatrix}_q + (-aq;q)_{k-1} \begin{bmatrix} N-k \\ k-1 \end{bmatrix}_q \right)$$

FACT:

$$(-aq;q)_\infty \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}(-a)^k =$$

$$1+\sum_{k=1}^{\infty} q^{k^2+\binom{k}{2}} a^k \cdot q^{k^2}(-a)^k$$

$$\times\left(q^k\frac{(-aq;q)_k}{(q;q)_k}+\frac{(-aq;q)_{k-1}}{(q;q)_{k-1}}\right)$$

$$\Big\downarrow \; a=-1$$

gives R-R

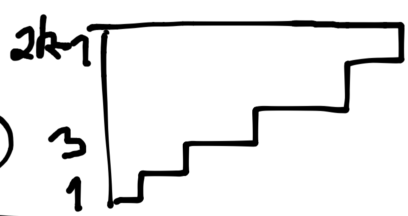
(with cancellation)

We'd like to prove this fact via an involution. How to set this up...

$$\underbrace{PD(a)}_{\substack{\text{partitions}\\ \text{with distinct}\\ \text{parts, parts}\\ \text{weight by } a}} \times \underbrace{RR(-a)}_{\substack{\text{Rogers-Ramanujan}\\ \text{partitions, each}\\ \text{part weighted by } (-a)}}$$

involution?

Want the fixed points to correspond to pairs

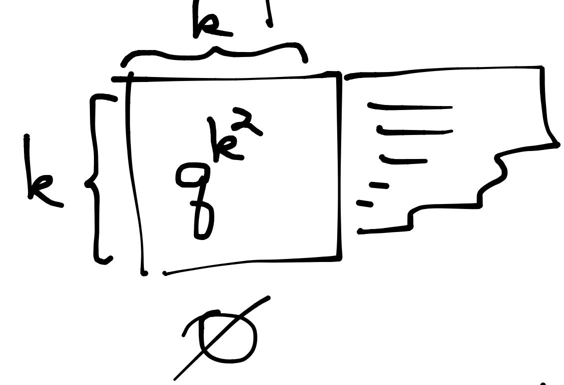
(λ , 

) with k = Durfee square size of λ

REU Problem 3(b): Find such an involution.

NOTE:

$$RR(-a) = \underbrace{SDS}_{\text{single Durfee square partitions}}(-a)$$



so one might try to work with these instead, and having fixed points corresponding to

$$\left\{ \left(\lambda, \underbrace{\boxed{}}_{k} \Big\} k \right) : k = \text{Durfee square size of } \lambda \right\}.$$