REU 2017 Day 7 S. Chepuri

Total positivity

DEFN: A **minor** of a matrix is the determinant of a square submatrix obtained by selecting certain rows and columns.

EXAMPLE $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$$|M_{\{2,3\},\{1,3\}}| = \left| \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix} \right| = 16 - 2 = 14$$

a minor of size 2

$$|M_{\{1\},\{2\}}| = |[2]| = 2$$ of size 1.

DEF'N: A totally positive (totally nonnegative) matrix is one that has all minors >0 (≥ 0)

- They have nice eigenvalues

- They are related to networks/plabic graphs/wiring diagrams/cluster algebras

NOTATION: $[n] := \{1, 2, \ldots, n\}$

Cauchy-Binet Formula:

For matrices A, B of sizes $n \times m$, $m \times n$ with $n \leq m$,

$$\det(AB) = \sum_{\substack{S \subset [m] \\ |S| = n}} \det(A[S]) \det(B[S])$$

where $A[S], B[S]$ are $n \times n$ submatrices corresponding to S.

EXAMPLE

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \begin{matrix} n=2 \\ m=3 \end{matrix}$$

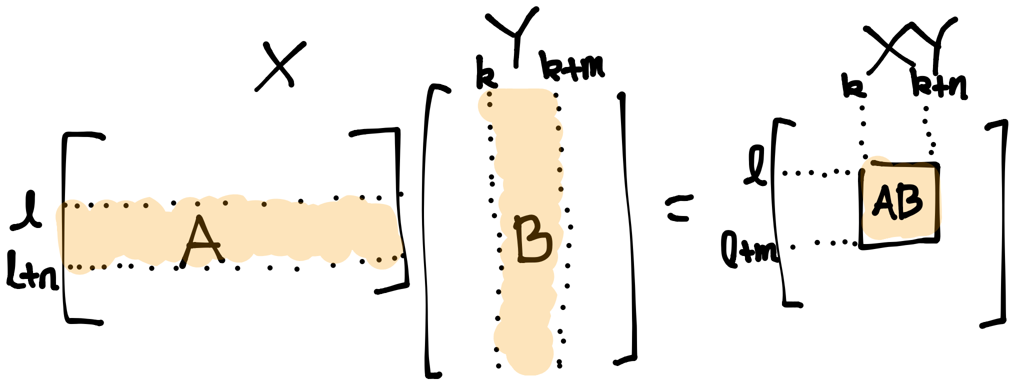
$$\begin{aligned} \det(AB) = &\det(A[12])\det(B[12]) \\ &+\det(A[13])\det(B[13]) \\ &+\det(A[23])\det(B[23]) \end{aligned}$$

$$\begin{aligned} = &\left|\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right|\left|\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right| + \left|\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}\right|\left|\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}\right| \\ &+ \left|\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\right|\left|\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\right| \end{aligned}$$

$$= (-2)^2 + (-4)^2 + (-2)^2 = 24$$

In general, if we want to know a minor of XY, we can think of A, B as submatrices of X, Y

Picture:



CONSEQUENCE

$n \times n$ totally positive matrices and $n \times n$ totally nonnegative matrices form semigroups under matrix multiplication.

LDU factorization:

Let X be a TNN nonsingular matrix.

Then we can write

$$X = L \quad D \quad U$$

L: lower unitriangular

$$\begin{bmatrix} 1 & & 0 \\ & 1 \ddots & \\ * & & 1 \end{bmatrix}$$

D: nonsingular diagonal

U: upper unitriangular

$$\begin{bmatrix} 1 & & * \\ & 1 \ddots & \\ 0 & & 1 \end{bmatrix}$$

In fact, $l_{ij} = \dfrac{|X_{[j-1]\cup\{i\},[j]}|}{|X_{[j],[j]}|}$

$$u_{ij} = \frac{|X_{[i],[i-1]\cup\{j\}}|}{|X_{[i],[i]}|}$$

$$d_{ii} = \frac{|X_{[i],[i]}|}{|X_{[i-1],[i-1]}|}$$

REU EXERCISE 14: Show the leading principal minors $|X_{[k],[k]}|$ are positive for nonsingular TNN X.

REU EXERCISE 15: Show the L, D, U in $X = LDU$ are also TNN.

DEF'N The Chevalley generators are $\{e_i(a)\}$, $\{f_i(a)\}$ $i=1,2,\dots,n-1$

$$\begin{array}{cc} & \begin{matrix} & & & i & i+1 & & \end{matrix} \\ \begin{matrix} \\ \\ \\ \text{row } i \\ i+1 \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & a & & \\ & & & 0 & 1 & & \\ & 0 & & & & 1 \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} & & & i & i+1 & & \end{matrix} \\ \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & 1 & 0 & & \\ & 0 & & a & 1 & & \\ & & & & & 1 \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}$$

THM: A TNN upper unitriangular matrix is a product of $e_i(a)$'s with $a \geq 0$.
Similarly for lower unitriangular TNN matrices and $f_i(a)$'s with $a \geq 0$.

THM: These relations hold:

(i) $e_i(a)e_{i+1}(b)e_i(c) = e_{i+1}\left(\frac{bc}{a+c}\right)e_i(a+c)e_{i+1}\left(\frac{ab}{a+c}\right)$

(ii) exact same, replacing e_i by f_i

(iii) $e_i(a)e_j(b) = e_j(b)e_i(a)$

(iv) $e_i(a)e_i(b) = e_i(a+b)$

(v) same for f_i's

(vi) same for f_i's

These give generators and relations for the semigroup of invertible TNN matrices.

DEF'N: A k-nonnegative matrix is a matrix where all minors of size $\leq k$ are nonnegative.

The $n \times n$ k-nonnegative matrices are a semigroup for same reason as before.

REU Problem 7:
What are the generators, relations for the semigroup of k-nonnegative matrices?

It might be easiest to start with $k=1$ or $k=n-1$.

REU Exercise 16:

For $n=2$, $k=1$ show the generators are $e_1(a)$'s, $f_1(a)$'s, diagonal matrices, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Another variation: restrict to upper unitriangular matrices.