Equality of Schur Supports of Ribbons

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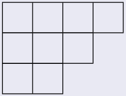
Overview

Schur Functions

Example/Definition (Young Diagram & Semistandard Young Tableau)

Partition $\lambda = (4, 3, 2)$



Young Diagram

1	1	2	3
2	2	3	
3	4		

SSYT

Definition

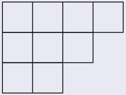
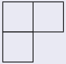
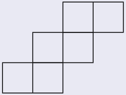
The Schur function s_λ of a partition λ is

$$s_\lambda(x_1, x_2, x_3, \ldots) = \sum_{\substack{T \,:\, \text{SSYT} \\ \text{of shape } \lambda}} x^T = \sum_T x_1^{t_1} x_2^{t_2} x_3^{t_3} \cdots$$

where t_i is the number of occurrences of i in T.

Skew Schur Functions

Example/Definition (Skew Shape)

$\lambda =$  $\mu =$  $\lambda/\mu =$ 

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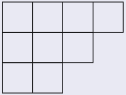
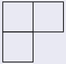
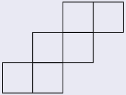
$\lambda =$  $\mu =$  $\lambda/\mu =$ 

Skew Schur functions are:

- defined analogously to straight Schur functions.
- Schur-positive, meaning

$$s_{\lambda/\mu} = \sum_{\nu} c^{\nu}_{\lambda,\mu} s_{\nu}$$

where ν is a straight partition, and $c^{\lambda}_{\mu,\nu} \geq 0$.

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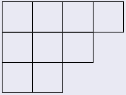
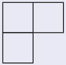
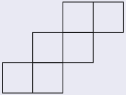
$\lambda =$  $\mu =$  $\lambda/\mu =$ 

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Definition

The *Schur support of a skew shape* λ/μ, denoted $[\lambda/\mu]$, is defined as

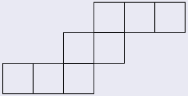
$$[\lambda/\mu] = \{\nu : \ c^{\lambda}_{\mu,\nu} > 0\}.$$

Ribbons

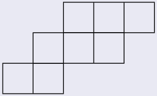
A ribbon is a skew shape which does not contain a 2×2 subdiagram.

Example

Ribbon :



Non-ribbon:



Given a sequence of integers, there's a **unique** ribbon with that sequence of row lengths. Thus, ribbons are uniquely determined by compositions of n (the total boxes).

The above example can be denoted as the ribbon $(3, 2, 3)$

Littlewood-Richardson Rule

This is a rule to check if a particular straight young diagram is present in the support.

Formal Definition

Let D be a skew shape. A partition $\lambda = (\lambda_1, \dots, \lambda_m)$ is in the support of s_D iff there is a valid LR-filling of D with content λ.

A filling of D is an LR-filling if:

- The tableau is semistandard.
- Every *initial reverse reading word* is *Yamanouchi*: $\#i$'s $\geq \#(i+1)$'s

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Example/Definition (Yamanouchi Property)

Reverse Reading Word: 1,1,1,2,2,3,2

This is Yamanouchi because there are at least as many 1's as 2's and as many 2's as 3's at every stage.

			1	1
	2	2	1	
2	3			

Littlewood Richardson Rule

Example of LR-rule

Reverse Reading Word: 1,1,2,2,1,3,2

This is Yamanouchi and semistandard, hence is a valid LR-filling

			1	1
	1	2	2	
2	3			

The content of the filling is (3,3,1), thus (3,3,1) is in the support of the ribbon: $(2,3,2)$.

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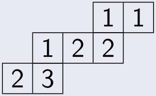
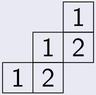


The content of the filling is (3,3,1), thus (3,3,1) is in the support of the ribbon: $(2, 3, 2)$.

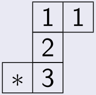
Proposition

Let $\alpha = (1, \alpha_2, \alpha_3)$ be a ribbon. Then, $\alpha' = (\alpha_2, 1, \alpha_3)$ and α don't have the same support.

Proof. When the row of length 1 is in the middle, there is no LR-filling with just 1's and 2's.



$\alpha = (1, 2, 2)$



$\alpha' = (2, 1, 2)$

Ribbons with Equal Support

[McNamara (2008)] gives that 2 ribbons can have the same support only if one is a permutation of the rows of the other.

Definition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. We use α_π to denote a ribbon formed by applying the permutation $\pi \in S_m$ to the row lengths of α.

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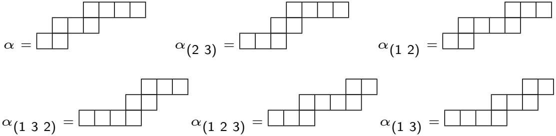

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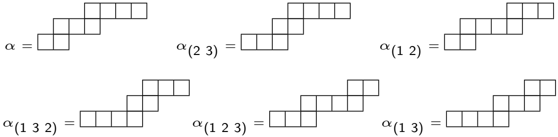

Definition

A ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is said to have *full equivalence class* if for any permutation $\pi \in S_m$, we have $[\alpha] = [\alpha_\pi]$.

Trivial Cases

Proposition

If a ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ *has* k *rows of length 1, where* $1 \leq k < m$*, then* α *does not have full equivalence class.*

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Remark

It is well known that rotating a ribbon by 180° preserves its support. It follows trivially that any ribbon with only two rows has full equivalence class.

For the rest of the presentation, we consider only ribbons with more than two rows and with no rows of length 1.

Sufficient Condition for Full E.C

Theorem (S, G, H, Tran, '17)

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon such that any subset of size three of $\{\alpha_i\}$ satisfies the **strict** triangle inequality ($\alpha_i < \alpha_j + \alpha_k$). Then α has full equivalence class.

Proof Idea: Given a ribbon with an LR-filling, show how to swap two adjacent row lengths while preserving the content, Yamanouchi property, and semistandardness of the filling.

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Proof Idea: Given a ribbon with an LR-filling, show how to swap two adjacent row lengths while preserving the content, Yamanouchi property, and semistandardness of the filling.

Proof Sketch:

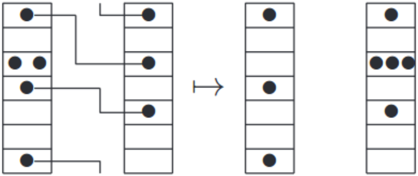
1. Use the *R*-matrix algorithm (described on the next slide) to swap adjacent row lengths while preserving content and the Yamanouchi property.
2. Show how to adjust the resulting filling to be semistandard.

R-Matrix Algorithm

Algorithm [Inoue et al. (2012), Section 2.2.3]

1. Represent the rows to be swapped as box-ball systems.

$$R\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 3 & 4 & 7 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline \end{array} \right) = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 3 & 3 & 5 \\ \hline \end{array}$$



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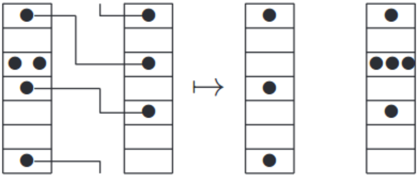
1. Represent the rows to be swapped as box-ball systems.
2. For each unconnected ball A on the right, find its partner B on the left which is an unconnected ball in the lowest position but higher than that of A; if there are no such balls, choose from the balls in the lowest position on the left. Connect A and B.

$$R\left(\boxed{1}\boxed{3}\boxed{3}\boxed{4}\boxed{7} \otimes \boxed{1}\boxed{3}\boxed{5} \right) = \boxed{1}\boxed{4}\boxed{7} \otimes \boxed{1}\boxed{3}\boxed{3}\boxed{3}\boxed{5}$$



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3. Shift all unconnected balls from the left to the right.

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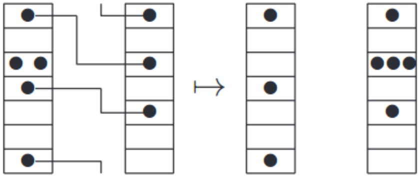

R-Matrix Properties

In this example, notice that

1. the Yamanouchi property is preserved.
2. the leftmost entry in the bottom row does not increase.

		1	3	3	4	7
1	3	5				

$\longrightarrow$

				1	4	7
1	3	3	3	5		

In fact, we prove that (1) and (2) hold in general. The remainder of the proof of the theorem ensures that we can move around the content within the ribbon so that the rightmost entry in the top row does not violate semistandardness.

Necessary Condition for Full E.C

Theorem (S, G, H, Tran, '17)

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ be a ribbon, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. If α has full equivalence class, then $N_j < \sum_{i=j+1}^{m} \alpha_i - (m - j - 2)$ for all $j \leq m - 2$, where

$$N_j = \max\{k | \sum_{i \leq j:\ \alpha_i < k} (k - \alpha_i) \leq m - j - 2\}.$$

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We prove the contrapositive by assuming

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and showing that there exists a content for an LR-filling of $\alpha_{(j\ j+1)}$ that is not the content of any LR-filling of α.

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Conjecture

The above necessary condition is sufficient for a ribbon to have full equivalence class.

Other Results

Proposition: 3 rows

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a ribbon. Then α has full E.C iff α_1, α_2 and α_3 satisfy the strict triangle inequality.

Proposition: 4 rows

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a ribbon such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. Then, α has full equivalence class iff

- $\alpha_1 < \alpha_2 + \alpha_3 + \alpha_4 - 2$
- $\alpha_2 < \alpha_3 + \alpha_4$

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