(1) 2015

# REU Day 9

## Monotone paths in zonotopes (Ref: PhD thesis of R. Edman)

① polytopes ~~[illegible]~~ (Ref: Ziegler's "Lectures on polytopes")

② functionals & monotone paths

(– motivation from LP)

– flip graphs

③ zonotopes

④ coherence

⑤ REU problem ~~[illegible]~~

⑥ Tools

- deletion / contraction
- duality

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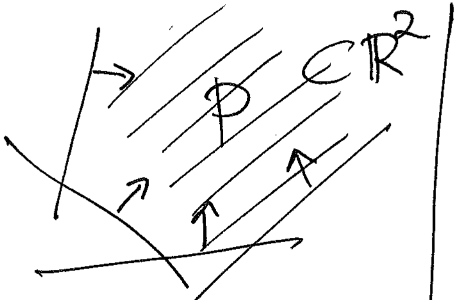
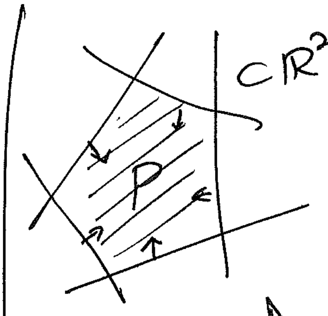
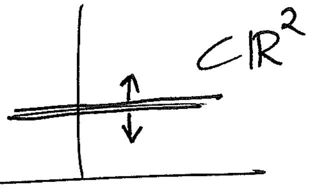
① A polyhedron $P \subset \mathbb{R}^d$ is a finite intersection

$$P = \bigcap_{i=1}^{t} \underbrace{H_i^+}_{\text{half-spaces}} \qquad H_i^+ = \{ x \in \mathbb{R}^d : \underset{\underset{a_1x_1+\dots+a_dx_d \in (\mathbb{R}^d)^*}{\shortparallel}}{f_i(x)} \le c_i \}$$

e.g.

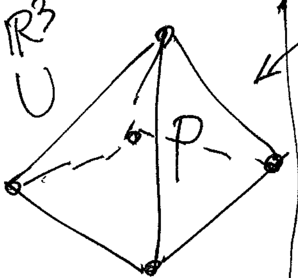






NOTE: Not every convex cone is a polyhedron:

cf. REU Problem

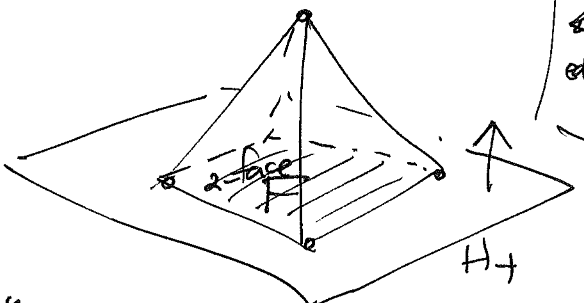


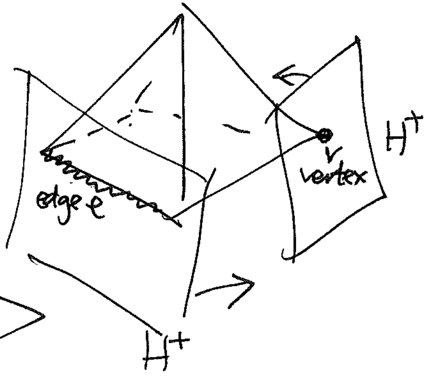
these two are polytopes := bounded polyhedra $\overset{\text{THM}}{=}$ convex hull of finitely many points $\{v_1, v_2, \dots, v_p\} \subset \mathbb{R}^d$

A face of a polytope/polyhedron is an intersection $F = P \cap H^{=}$

where $H^+ = \{x : f(x) \le c\}$ is any supporting ~~[illegible]~~ halfspace i.e. $P \subset H^+$

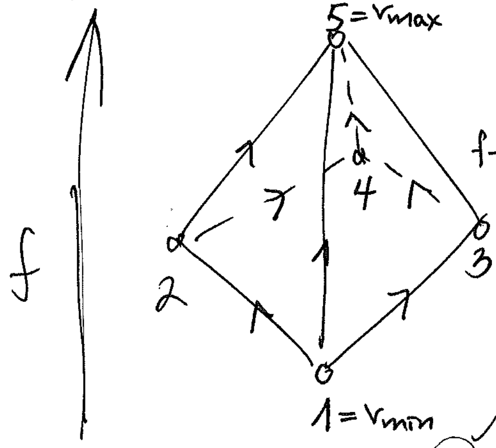
and $H^{=} := \{x : f(x) = c\}$





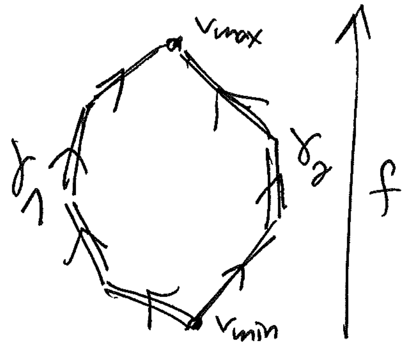
Monotone paths

② ~~Say~~ DEF'N: a (linear) functional $f \in \mathbb{R}^*$ is (edge-)generic on P if it's not constant on an edge of P, so it orients them all:



Call a path $\gamma$ in P ~~[illegible]~~ from $v_{min}$ to $v_{max}$ along edges f-directed ~~[illegible]~~ f-monotone path, ~~always~~

EXAMPLE: A polygon P will have only 2 such $\gamma$:



? -OR-

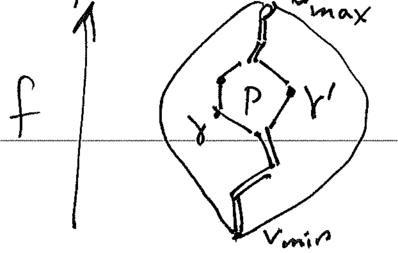
Insert LP motivation

~~Say 2 [illegible] f-monotone paths $\gamma_1, \gamma_2$ are adjacent along a flip~~

DEF'N: The flip graph $G(P,f) = (V, E)$

$V = \{$all f-monotone paths $\gamma\}$

$E = \{\{\gamma, \gamma'\} : \gamma, \gamma'$ differ only along one polygonal ~~face~~ of $P\}$



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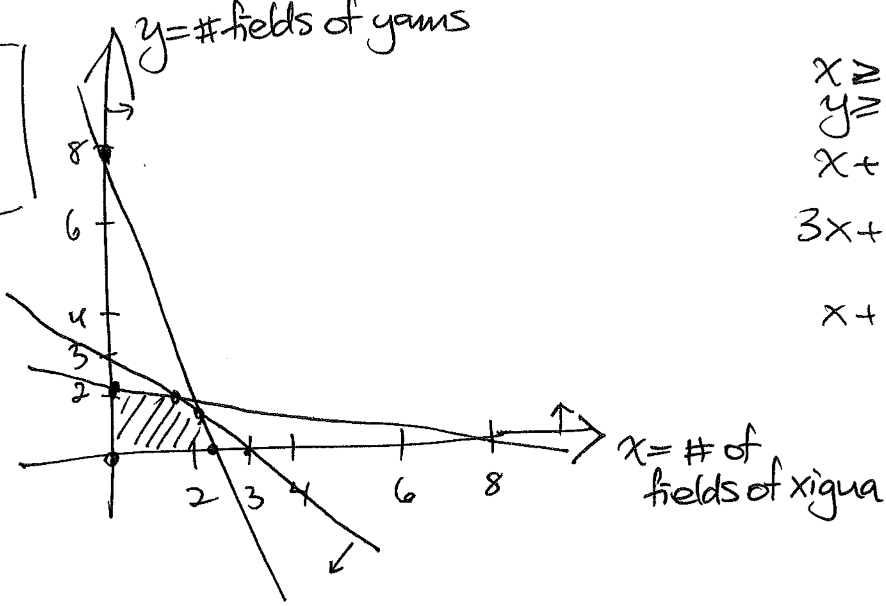
MOTIVATION for f-monotone paths:

Linear programming solves an optimization problem using monotone paths ...

What to plant?



$x \geq 0$

$y \geq 0$

$x+y \leq 3$ (← # of fields available)

$3x+y \leq 8$ (field of xigua takes 3 gallons of water yams 1; 8 available)

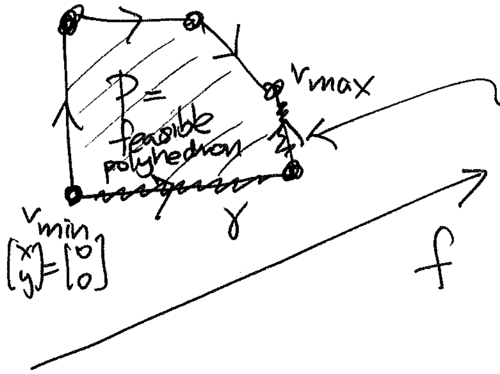
$x+3y \leq 8$ (yam seeds are $3 xigua are $1; $8 available)

maximizing profit

$f(x,y)=\underset{\sim}{c_1}x+\underset{\sim}{c_2}y \in (\mathbb{R}^2)^*$

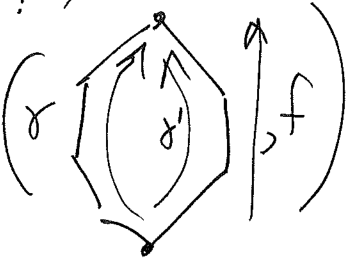
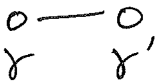
$ profit per xigua field

$ profit per yams field

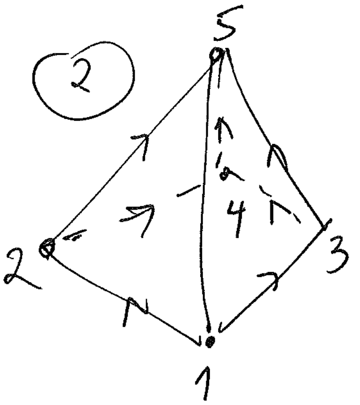


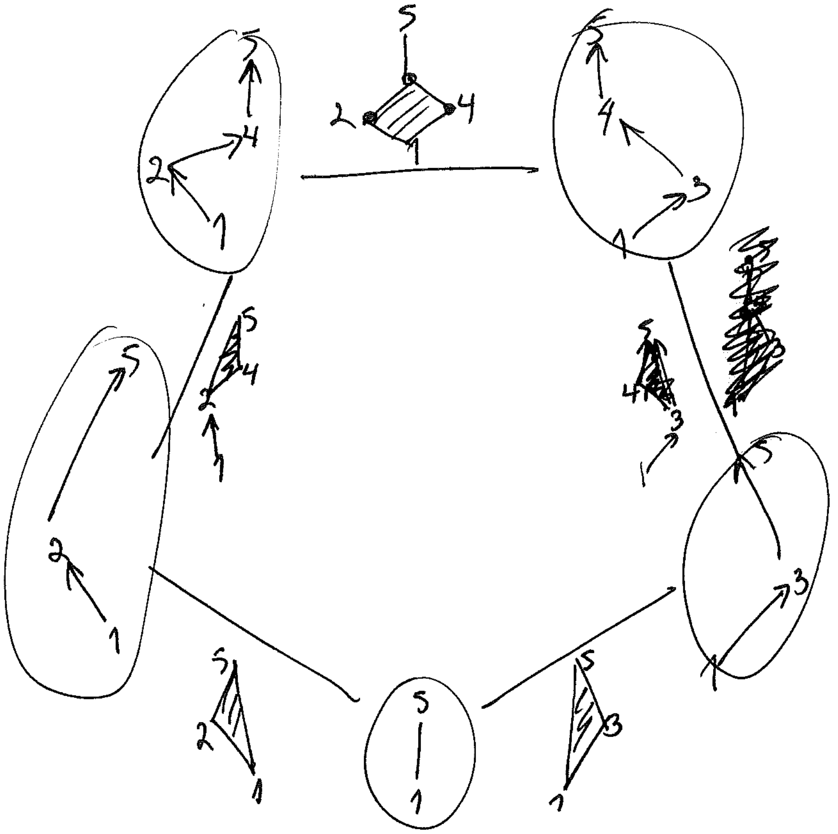
pivoting rules tell you where to go next

## Flip graph $G(P,f)$ EXAMPLES:

① $G\Big(\gamma$  $, f\Big) =$  boring!

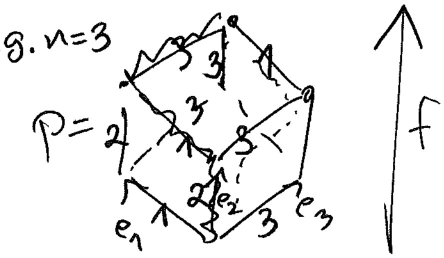
②





③ n-cubes $P = [0,1]^n$

e.g. $n=3$

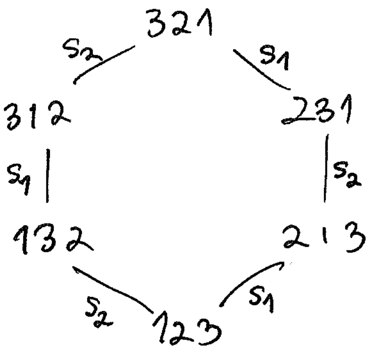
$P =$ 

$\{\text{monotone paths } \gamma\} \longleftrightarrow \{\text{permutations } w_1 \cdots w_n \text{ in } S_n \text{ of } \{1,2,\dots,n\}\}$

$\gamma \longleftrightarrow w=(2,1,3)$

$|$

$\gamma' \longleftrightarrow w=(2,3,1)$



$G(P,f) = \text{Cayley graph}\,(\overset{S_n}{\overset{\parallel}{W}}, \overset{\{s_1,\dots,s_{n-1}\}}{\overset{\parallel}{S}})$

(= 1-skeleton of <u>permutohedron</u>

i.e. vertices $V = \overset{W=S_n}{\text{permutations}}$

edges $E = \{\{w, ws\} : s \in S\}$

(5)

## ③ Zonotopes $P = Z(A)$ where $A = d\left\{\overbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}}^{n}\right. \in \mathbb{R}^{d\times n}$

$:=$ image of $n$-cube $[0,1]^n \subset \mathbb{R}^n$

under a linear map $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^d$

$$e_1 \longmapsto a_1$$
$$\vdots \qquad \vdots$$
$$e_n \longmapsto a_n$$

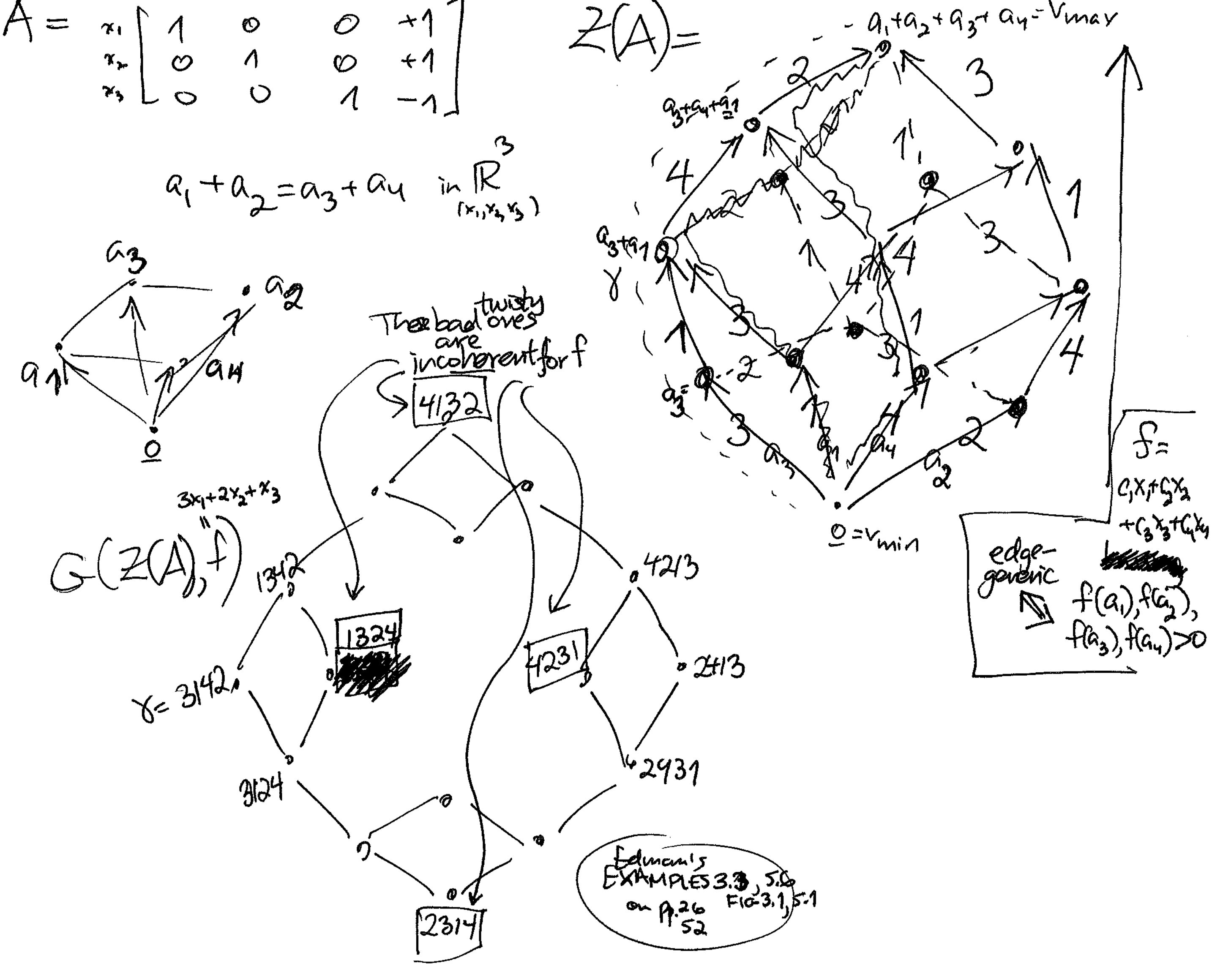
$$[0,1]^n \twoheadrightarrow \underset{\overset{\shortparallel}{P}}{Z(A)} = \left\{\sum_{i=1}^{n} c_i a_i : c_i \in [0,1]\right\}$$

EXAMPLE:

$$A = \begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline x_1 & 1 & 0 & 0 & +1 \\ x_2 & 0 & 1 & 0 & +1 \\ x_3 & 0 & 0 & 1 & -1 \end{array}$$

$d=3$
$n=4$

$a_1 + a_2 = a_3 + a_4$ in $\mathbb{R}^3_{(x_1,x_2,x_3)}$

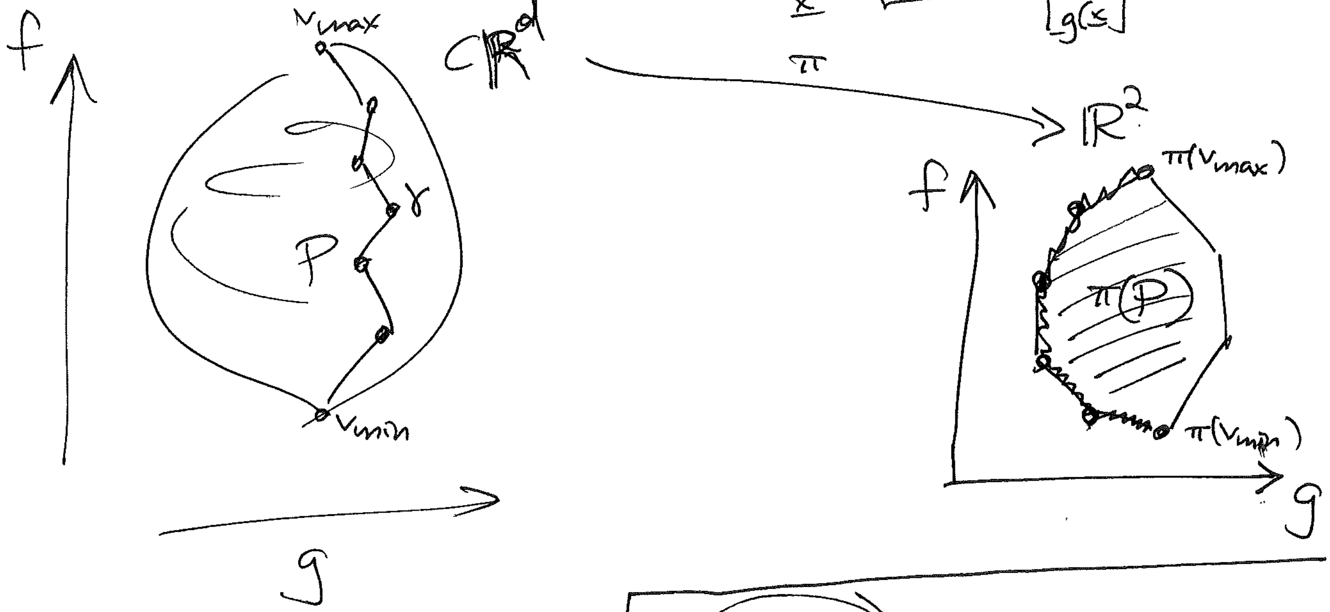


Edman's EXAMPLES 3.3, 5.6 on p. 26, 52 FIG 3.1, 5.1

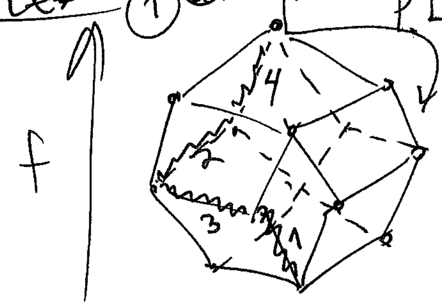
## ④ Coherence

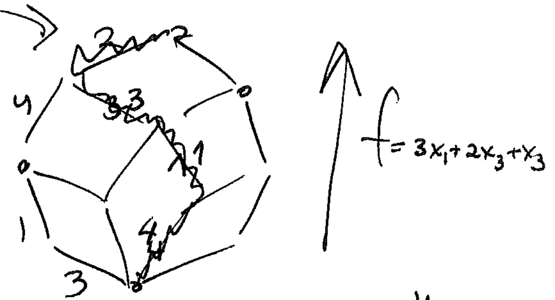
DEF'N: $\gamma$ an $f$-monotone path on $P$ is coherent if $\exists\, g \in (\mathbb{R}^d)^*$ so that $\pi(\gamma)$ is on the ~~upper~~ boundary of $\pi(P)$

for $\begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R}^2 \\ x & \longmapsto & \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} \\ & \pi & \end{array}$



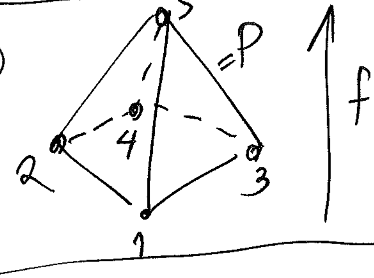
EXAMPLES: ① $\boxed{1324}$, $\boxed{4132}$



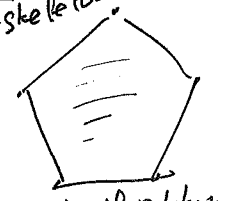


and their opposites are incoherent, but removing them from $G(P,f)$ gives a 12-gon:

②



had all $\gamma$ coherent $\rightsquigarrow$ $G(P,f) =$ 1-skeleton of



2-dim'l polytope

③ n-cubes $\left(\underset{\overset{\shortparallel}{P}}{[0,1]^n}, f\right)$ have all $\gamma$ coherent $\rightsquigarrow$ $G([0,1]^n, f) =$ permutohedron $=$ skeleton of an $(n-1)$-dim'l polytope

THM (Billera & Sturmfels) "Fiber polytopes" 1992: For any edge-generic $f$ on any $d$-polytope $P$, $G(P,f)$ restricted to $\{\text{coherent } \gamma\}$ is the 1-skeleton of a $(d-1)$-polytope.

⑤ (Too hard a) PROBLEM ? : For which $(\underset{\underset{\text{polytopes}}{\shortparallel}}{P}, f)$

are all f-monotone paths γ coherent ?

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Not too hard...

## REU Problem 8

Which zonotopes $(\underset{\underset{Z(A)}{\shortparallel}}{P}, f)$ have all f-monotone paths coherent ?

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- n-cubes have $\underset{n=d}{A} = \overbrace{\begin{bmatrix}1 & & 0\\ & 1 & \\ 0 & & \ddots\ 1\end{bmatrix}}^{n}$ so $n-d=0$, and all γ coherent ✓
- R. Edman: Nice, easy answers for $n-d=1$
  (?) $n-d=2$

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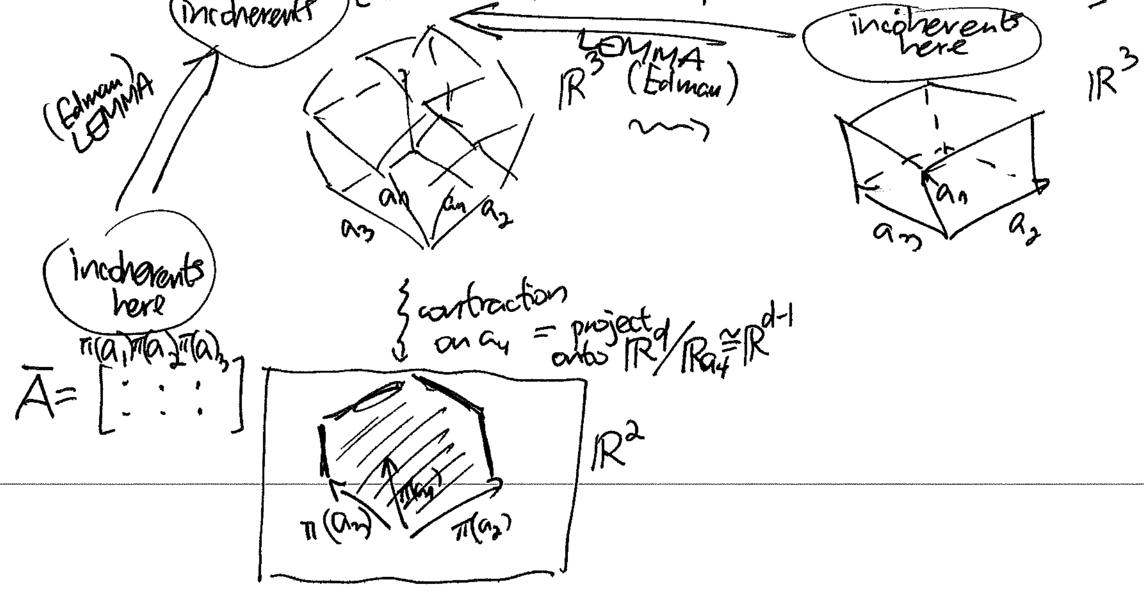
⑥ Tools

- Deletion/contraction

$$A=\begin{array}{c}\begin{array}{cccc}a_1 & a_2 & a_3 & a_4\end{array} \subset \mathbb{R}^d\\ \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}\end{array} \xrightarrow[\text{of } a_4]{\text{deletion}} \bar{A}=\begin{array}{c}\begin{array}{ccc}a_1 & a_2 & a_3\end{array} \subset \mathbb{R}^d\\ \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array}$$

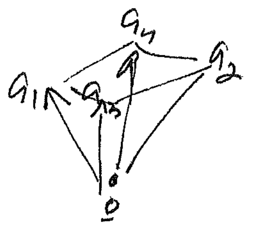


Thus one needs to find $\overset{\text{(minor-deletion/contraction)}}{\text{minimal}}$ obstructions.

For this, $\overset{\text{(oriented-matroid)}}{\text{duality}}$ is very useful:

$$A = d\left\{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}\right. \quad = \quad d=3\left\{\overbrace{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}}^{n=4}\right.$$

$\subset \mathbb{R}^3$

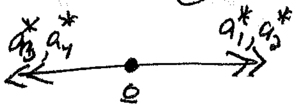


$a_1 + a_2 = a_3 + a_4$

$\Updownarrow$ dual means $\mathrm{Row}(A)^\perp = \ker(A) = \mathrm{Row}(A^*)$ inside $\mathbb{R}^n$

$$A^* = \underbrace{n-d}_{\text{corank}}\left\{\overbrace{\begin{bmatrix} | & & | \\ a_1^* & \cdots & a_n^* \\ | & & | \end{bmatrix}}^{n}\right. \quad = \quad \begin{bmatrix} a_1^* & a_2^* & a_3^* & a_4^* \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$\subset \mathbb{R}^1$



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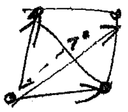
- when corank $n-d=0$ then $Z(A) = [0,1]^n$ and all $\gamma$ coherent.



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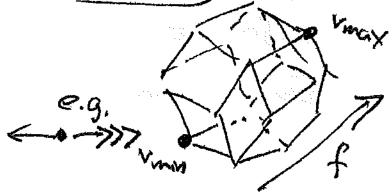
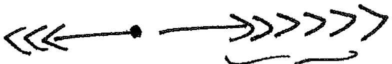
Generalize ?!

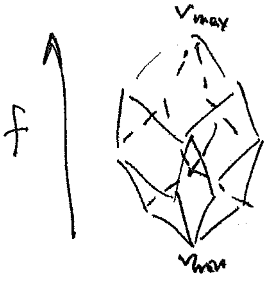
- THM (Edman) When $n-d=1$, $A^* \subset \mathbb{R}^1$
  $\|$

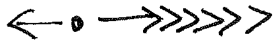




≥2 ≥2

has incoherents

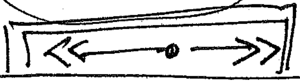




only 1

all coherent

i.e.



is a minimal obstruction

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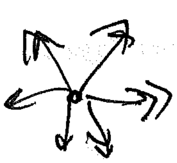
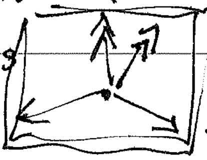
- THM (Edman)
  When $n-d=2$, $A^* \subseteq \mathbb{R}^2$



$(Z(A), f)$ ~~one~~ has ~~[illegible]~~ all coherent $\gamma$ $\iff$ $A^*$ avoids

as a contraction

and avoids



as a deletion

← new minimal obstructions!

# DUALITY EXERCISES

**EXERCISE #25.** Given $A \in \mathbb{R}^{d \times n}$ of rank $d$

$A^* \in \mathbb{R}^{(n-d) \times n}$ of rank $n-d$

with $\mathrm{Row}(A)^\perp = \mathrm{Row}(A^*)$,

show there exists some $c \in \mathbb{R} - \{0\}$ with the property

that every decomp. $\{1, 2, \ldots, n\} = \underbrace{B}_{\text{size } d} \sqcup \underbrace{B^c}_{\text{size } n-d}$

has

$$\underbrace{\det\left(A\big|_{\text{columns } B}\right)}_{d \times d} \Big/ \underbrace{\det\left(A^*\big|_{\text{columns } B^c}\right)}_{(n-d)\times(n-d)} = c$$

Thus the Oriented matroid $M(A) = \{\overset{+/-/0}{\text{signs}} \text{ of } \det(A|_{\text{cols } B})\}$ determines that of $M(A^*)$ and vice-versa.

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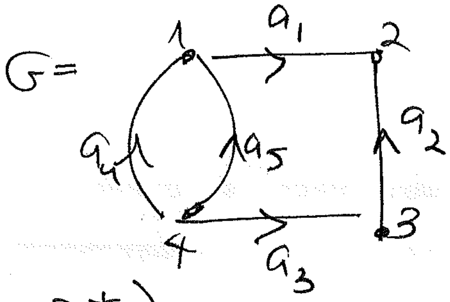
**EXERCISE #26.** Given $G = (V, E)$ a connected plane graph, orient it to get $A$ as before

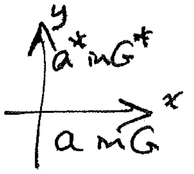
e.g.

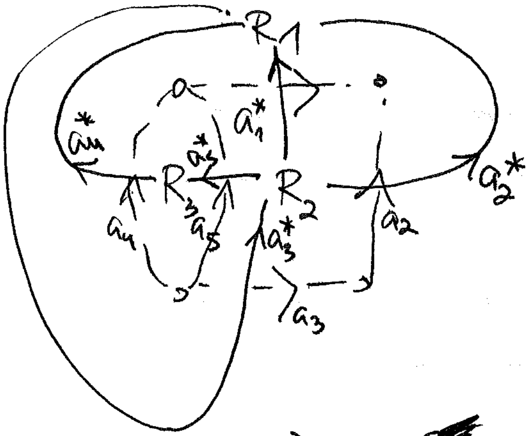
$$A = \begin{array}{c|ccccc} & a_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & -1 & 0 & 0 & +1 & +1 \\ 2 & +1 & +1 & 0 & 0 & 0 \\ 3 & 0 & -1 & +1 & 0 & 0 \\ 4 & 0 & 0 & -1 & -1 & -1 \end{array}$$

$G =$ 

~~Show that~~ and its plane dual $G^* = (V^*, E^*)$

$V^*$ = regions of $G$, $E^*$ = crossing edges, directed like this:





$$A_* = \begin{array}{c|ccccc} & a_1^* & a_2^* & a_3^* & a_4^* & a_5^* \\ R_1 & +1 & +1 & -1 & +1 & 0 \\ R_2 & \text{[illegible]}1 & -1 & +1 & 0 & -1 \\ R_3 & 0 & 0 & 0 & -1 & +1 \end{array}$$

Show that (a) ~~has~~ $\mathrm{Row}(A)^\perp = \mathrm{Row}(A^*)$

(b) A subset $\underset{\{a_1, \ldots, a_r\}}{T} \subset E$ ~~forms~~s a spanning tree in $G$

$\Longleftrightarrow T^* := \{a_1^*, \ldots, a_n^*\} \setminus \{a_1^*, \ldots, a_r^*\}$ ~~gives~~ forms one in $G^*$