REU 2017 Day 4 E. del Mas
Type B Reflection factorizations

I. Complex reflection groups
II. REU Problem 4
III. Representation theory

Chapuy-Stump

Complex reflections

$V = \mathbb{C}^n$, $r \in GL(V) = GL_n(\mathbb{C})$

DEF'N: r is a **reflection** if

i) it has finite order

ii) its fixed space $\ker(r-1) \subset V$ is a hyperplane i.e. has dim $n-1$.

EXAMPLE

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL(\mathbb{C}^2)$ is a reflection fixing $\mathbb{C}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a reflection fixing $\mathbb{C}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

DEF'N: $W \subset GL(V)$ is a complex reflection group if it is finite and generated by reflections.

EXAMPLE:

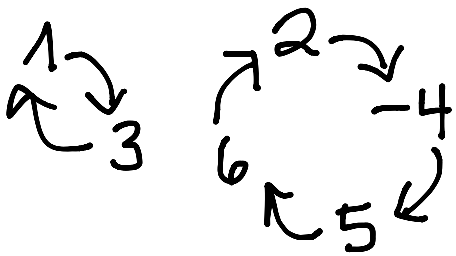
$$W = \{ \text{signed } n\times n \text{ permutation matrices} \} = B_n$$

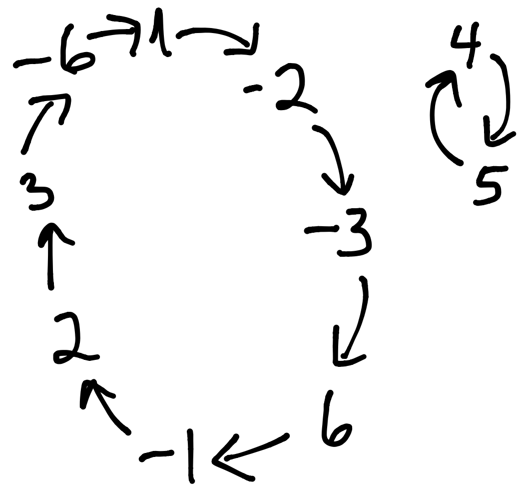
$n=4$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in B_4$$

$n=6$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & -4 & 1 & -5 & 6 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -2 & 3 & -6 & 5 & 4 & -1 \end{pmatrix}$$





$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A set of generating reflections for B_n:

$$\left\{ \begin{bmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}} & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & O \\ & \boxed{\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}} & & \\ & & 1 & \\ O & & & \ddots \\ & & & & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & & & O \\ & \ddots & & \\ & & 1 & \\ O & & & \boxed{\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}} \end{bmatrix}, \begin{bmatrix} -1 & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{bmatrix} \right\}$$

EXAMPLE: B_n is the special case $G(2,1,n)$ of the infinite family of complex reflection groups $G(r,p,n) \subset GL_n(\mathbb{C})$

$r \geq 2$

$n, p \geq 1$

$p \mid r$

$$G(r,p,n) = \left\{ \begin{array}{l} n \times n \text{ monomial matrices with} \\ \text{nonzero entries all } p^{th} \text{ roots of } 1, \\ \text{and product of nonzero entries} \\ \text{are an } \left(\frac{r}{p}\right)^{th} \text{ root of } 1 \end{array} \right\}$$

REMARK: $G(r,p,n)$ is well-generated

generated by $n = \dim V$ reflections

$\iff p=1$ or $p=r$

DEF'N: $R = \{\text{all reflections in } W\}$

$R^* = \{\text{all reflecting hyperplanes in } W\}$

$h := \frac{|R| + |R^*|}{n}$ is the Coxeter number of W

DEF'N: $w \in W$ is called ζ-regular if w has an eigenvector not fixed by any $r \in R$.

For any h^{th} root-of-1 ζ, there exists a ζ regular element.

DEF'N: $w \in W$ that is ζ-regular for ζ an h^{th} root-of-1 is called a Coxeter element of W.

PROP: $w \in W$ is a Coxeter element

$\iff$ it has ζ as an eigenvalue

EXAMPLE:

In B_n, $w = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & & & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix}$ is a Coxeter element.

REU EXERCISE 9:
Find all Coxeter elements in B_3
(there are 8)

Since W is generated by reflections we can write any element $w \in W$

$$w = r_1 r_2 \cdots r_\ell, \quad r_i \in R$$

This is a *(reflection) factorization* of w.

A general question:

Given a Coxeter element $c \in W$, how many reflection factorizations of length ℓ does it have, i.e. what is

$$f_\ell := \{(r_1, \ldots, r_\ell) : r_1 r_2 \cdots r_\ell = c\} ?$$

EXAMPLE: For S_n, $c = (1\,2 \cdots n)$ and one can use Cayley's formula n^{n-2} for the number of trees on n labeled vertices $\{1, 2, \ldots, n\}$ to show that $f_{n-1} = n^{n-2}$.

THM 1 (Chapuy–Stump)

For a Coxeter element w in any well-generated complex reflection group,

$$\mathrm{FAC}_w(t) := \sum_{\ell \geq 0} \frac{t^\ell}{\ell!} f_\ell = \frac{1}{|W|}\left(e^{\frac{t|R|}{n}} + e^{-\frac{t|R^*|}{n}} \right)^n$$

REU EXERCISE 10:

Let $W = G(r,1,1) = C_r$ = cyclic group of r^{th} roots-of-1

$$\left.\begin{array}{l} R = \{\zeta, \zeta^2, \ldots, \zeta^{r-1}\} \\ R^* = \{0\} \end{array}\right\} \Rightarrow h = \frac{|R| + |R^*|}{n} = \frac{r-1+1}{1} = r$$

(a) Find and prove a recurrence for f_ℓ

(b) Prove THM 1 for $W = C_r$.

In type B_n, THM 1 says

$$FAC_W(t) = \frac{1}{2^n n!}\left(e^{\frac{tn^2}{n}} - e^{-\frac{tn^2}{n}}\right)^n$$

$$= \frac{1}{2^n n!}\left(e^{tn} - e^{-tn}\right)^n$$

REMARK: Lewis-Reiner-Stanton working with $W = GL_n(\mathbb{F}_q) = GL(V)$ and specifying a sequence $(\alpha_1, \ldots, \alpha_\ell) \in (\mathbb{F}_q^\times)^\ell$ showed that

$\#\left\{\begin{array}{c}\text{factorizations} \\ c = r_1 r_2 \cdots r_\ell \text{ with} \\ \det(r_i) = \alpha_i\end{array}\right\}$ depends only on how many $\alpha_i = 1$.

REU PROBLEM 4: In type B_n,

generalize f_ℓ in the following way:

$$f_{\ell,m} := \#\left\{\begin{array}{l}\text{factorizations} \\ c = r_1 r_2 \cdots r_\ell \text{ with the first} \\ m \text{ of the } r_i \text{ lying in} \\ \text{the "sign change" conjugacy} \\ \text{class, i.e. } \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} \\ \text{and the rest of the } r_i \text{ in} \\ \text{the transposition conjugacy} \\ \text{class.} \end{array}\right\}$$

any Coxeter element

(should be the same answer for any choice of the m positions out of ℓ)

(a) Generalize THM 1 for B_n to compute
$$\sum_{\ell \geq 0} f_{\ell,m} \frac{t^\ell}{\ell}$$
for each fixed m.

(b) Generalize to $G(p,1,n)$, $G(p,p,n)$.

REPRESENTATION THEORY

To calculate f_ℓ, Chapuy–Stump used the group algebra.

DEF'N: For a finite group W, let $\mathbb{C}[W]$ be the algebra over $\mathbb{C}$ with $\mathbb{C}$-basis $\{v_w\}_{w\in W}$ and $v_w \cdot v_u := v_{wu}$

EXAMPLE In $\mathbb{C}[S_4]$,

$$\left(e^{\frac{2\pi i}{10}} v_{(12)} + v_{(243)}\right)\left(e^{\frac{2\pi i}{5}} v_{(13)}\right)$$

$$= e^{\frac{3\pi i}{5}} v_{(12)(13)} + e^{\frac{2\pi i}{5}} v_{(243)(13)}$$

$$= e^{\frac{3\pi i}{5}} v_{(132)} + e^{\frac{2\pi i}{5}} v_{(1243)}$$

$W \circlearrowright \mathbb{C}[W]$
"acts on"

$$w \cdot \left(\sum_{g \in W}' \lambda_g v_g \right) = \sum_{g \in W}' \lambda_g v_{wg}$$

gives a (permutation) representation

$$\begin{aligned} W &\xrightarrow{\rho} GL(\mathbb{C}^{|W|}) \\ w &\longmapsto \begin{bmatrix} \text{permutation matrix for} \\ g \mapsto wg \end{bmatrix} \end{aligned}$$

$\mathbb{C}[W]$ turns out to contain every irreducible W-representation

THM 2: $$\mathbb{C}[W] = \bigoplus_{\lambda \in \Lambda} \dim(\lambda)\, V^\lambda$$

an indexing set for the irreducible W-reps

Note that $\mathbb{C}[W]$ has character

$$\chi_\rho(w) = \text{Trace of } \rho(w) = \begin{cases} |W| & \text{if } w = 1 \\ 0 & \text{if } w \neq 1 \end{cases}$$

Define $R := \sum_{r\in R} v_r \in \mathbb{C}[W]$

so $$R^\ell = \sum_{(r_1,\dots,r_\ell)\in R} v_{r_1 r_2 \cdots r_\ell}$$

and f_ℓ = coefficient of v_c in R^ℓ

$= $ coefficient of v_1 in $R^\ell v_{c^{-1}}$

$$= \frac{1}{|W|}\underbrace{\chi_\rho\left(R^\ell v_{c^{-1}}\right)}$$

extend $\chi_\rho : W \to \mathbb{C}$ linearly to a function $\chi_\rho : \mathbb{C}[W] \to \mathbb{C}$

THM 2

$$= \frac{1}{|W|}\sum_{\lambda\in\underline{\Lambda}} \dim(\lambda)\,\chi_\lambda\left(R^\ell v_{c^{-1}}\right)$$

We claim $R \in Z(\mathbb{C}[W])$ = center of $\mathbb{C}[W]$

because of...

REU EXERCISE 11:

For a finite group G with conjugacy class $T_1, T_2, \ldots, T_m$, show that $Z(\mathbb{C}[G])$ has $\mathbb{C}$-basis $\left\{ \sum_{g \in T_i} t_g \right\}_{i=1,2,\ldots,m}$

THM 3: An element $z \in \mathbb{C}[W]$ lies in $Z(\mathbb{C}[W]) \iff$ in every W-irreducible z acts a scalar matrix $\begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma \end{bmatrix} = \gamma \cdot I$

$\Rightarrow$ R acts in the W-irreducible λ via some $\gamma_\lambda I$

$\Rightarrow \chi_\lambda(R) = \gamma_\lambda \dim(\lambda)$ $\Rightarrow$ $\boxed{\gamma_\lambda = \dfrac{\chi_\lambda(R)}{\dim(\lambda)}}$

Continuing…

$$f_\ell = \frac{1}{|W|} \sum_{\lambda\in\Lambda}{}' \dim(\lambda)\, \chi_\lambda(R^\ell v_c^{-1})$$

$$= \frac{1}{|W|} \sum_{\lambda\in\Lambda}{}' \dim(\lambda)\, \underbrace{\gamma_\lambda \gamma_\lambda \cdots \gamma_\lambda}_{\ell \text{ times}} \cdot \chi_\lambda(c^{-1})$$

$$= \frac{1}{|W|} \sum_{\lambda\in\Lambda}{}' \dim(\lambda) \left(\frac{\chi_\lambda(R)}{\dim(\lambda)}\right)^\ell \chi_\lambda(c^{-1})$$

$$\boxed{f_\ell = \frac{1}{|W|} \sum_{\lambda\in\Lambda} \dim(\lambda)^{1-\ell}\, \chi_\lambda(R)^\ell\, \chi_\lambda(c^{-1})}$$

Very useful for Chapuy & Stump's calculation.

– an instance of Frobenius's formula