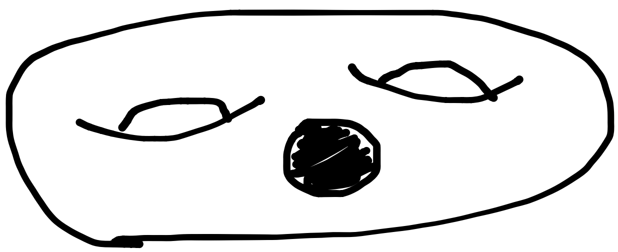
REU Day 5

P. Pylyavskyy

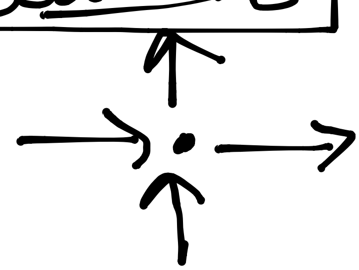
Networks on surfaces

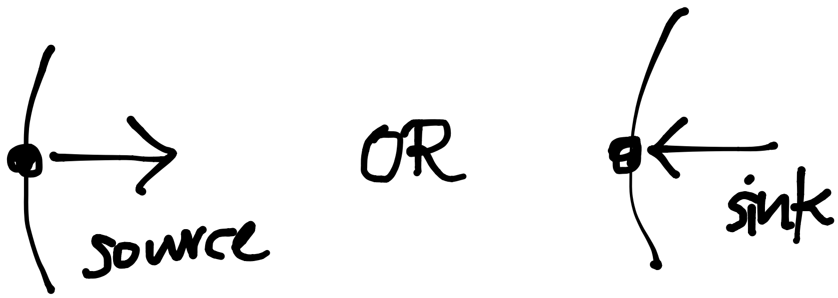


surface, with boundary

We'll want networks on them in which each internal vertex

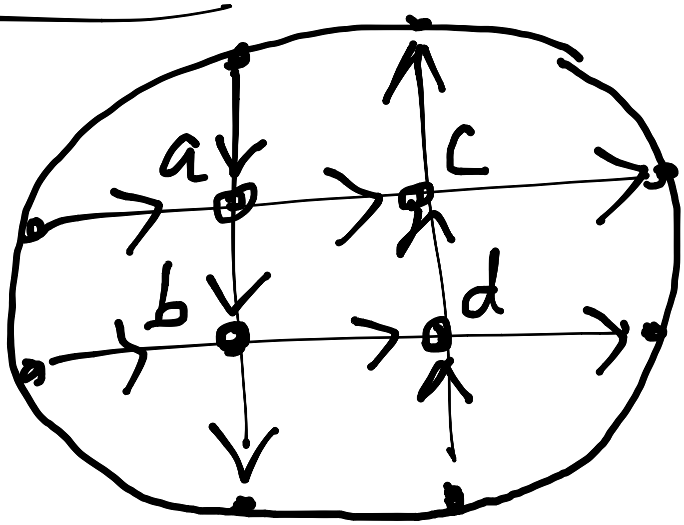
not on the boundary

looks like this:  , and each boundary vertex looks like this:



Internal vertices are given (variable) weights.

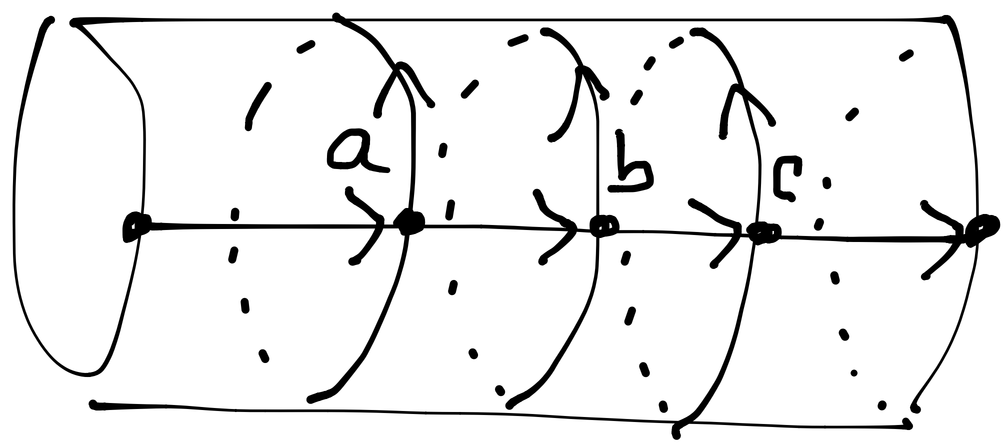
EXAMPLE 1:



surface = disk

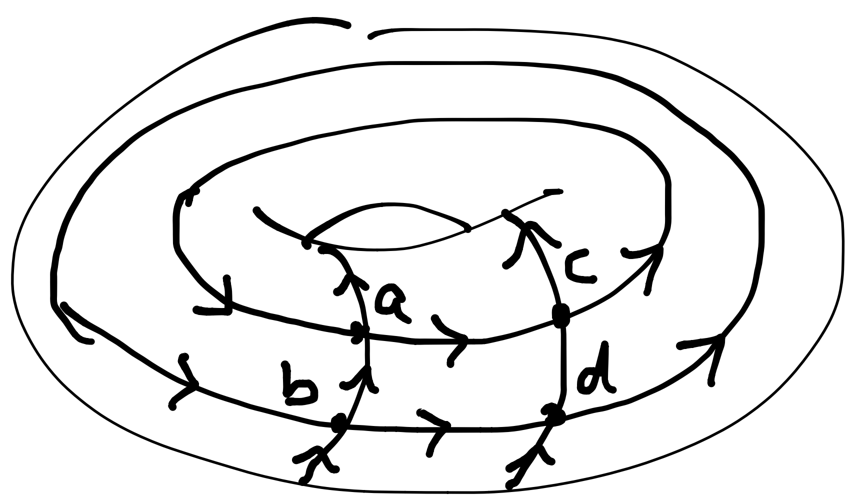
EXAMPLE 2:

surface = cylinder

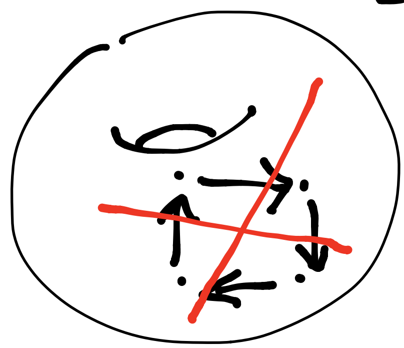


EXAMPLE 3:

surface = torus



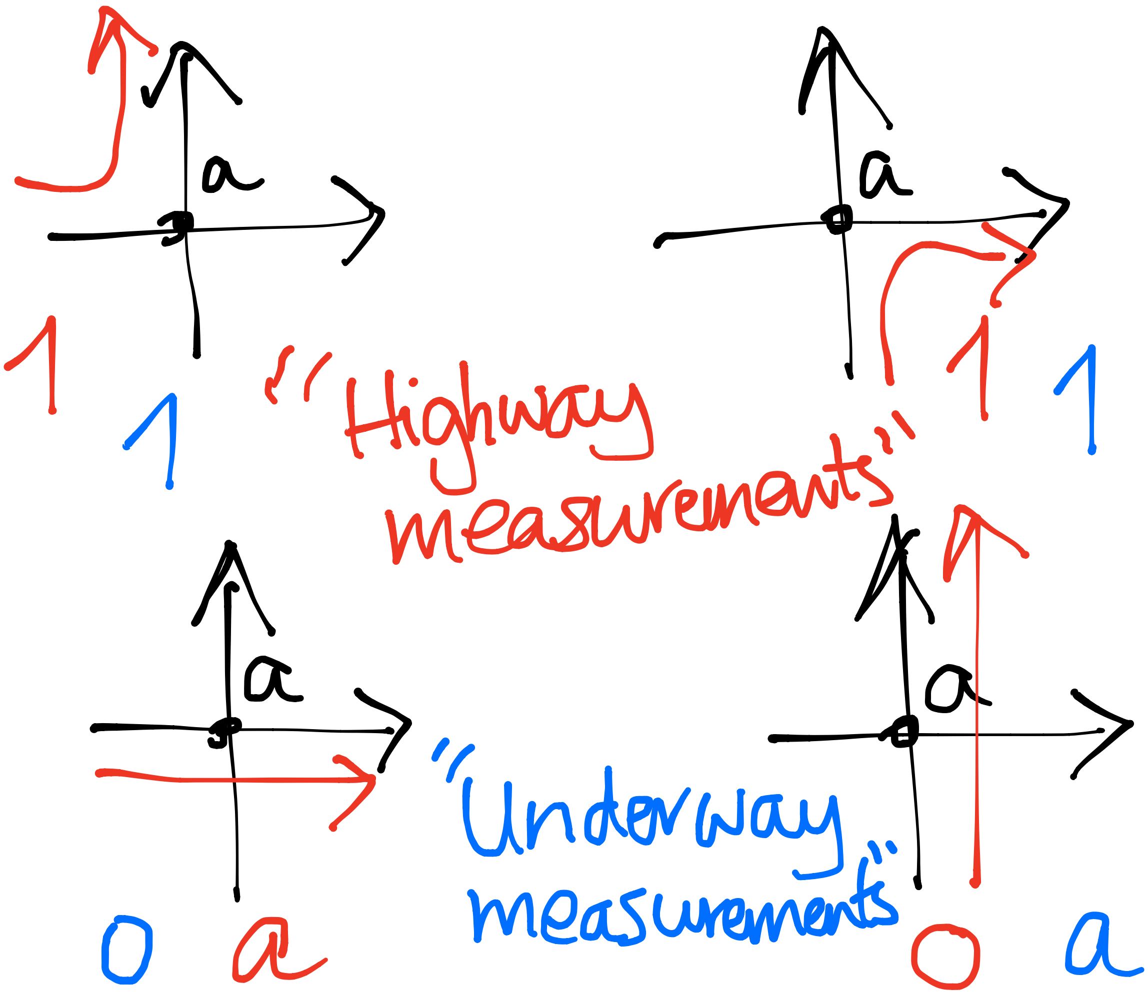
No directed contractible cycles are allowed:



Measurements

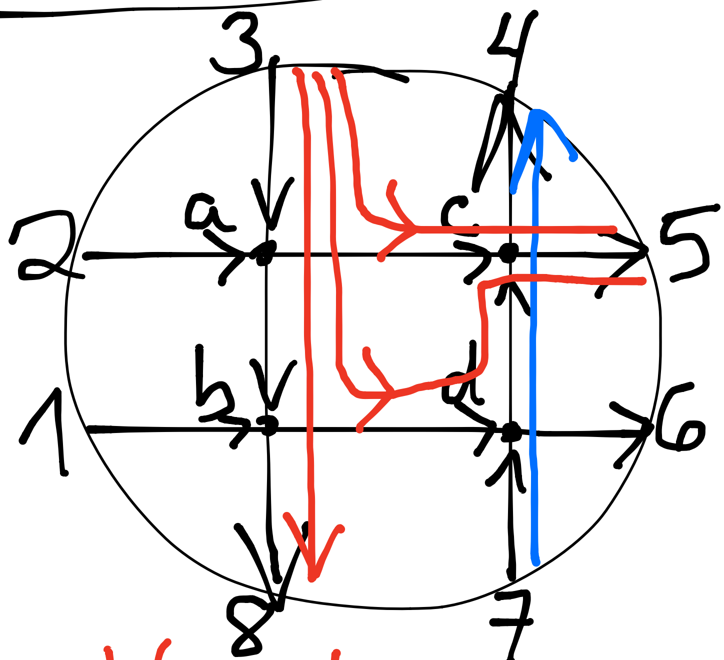
- are associated to homotopy classes of walks from sources to sinks, or closed (non-contractible) walks.

Here is how you pick up the variable weights along walks...



distinguishable because the surface is assumed orientable.

EXAMPLE 1:



weight: ab, a+c, b, d,

$3 \to 8$, $3 \to 5$, $2 \to 8$, $2 \to 6$

ad

$3 \to 6$

Highway

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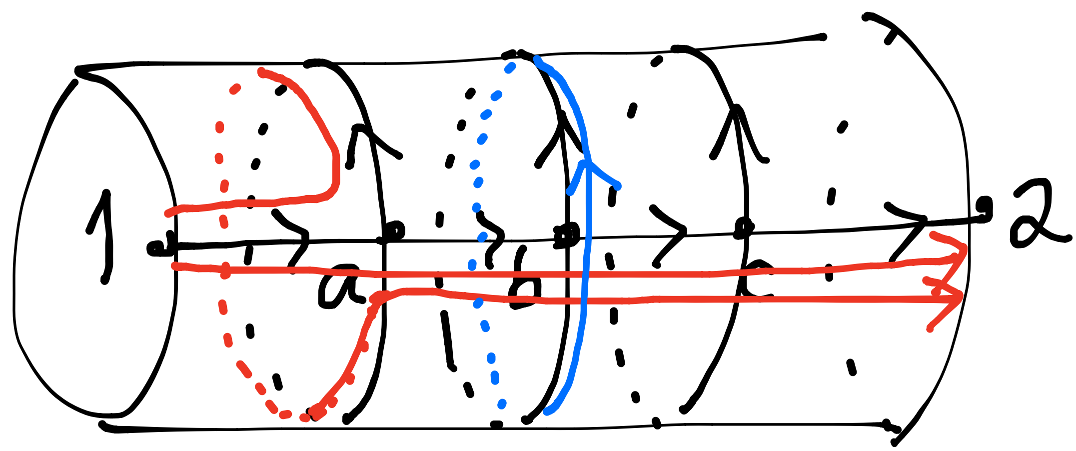
weight: cd, a+c, b, d, bc

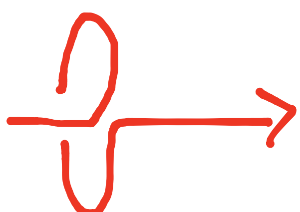
$7 \to 4$, $2 \to 4$, $1 \to 5$, $7 \to 5$, $1 \to 4$.

Underway

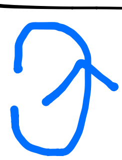
EXAMPLE 2:



$\longrightarrow$ has weight abc

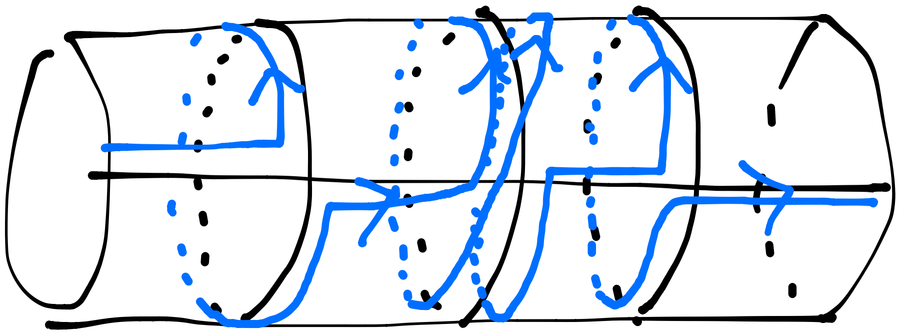
 has weight $ab+ac+bc$

 has weight $a+b+c$

 has weight $a+b+c$

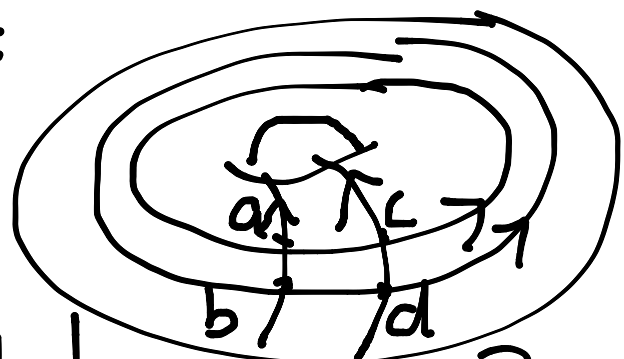
 $\dfrac{a^2+b^2+c^2}{2}$

 $\dfrac{a^3+b^3+c^3}{3}$

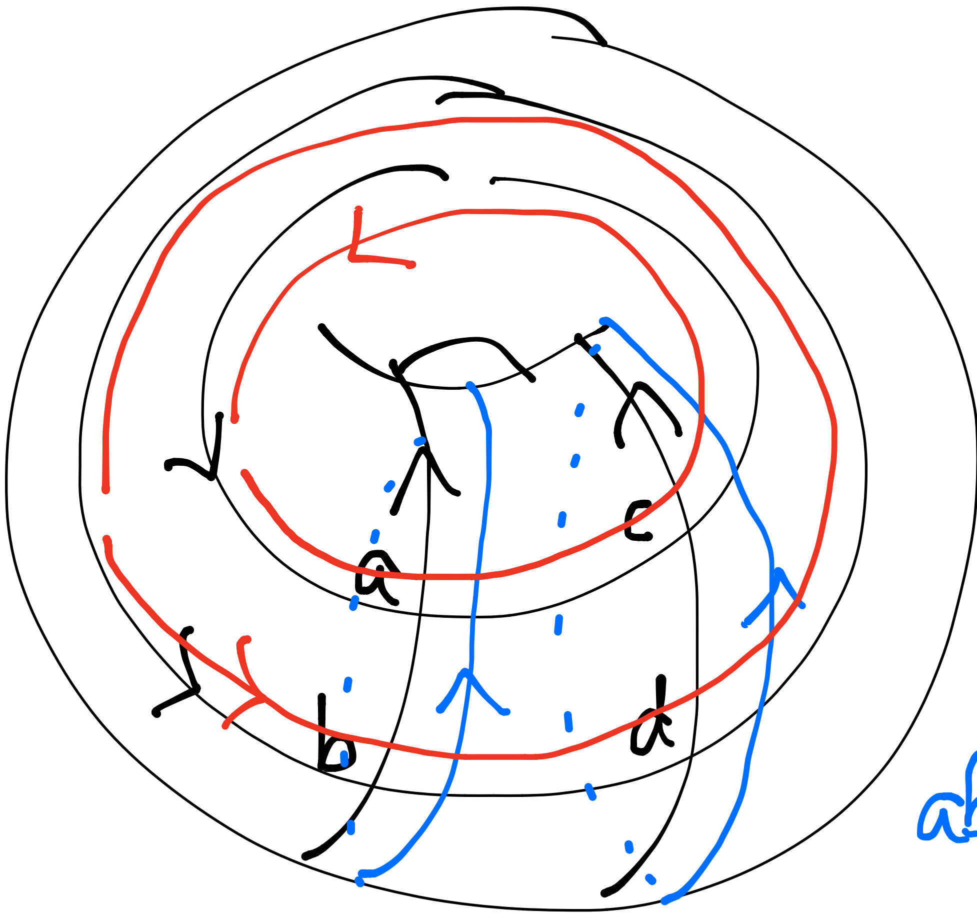


$$a^2+b^2+c^2+ab+ac+bc$$

EXAMPLE 3:

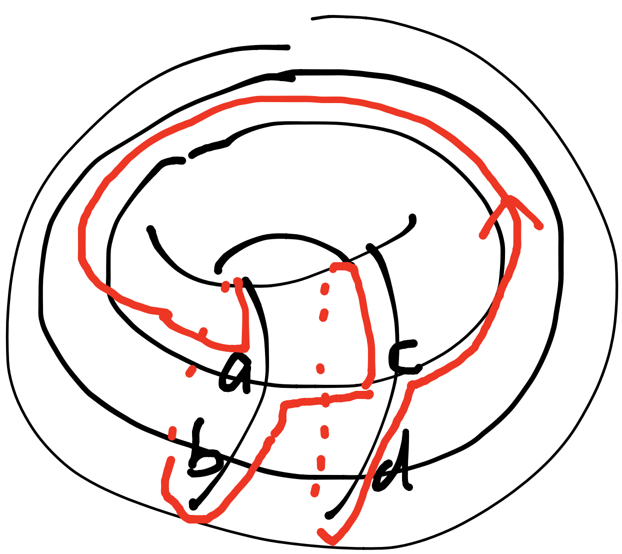


#times around ↺ \ #times around	0	1	2	3
0	dull	$ab+cd$	$\frac{a^2b^2+c^2d^2}{2}$	...
1	$ac+bd$	dull	$bd+ac+ab+cd$	
2	$\frac{a^2c^2+b^2d^2}{2}$	$ab+bd+ac+cd$	dull	
3	⋮			dull



ab+cd

ac+bd



bd+ac
+ab+cd

CONJECTURE:

In characteristic zero, the algebra generated by the highway measurements is the same as the algebra generated by the underway measurements.

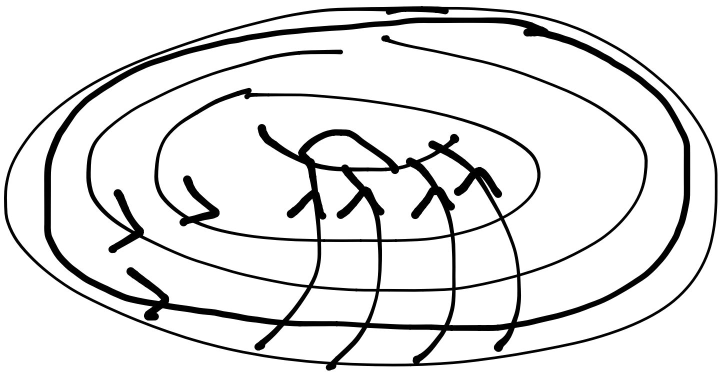
REU Problem 5a:

Prove this CONJECTURE for an (n, m, k) torus network

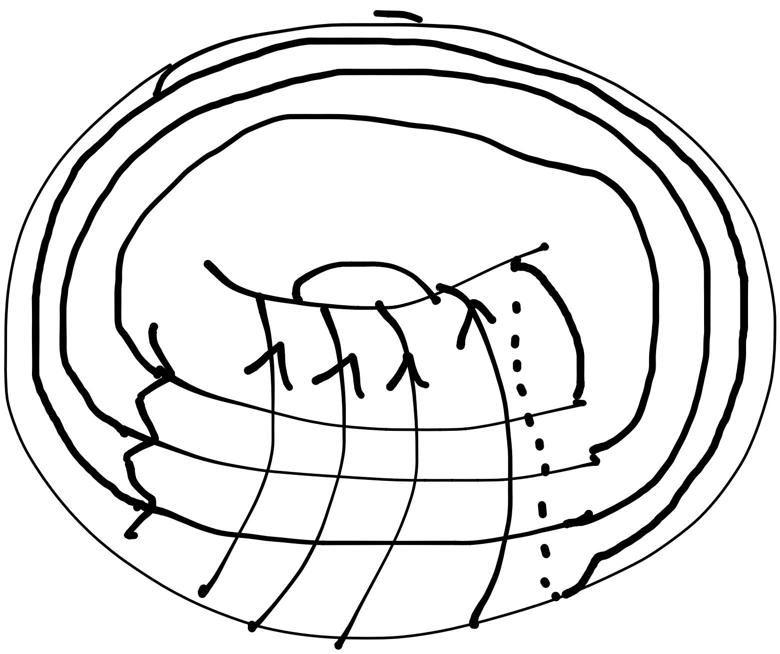
n horizontal cycles

m vertical cycles

$\frac{k}{n}$ Dehn twists before reconnecting



(n, m, k)
(3, 4, 0)

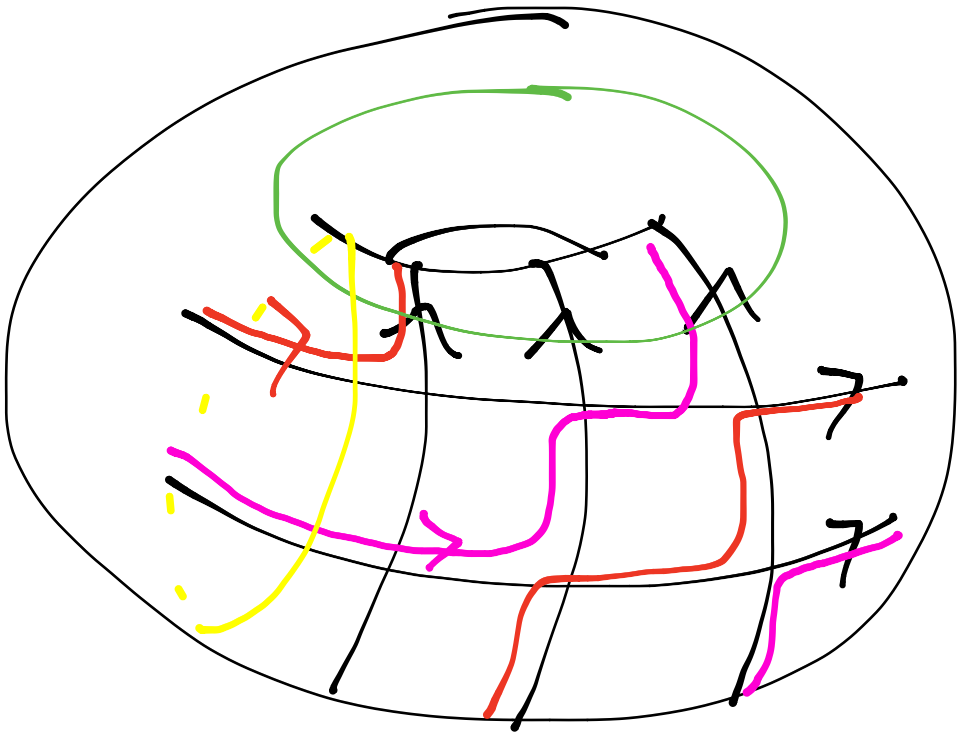


(n, m, k)
(3, 4, 1)

Now for more than one path....

Say 2 paths are <u>noncrossing</u> if they have no common edges
(so a closed path that goes around twice crosses itself!)

The 2^{nd} kind of measurement sums over families of noncrossing paths with <u>fixed</u> <u>overall</u> <u>topology</u>.



Let a = # crossings with 

b = # crossings with

EXERCISE 14: Determine for which (a,b) a measurement <ins>exists at all</ins>.

EXERCISE 1.5(a): Prove one gets the same measurement here using highway or underway rules.

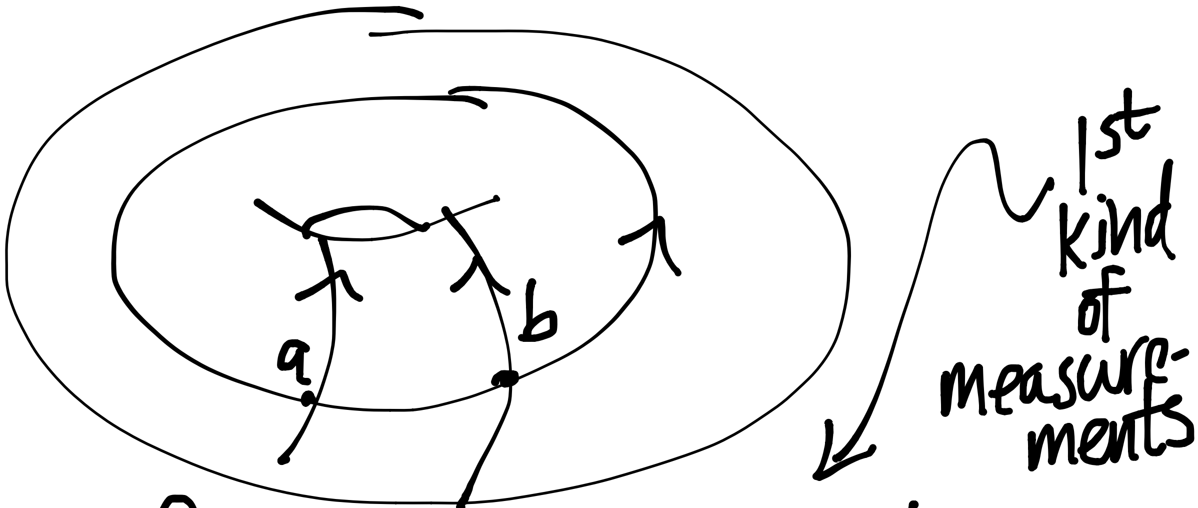
EXERCISE 1.5(b):

Prove the measurements of the 1st and 2nd kinds generate the same algebra. Can you give formulas expressing the generators in terms of each other?

REU Problem 5(c):

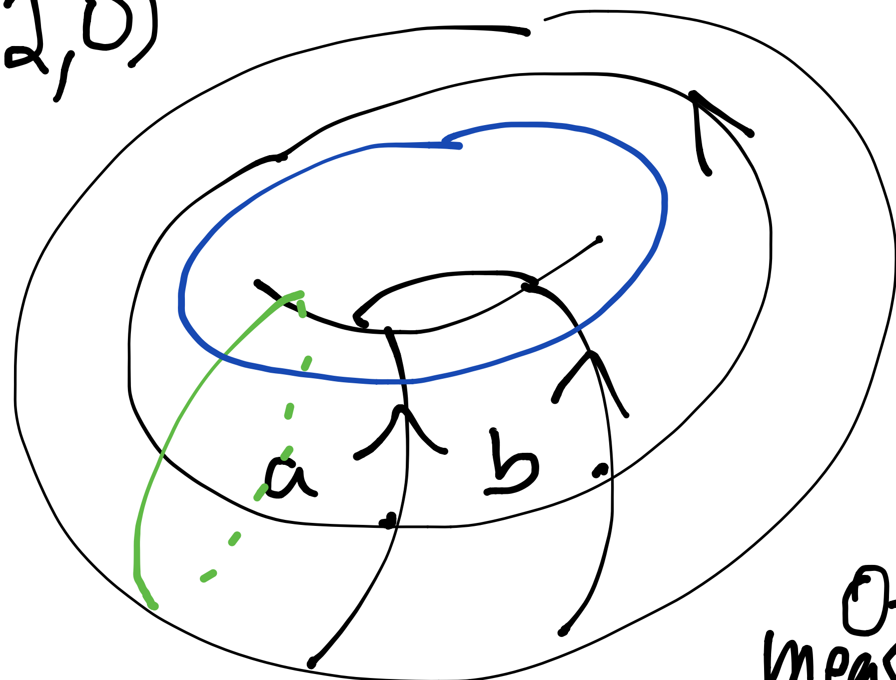
Prove that the nonconstant measurements of the 2nd kind are algebraically independent.

EXAMPLE: $(n,m,k) = (1,2,0)$



# ↻ \ # ↑	0	1	2	3
0	$*$	$a+b$	$\frac{a^2+b^2}{2}$	$\frac{a^3+b^3}{3}$
1	ab	$a+b$	$*$	$a+b$
2		ab^2+a^2b		

(1,2,0)



2nd kind of measurements

	0	1	2	3
0	1	0	0	0
1	ab	a+b	1	0
2	0	0	0	0

Note that $ab, a+b$

generates the same algebra as

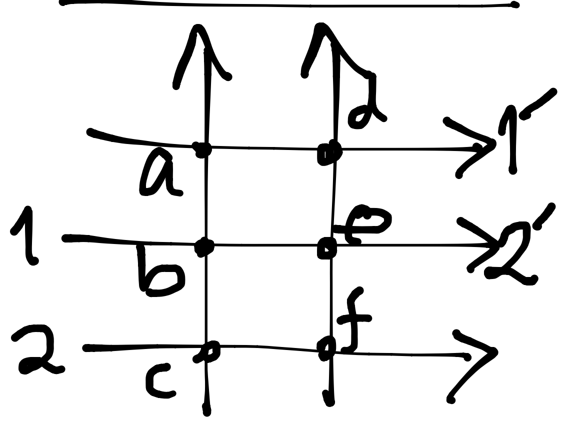
$a+b, \frac{a^2+b^2}{2}, \frac{a^3+b^3}{3}, \ldots$

which is also the same algebra

generated by

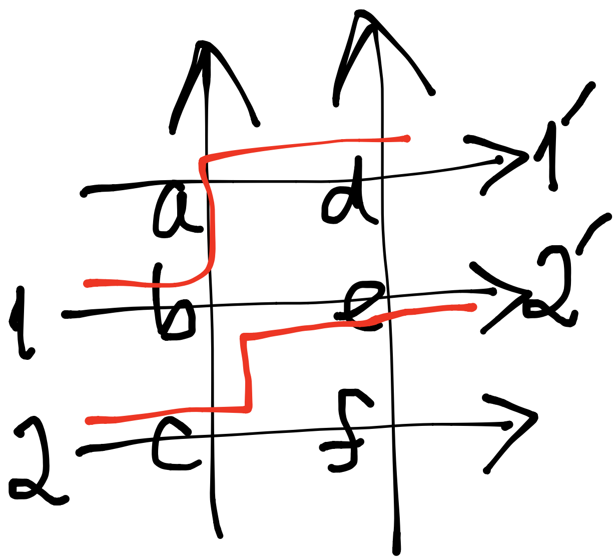
$ab, a+b, ab^2+a^2b, \ldots$

The disk case



2^{nd} kind of highway measurement

$(1,2) \to (1',2')$



$dc + bc + de$

What about 1^{st} kind of measurements?

"Lindström Lemma" (Gessel-Viennot method)

$1 \to 1'$ $b+d$

$1 \to 2'$ be

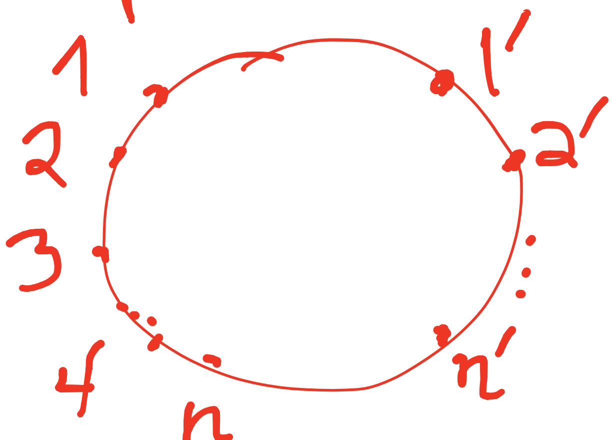
$2 \to 1'$ 1

$2 \to 2'$ ce

$$dc+bc+de = \det \begin{array}{c c} & \begin{array}{cc} 1' & 2' \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} b+d & be \\ 1 & c+e \end{bmatrix} \end{array}$$

EXERCISE 16: Prove this,

i.e. when the sources $1, 2, \ldots, n$ and sinks $1', 2', \ldots, n'$ are separated on the disk boundary



the 2^{nd} kind boundary measurement $(1, 2, \ldots, n) \to (1', 2', \ldots, n') = \det[a_{ij}]$

where a_{ij} is the 1^{st} kind boundary measurement $i \to j'$.