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Wythoff's Construction for Coxeter Groups

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0. INTRODUCTION

Suppose that V is an n-dimensional real vector space, G a finite subgroup of $\mathrm{GL}(V)$, and P a finite subset of V. Let $[G, P]$ be the convex cone in V generated by vectors of the form $g(p)$, where $g \in G$ and $p \in P$. Then $[G, P]$ is a finitely generated cone invariant under G. If $[G, P]$ is pointed as well, a bounded section of $[G, P]$ by an affine hyperplane in V is a polytope, said by Coxeter to be obtained by "Wythoff's construction" [3].

If G_p is the stabiliser of $p \in P$ in G, one might ask if the face lattice of $[G, P]$ can be described in terms of G and its subgroups G_p. This is possible under certain assumptions on V and P if G is a Coxeter group. The interest of this example lies in the fact that highly symmetrical examples can be obtained, such as the regular polytopes and many "semi-regular" ones. Similar ideas for euclidean and hyperbolic Coxeter groups G lead to tesselations of those spaces [4–6].

Suppose now that $W = W_S$ is an arbitrary Coxeter group, with S finite, and $\mathscr{T} = (T_1, ..., T_k)$ is a family of subsets of S. We construct in Sections 1–3 an abstract "shadow lattice" $L(W, \mathscr{T})$, which is closely related to the concept of "shadows" introduced by Tits [10]. Any interval of such a shadow lattice is again a shadow lattice, although possibly with a different number of elements in the family $\mathscr{T}$. This fact is the principal advantage of not restricting oneself to the case $k = 1$, as it makes inductive proofs possible.

For finite W, we show that $L(W, \mathscr{T})$ is isomorphic to the face lattice of a suitable polytope $[W, P]$, where $P = \{p_1, ..., p_k\}$ is a set of points in V and W_{T_i} is the stabiliser of p_i in W. Alternatively, $L(W, \mathscr{T})$ can then be interpreted as a spherical tesselation. When W is euclidean or hyperbolic in the sense of [1], $L(W, \mathscr{T})$ is isomorphic to a tesselation of the corresponding space.

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The validity of such isomorphisms was recognised by Coxeter in [3–6], as well as by Tits [12], but no proof has appeared in print to the best of our knowledge. The principal aim of this paper is to supply in Section 5 a complete proof of these facts. We sketch in Section 7 the analogous theory for the rotation subgroup W^+ of W, which has also been discussed by Coxeter.

In an earlier paper of the author [7], a euclidean sphere packing was associated to every hyperbolic group W of "level 2." However, the maximality of such a packing depended on a certain conjecture about the Tits cone of W [7, Theorem 3.3], which is proved here in Section 6 for all such W.

The author is very grateful to R. Scharlau for pointing out that much of our theory in Sections 1–2 was a special case of Tits's work on shadows in [10], as well as other helpful remarks. His forthcoming paper [8] deals extensively with geometrical realisations of the shadow concept from a more general point of view.

As in [8], we say that a partially ordered set X is *pure* of *dimension* n if any two maximal flags in X have the same cardinality $n+1$. If $x_0 < \cdots < x_d = x < \cdots < x_n$ is a maximal flag containing some $x \in X$, then d, the *dimension* of x, is independent of the choice of the flag.

Let X be a partially ordered set in which every flag is finite, any two elements have a greatest lower bound, and a largest element exists. Then any family of elements of X has a greatest lower bound, which is equal to the greatest lower bound of a finite subfamily. It follows that any family of elements of X also has a least upper bound, so that X is a complete lattice.

1. T-SHADOWS

Let $W = W_S$ be a Coxeter group, with S finite. If X and Y are subsets of S and $a \in W$, the double coset $W_X a W_Y$ contains a unique element of minimal length, called the (X, Y)-*reduction* of a. When $w = a$, one says that a is (X, Y)-*reduced* [1, IV.1, Exer. 1, 3]. If $(s_1, ..., s_q)$ is a reduced decomposition of some $w \in W$, the set $\{s_1, ..., s_q\}$ is independent of the choice of such a decomposition and is denoted by S_w [1, IV.1.8]. For a subset X of S, $X^\perp$ denotes the set of those $s \in S$ which commute with every element of X. The Coxeter graph of W is denoted by Γ.

The following two results follow immediately from Theorem 3 of Tits [11].

PROPOSITION 1.1. *Suppose that $s \in S$ commutes with $w \in W$. Then either $s \in S_w$ or $s \in S_w^\perp$.*

PROPOSITION 1.2. *Suppose that $(s_1, ..., s_q)$ is a path in Γ such that all the s_i are distinct. Then $(s_1, ..., s_q)$ is the unique reduced decomposition of $w = s_1 \cdots s_q$.*

If T is a fixed subset of S, a *T-shadow* is a subset of W of the form $gW_X W_T$ for some $g \in W$ and $X \subset S$. The set of all T-shadows in W can be viewed as a "shadow geometry" in the sense of Tits [10, Chap. 12]. The points of such a geometry are the left cosets hW_T of W_T in W, while its subspaces are T-shadows of sets of the form gW_X. In this context, the T-shadow of gW_X is usually defined as the set of all points hW_T which intersect gW_X. However, since this amounts to saying that the coset hW_T is equal to gwW_T for some $w \in W_X$, the T-shadow of gW_X can be identified with the union of such cosets, as we have chosen to do.

A general result of Tits [10, Cor. 12.9] implies that the intersection of two T-shadows is either empty or else another T-shadow. In our case, we have the following explicit formula:

THEOREM 1.3. *Suppose that $g^{-1}h = xwy$, where $x \in W_X$, $y \in W_Y$, and w is (X, Y)-reduced. Then $gW_X W_T$ and $hW_Y W_T$ have a nonempty intersection if and only if $w \in W_T$, in which case*

$$gW_X W_T \cap hW_Y W_T = gxW_Z W_T,$$

where $Z = X \cap Y \cap S_w^{\perp}$.

The proof depends on an extension of Proposition 1.1:

LEMMA 1.4. *Suppose that w is (X, Y)-reduced and $u \in W_X$, $v \in W_Y$ are such that $uw = wv$. Then $u \in W_Z W_{S_w}$, where $Z = X \cap Y \cap S_w^{\perp}$.*

Proof. We argue by induction on the length $l(u)$. Let $s \in X$ be such that $l(su) < l(u)$. Then $l(swv) = l(suw) = l(su) + l(w) < l(u) + l(w) = l(uw) = l(wv)$, since w is (X, Y)-reduced, while $l(sw) > l(w)$.

Let $(s_1, ..., s_q)$ be a reduced decomposition of w and $(s_{q+1}, ..., s_p)$ a reduced decomposition of v; then $(s_1, ..., s_p)$ is a reduced decomposition of wv. As $l(swv) < l(wv)$, the exchange condition shows that there exists an integer j, with $1 \leqslant j \leqslant p$, for which $ss_1 \cdots s_{j-1} = s_1 \cdots s_j$. Furthermore, $j \geqslant q+1$ since $l(sw) > l(w)$, so that $sws_{q+1} \cdots s_{j-1} = ws_{q+1} \cdots s_j$, or $sw = wt$, where $t = s_{q+1} \cdots s_j s_{j-1} s_j \cdots s_{q+1} \in W_Y$. Since $l(sw) = l(w) + 1$, t is of length 1 and therefore contained in Y.

The equation $suw = wtv$ shows by induction that $su \in W_Z W_{S_w}$. On the other hand, $sw = wt$ implies either that both s and t belong to S_w, or that $s = t$ belongs to Z by Proposition 1.1. In either case, it follows that $u \in W_Z W_{S_w}$, since elements of Z commute with elements of S_w. ∎

Proof of Theorem 1.3. By multiplying with $(gx)^{-1}$, it suffices to show that $W_X W_T \cap wW_Y W_T = W_Z W_T$.

Since elements of Z commute with w, an element $c \in W_Z$ is equal to wcw^{-1}, which belongs to $wW_Y W_T$ if $w \in W_T$; therefore in this case $W_Z W_T$ is contained in $W_X W_T \cap wW_Y W_T$, as $Z \subset X$.

Conversely, if $W_X W_T \cap wW_Y W_T \neq \emptyset$, an element $c \in W_X W_T \cap wW_Y W_T$ can be written as $c = uk = wvm$, with $u \in W_X$, $v \in W_Y$, and $k, m \in W_T$, so that $u^{-1}wv = km^{-1} \in W_T$. Let $(s_1, ..., s_q, s_{q+1}, ..., s_p, s_{p+1}, ..., s_r)$ be a reduced decomposition of $u^{-1}wv$ such that $(s_{q+1}, ..., s_p)$ is a reduced decomposition of w and set $s_1 \cdots s_q = a$, $s_{p+1} \cdots s_r = b^{-1}$. Since $u^{-1}wv \in W_T$, it follows that $w \in W_T$, while $a \in W_{X \cap T}$ and $b \in W_{Y \cap T}$. The equation $(ua)w = w(vb)$ shows by Lemma 1.4 that $ua \in W_Z W_{S_w}$ and hence $c = uk \in W_Z W_T$, since S_w is contained in T. ∎

2. T-Minimal Sets

It may happen that $W_X W_T = W_Y W_T$ for two subsets X, Y of S; then X and Y are said to be *T-equivalent*. We shall see that each T-equivalence class contains a unique minimal (as well as a maximal) member, which may be used to represent the class.

A set $X \subset S$ is called *T-minimal* if every element of $X \cap T$ is connected by a path in Γ contained in X to an element of $X \backslash T$. Every subset X of S contains a largest T-minimal subset X_T, consisting of $X \backslash T$ and those $s \in X \cap T$ which are connected to an element of $X \backslash T$ by a path in X. The *T-complement* X^T of X is the set of those elements of T which are not joined by an edge in Γ to any element of X_T, i.e., those which commute with elements of X_T. It is clear that X^T is disjoint from X_T and that X is contained in $X\langle T \rangle = X_T \cup X^T$, the *T-completion* of X. These concepts are also considered by Scharlau [8] in slightly different language. He calls $S \backslash X_T$ the "T-closure" of $S \backslash X$ and, following Tits, $S \backslash X\langle T \rangle$ the "T-reduction" of $S \backslash X$.

An inclusion $X \subset Y$ implies that $X_T \subset Y_T$ and $X^T \supset Y^T$.

Proposition 2.1. *A subset Y of S is T-equivalent to X if and only if $X_T \subset Y \subset X\langle T \rangle$.*

Proof. If $W_X W_T = W_Y W_T$, we have $X \subset Y \cup T$ and $Y \subset X \cup T$, so that $X \backslash T = Y \backslash T$. Every element $s \in X_T$ occurs as the initial member s_1 of a path $(s_1, ..., s_n)$, consisting of distinct elements of X, in which $s_n \in X \backslash T$. By Proposition 1.2, $w = s_1 \cdots s_n$ has the unique reduced decomposition $(s_1, ..., s_n)$. Since $s_n \notin T$, w can belong to $W_Y W_T$ only if $s_i \in Y$ for $1 \leqslant i \leqslant n$, so that $X_T \subset Y$. Similarly, we conclude that $Y_T \subset X$ and hence $X_T = Y_T$, which implies that $X\langle T\rangle = Y\langle T\rangle$, so that $Y \subset X\langle T\rangle$.

Conversely, if $X_T \subset Y \subset X\langle T\rangle$, elements of $Y \backslash X_T$ belong to T and commute with elements of X_T, so that both X and Y are T-equivalent to X_T. ∎

We can now give a criterion for inclusion between two T-shadows, which is again a special case of a result of Tits [10, Theorem 12.15].

PROPOSITION 2.2. *We have $gW_X W_T \subset hW_Y W_T$ if and only if $X_T \subset Y_T$ and $S_w \subset X^T$, where w is the (X, Y)-reduction of $g^{-1}h$.*

Proof. By Theorem 1.3, $gW_X W_T \cap hW_Y W_T = gW_X W_T$ holds if and only if $w \in W_T$ and $W_Z W_T = W_X W_T$, i.e., $X_T \subset Z \subset X\langle T\rangle$ by Proposition 2.1, where $Z = X \cap Y \cap S_w^\perp$. However, X_T can be contained in Z only if $X_T \subset Y_T$ and elements of S_w commute with elements of X_T, so that S_w is contained in X^T, since $S_w \subset T$. The converse is clear. ∎

COROLLARY 2.3. *We have $gW_X W_T = hW_Y W_T$ if and only if $X_T = Y_T$ (so that $X\langle T\rangle = Y\langle T\rangle$) and $gW_{X\langle T\rangle} = hW_{Y\langle T\rangle}$.*

Proof. It follows from Proposition 2.2 that $gW_X W_T = hW_Y W_T$ if and only if $X_T = Y_T$ and $S_w \subset X^T = Y^T$, where w is the (X, Y)-reduction of $g^{-1}h$. However, $g^{-1}h$ then belongs to $W_{X\langle T\rangle}$, as both X and Y are contained in $X\langle T\rangle$, so that $gW_{X\langle T\rangle} = hW_{Y\langle T\rangle}$. Conversely, if this is true, then $w \in W_{X^T}$, so that $S_w \subset X^T$. ∎

If T contains a component S' of S, the elements of S' cannot be contained in any T-minimal subset of S. The union of the remaining components of S is therefore equal to S_T.

It is easy to see that a union of T-minimal subsets of S is still T-minimal, but that this need not be true of an intersection. However, since $(X \cap Y)_T$ serves as the greatest lower bound of X and Y, the T-minimal subsets of S do form a finite lattice.

PROPOSITION 2.4. *Suppose that X and Y are T-minimal subsets of S, with $X \subsetneq Y$. For $s \in Y \backslash X$, the set $X \cup \{s\}$ is T-minimal if and only if either $s \notin T$ or $s \in T \backslash X^T$. Moreover, such an element s always exists in $Y \backslash X$.*

Proof. The first statement follows from the definition of T-minimality. For the second, note that if $Y \backslash X$ is contained in X^T, we have $Y \subset X\langle T\rangle$ and thus $Y = X$ since both X and Y are T-minimal. ∎

To construct the lattice of T-minimal subsets of S in a specific example, one starts with the empty set and adds elements of S one at a time in accordance with Proposition 2.4 until S_T is reached.

PROPOSITION 2.5. *Suppose that T_1, T_2 are fixed subsets of S. Then, for any $X \subset S$,*

(a) *if $T_1 \subset T_2$, then $X_{T_1} \supset X_{T_2}$ and $X^{T_1} \subset X^{T_2}$;*

(b) $X_{T_1 \cap T_2} = X_{T_1} \cup X_{T_2}$;

(c) $X^{T_1 \cap T_2} = X^{T_1} \cap X^{T_2}$.

Proof. (a) If $s \in X_{T_2}$, then either $s \in X \backslash T_1$ or $s \in X \cap T_1$ and is joined by a path in X to an element of $X \backslash T_2 \subset X \backslash T_1$, so that $s \in X_{T_1}$. Second, if $t \in X^{T_1}$, then $t \in T_2$ and commutes with elements of $X_{T_2} \subset X_{T_1}$, so that $t \in X^{T_2}$.

(b) Since $T_1 \cap T_2$ is contained in T_1 and T_2, $X_{T_1 \cap T_2}$ contains $X_{T_1} \cup X_{T_2}$ by (a). Conversely, an element $s \in X \backslash (T_1 \cap T_2)$ belongs either to $X \backslash T_1$ or $X \backslash T_2$, which implies that $X_{T_1 \cap T_2} \subset X_{T_1} \cup X_{T_2}$.

(c) It follows from (a) that $X^{T_1 \cap T_2}$ is contained in $X^{T_1} \cap X^{T_2}$. An element $t \in X^{T_1} \cap X^{T_2}$ is not joined to any element of X_{T_1} or X_{T_2}, i.e., to any element of $X_{T_1 \cap T_2}$ by (b), so that $X^{T_1} \cap X^{T_2}$ is contained in $X^{T_1 \cap T_2}$. ∎

3. THE LATTICE $L(W, \mathscr{T})$

We extend the concept of a T-shadow to a finite family $\mathscr{T} = (T_1, ..., T_k)$ of subsets of S. Form the set of all triples $[g, X, \mathscr{Q}]$, where $g \in W$, $X \subset S$, and $\mathscr{Q}$ is a subfamily of $\mathscr{T}$, preordered by the relation

$$[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}] \Leftrightarrow \mathscr{Q} \subset \mathscr{R}$$

and

$$gW_X W_T \subset hW_Y W_T \qquad \text{for all} \quad T \in \mathscr{Q}. \tag{3.1}$$

Call triples $[g, X, \mathscr{Q}]$ and $[h, Y, \mathscr{R}]$ equivalent if $[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}]$ and $[h, Y, \mathscr{R}] \leqslant [g, X, \mathscr{Q}]$, and denote the partially ordered set of equivalence classes of triples by $L(W, \mathscr{T})$. We retain the symbol $[g, X, \mathscr{Q}]$ for the equivalence class of $[g, X, \mathscr{Q}]$ and call it a *$\mathscr{T}$-shadow*. The group W acts on $L(W, \mathscr{T})$ by the rule

$$w \cdot [g, X, \mathscr{Q}] = [wg, X, \mathscr{Q}],$$

which preserves the order in $L(W, \mathscr{T})$.

All triples $[g, X, \mathscr{Q}]$ with $\mathscr{Q} = \varnothing$ form a single class, denoted by $\varnothing$, which is the least element of $L(W, \mathscr{T})$. On the other hand, triples of the form $[g, S, \mathscr{T}]$, for $g \in W$, belong to the class of $[1, S, \mathscr{T}]$, which is the largest element of $L(W, \mathscr{T})$. When $\mathscr{T}$ consists of a single subset T, one can identify a class $[g, X, (T)]$ with the T-shadow $gW_X W_T$.

The intersection formula for T-shadows extends to

THEOREM 3.1. *Suppose that* $g^{-1}h = xwy$, *where* $x \in W_X$, $y \in W_Y$, *and* w *is* (X, Y)*-reduced. Then* $[g, X, \mathscr{Q}]$ *and* $[h, Y, \mathscr{R}]$ *have* $[gx, Z, \mathscr{S}]$ *as their greatest lower bound in* $L(W, \mathscr{T})$, *where* $Z = X \cap Y \cap S_w^{\perp}$ *and* $\mathscr{S}$ *consists of those* T *in* $\mathscr{Q} \cap \mathscr{R}$ *for which* $w \in W_T$.

Proof. If $[f, U, \mathscr{P}] \leqslant [g, X, \mathscr{Q}]$ and $[f, U, \mathscr{P}] \leqslant [h, Y, \mathscr{R}]$ for some element $[f, U, \mathscr{P}]$ in $L(W, \mathscr{T})$, then $\mathscr{P} \subset \mathscr{Q} \cap \mathscr{R}$ and $fW_U W_T \subset gW_X W_T \cap hW_Y W_T$ for all $T \in \mathscr{P}$. By Theorem 1.3, this is possible if and only if $w \in W_T$ and $fW_U W_T \subset gxW_Z W_T$ for all T in $\mathscr{P}$, i.e., if $\mathscr{P} \subset \mathscr{S}$ and $[f, U, \mathscr{P}] \leqslant [gx, Z, \mathscr{S}]$. ∎

Let $X_{\mathscr{T}} = \bigcup_{i=1}^k X_{T_i}$, $X^{\mathscr{T}} = \bigcap_{i=1}^k X^{T_i}$, $X\langle \mathscr{T} \rangle = X_{\mathscr{T}} \cup X^{\mathscr{T}}$ and call X *$\mathscr{T}$-minimal* if $X = X_{\mathscr{T}}$. If $T = \bigcap_{i=1}^k T_i$, it follows from Proposition 2.5 that $X_{\mathscr{T}} = X_T$, $X^{\mathscr{T}} = X^T$, and $X\langle \mathscr{T} \rangle = X\langle T \rangle$, so that X is $\mathscr{T}$-minimal if and only if X is T-minimal. This shows that the properties of T-minimal sets established in Section 2 extend to $\mathscr{T}$-minimal sets. On the other hand, if $\mathscr{Q} \subset \mathscr{R}$, then $X_{\mathscr{Q}} \subset X_{\mathscr{R}}$ and $X^{\mathscr{Q}} \supset X^{\mathscr{R}}$ for all $X \subset S$. In particular, if X is $\mathscr{Q}$-minimal, then X is also $\mathscr{R}$-minimal.

It follows from Proposition 2.2 that the preorder (3.1) can also be described as

$$[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}] \Leftrightarrow \mathscr{Q} \subset \mathscr{R},\ X_{\mathscr{Q}} \subset Y \quad \text{and} \quad S_w \in X^{\mathscr{Q}}, \tag{3.2}$$

where w is the (X, Y)-reduction of $g^{-1}h$. We then conclude as in Corollary 2.3 that $[g, X, \mathscr{Q}]$ and $[h, Y, \mathscr{R}]$ are equivalent if and only if $\mathscr{Q} = \mathscr{R}$, $X_{\mathscr{Q}} = Y_{\mathscr{R}}$, and $gW_{X\langle \mathscr{Q} \rangle} = hW_{Y\langle \mathscr{R} \rangle}$. Each $\mathscr{T}$-shadow can therefore be represented by a unique triple $[g, X, \mathscr{Q}]$ in which X is $\mathscr{Q}$-minimal and g is $(\varnothing, X\langle \mathscr{Q} \rangle)$-reduced. We shall assume that such a triple is normally chosen. The pair $(X, \mathscr{Q})$ is then called the *type* of the class $[g, X, \mathscr{Q}]$, while

$$\dim[g, X, \mathscr{Q}] = \operatorname{card}(X) + \operatorname{card}(\mathscr{Q})$$

is the *dimension* of $[g, X, \mathscr{Q}]$ (or of $(X, \mathscr{Q})$). Classes of type $(X, \mathscr{Q})$ in $L(W, \mathscr{T})$ correspond to left cosets of $W_{X\langle \mathscr{Q} \rangle}$ in W.

The types themselves form a set $M(S, \mathscr{T})$, partially ordered by

$$(X, \mathscr{Q}) \leqslant (Y, \mathscr{R}) \Leftrightarrow X \subset Y \quad \text{and} \quad \mathscr{Q} \subset \mathscr{R}.$$

If $[g, X, \mathscr{Q}] < [h, Y, \mathscr{R}]$ in $L(W, \mathscr{T})$, then $(X, \mathscr{Q}) < (Y, \mathscr{R})$ in $M(S, \mathscr{T})$, so that $\dim[g, X, \mathscr{Q}] < \dim[h, Y, \mathscr{R}]$. This shows that a flag in $L(W, \mathscr{T})$ has at most $n + 1$ elements, where

$$n = \operatorname{card}(S_{\mathscr{T}}) + k \tag{3.3}$$

is the dimension of the largest element $[1, S_{\mathscr{T}}, \mathscr{T}]$ in $L(W, \mathscr{T})$. (Note that $S_{\mathscr{T}}$ is obtained from S by omitting those components of S which are contained in every T in $\mathscr{T}$.) Therefore $L(W, \mathscr{T})$ is a lattice by the general discussion in Section 0; a lattice of this kind will be called a *shadow lattice*.

Any two elements $(X, \mathscr{Q})$ and $(Y, \mathscr{R})$ of $M(S, \mathscr{T})$ have $(X \cup Y, \mathscr{Q} \cup \mathscr{R})$ as their least upper bound in $M(S, \mathscr{T})$, since both X and Y are $\mathscr{Q} \cup \mathscr{R}$)-minimal, while their greatest lower bound is equal to $((X \cap Y)_{\mathscr{Q} \cap \mathscr{R}}, \mathscr{Q} \cap \mathscr{R})$. If $(X, \mathscr{Q}) \leqslant (Y, \mathscr{R})$, one can construct a flag which reaches from $(X, \mathscr{Q})$ to $(X, \mathscr{R})$ in $\operatorname{card}(\mathscr{R}) - \operatorname{card}(\mathscr{Q})$ steps and then from $(X, \mathscr{R})$ to $(Y, \mathscr{R})$ in $\operatorname{card}(Y) - \operatorname{card}(X)$ steps by using Proposition 2.4, since X is necessarily $\mathscr{R}$-minimal. The total number of steps is equal to $\dim(Y, \mathscr{R}) - \dim(X, \mathscr{Q})$, which shows that $M(S, \mathscr{T})$ is a pure lattice of dimension n, given by (3.3). It is called the *type lattice* of $L(W, \mathscr{T})$. Note, however, that although the type map $[g, X, \mathscr{Q}] \to (X, \mathscr{Q})$ preserves order, it is not a lattice homomorphism.

The following result shows that any interval of a shadow lattice is again a shadow lattice.

THEOREM 3.2. *Suppose that $[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}]$ in $L(W, \mathscr{T})$. Then the sublattice of $L(W, \mathscr{T})$ consisting of all $[f, U, \mathscr{P}]$ in $L(W, \mathscr{T})$ such that $[g, X, \mathscr{Q}] \leqslant [f, U, \mathscr{P}] \leqslant [h, Y, \mathscr{R}]$ is isomorphic to $L(W_Z, \mathscr{S})$, where $Z = X^{\mathscr{Q}} \cap Y$ and $\mathscr{S}$ is the family of sets of the form $T \cap Z$ for each $T \in \mathscr{R} \setminus \mathscr{Q}$, together with sets of the form $\{s\}^Z$ for each $s \in Y \setminus X\langle \mathscr{Q} \rangle$.*

Proof. Write $g^{-1}h$ as xwy, where $x \in W_X$, $y \in W_Y$ and w is (X, Y)-reduced. Then $[g, X, \mathscr{Q}] = u \cdot [1, X, \mathscr{Q}]$ and $[h, Y, \mathscr{R}] = u \cdot [1, Y, \mathscr{R}]$ if $u = gxw = hy^{-1}$, with $[1, X, \mathscr{Q}] \leqslant [1, Y, \mathscr{R}]$. Multiplying by u^{-1} allows us to assume that $g = h = 1$.

The element $[f, U, \mathscr{P}]$ lies between $[1, X, \mathscr{Q}]$ nd $[1, Y, \mathscr{R}]$ if and only if $\mathscr{Q} \subset \mathscr{P} \subset \mathscr{R}$, $X \subset U \subset Y$ and $f \in W_Z$. The first two conditions are clear from (3.2). Furthermore, the (X, U)-reduction w of f is contained in $W_{X^{\mathscr{Q}}}$. Writing $f = uwv$ for some $u \in W_X$ and $v \in W_U$, we have $f = w(uv)$ since elements of $X = X_{\mathscr{Q}}$ commute with those of $X^{\mathscr{Q}}$. As $X \subset U$ and f is assumed to be $(\varnothing, U\langle \mathscr{Q} \rangle)$-reduced, we must have $f = w \in W_{X^{\mathscr{Q}}}$. Second, the (U, Y)-reduction w' of f^{-1} is contained in $W_{U^{\mathscr{P}}}$. Writing $f^{-1} = u'w'v'$ for some $u' \in W_U$ and $v' \in W_Y$, we have $f = (v')^{-1} (u'w')^{-1}$, with $(u'w')^{-1} \in W_{U\langle \mathscr{P} \rangle}$, so that again $f = (v')^{-1} \in W_Y$. Hence $f \in W_{X^{\mathscr{Q}}} \cap W_Y = W_Z$. The converse is immediate.

For such a class $[f, U, \mathscr{P}]$, let U^* be the subset $U \cap Z$ of Z and $\mathscr{P}^*$ the subfamily of $\mathscr{S}$ consisting of sets $T \cap Z$ for $T \in \mathscr{P} \setminus \mathscr{Q}$ and $\{s\}^Z$ for $s \in U \setminus X\langle \mathscr{Q} \rangle$. Then U^* is $\mathscr{P}^*$-minimal. Indeed, if $u \in U^*$, there exists a path $u = u_1, ..., u_n$ in Γ contained in U such that $u_n \notin T$ for some $T \in \mathscr{P}$ by the $\mathscr{P}$-minimality of U. Let k be the last index for which $u_1, ..., u_k$ all belong to

U^*. If $k=n$, then $T \notin \mathcal{Q}$, since $u_n \in X^{\mathcal{Q}}$, which is contained in every $T \in \mathcal{Q}$. As u_n certainly does not belong to $T \cap Z$, we have $u \in U^*_{T \cap Z}$. On the other hand, if $k<n$, then $s = u_{k+1}$ does not belong to X, since $u_k \in X^{\mathcal{Q}}$ commutes with elements of X, nor by assumption to $X^{\mathcal{Q}}$ and therefore not to $X\langle \mathcal{Q} \rangle$. Since $u_k \in U^* \backslash \{s\}^Z$, it follows that $u \in U^*_{\{s\}Z}$, with $s \in U \backslash X\langle \mathcal{Q} \rangle$. In particular, $Z = Y^*$ is $\mathcal{S} = \mathcal{R}^*$-minimal. It is easy to verify that $(U^*)^{\mathcal{P}^*}$ is equal to $U^{\mathcal{P}} \cap Z$, so that $U^*\langle \mathcal{P}^* \rangle = U\langle \mathcal{P} \rangle \cap Z$.

We associate to $[f, U, \mathcal{P}]$ the class $[f, U^*, \mathcal{P}^*]$ in $L(W_Z, \mathcal{S})$. If $[f_1, U_1, \mathcal{P}_1] \leqslant [f_2, U_2, \mathcal{P}_2]$, then $[f_1, U_1^*, \mathcal{P}_1^*] \leqslant [f_2, U_2^*, \mathcal{P}_2^*]$. Conversely, if the latter relation holds, $\mathcal{P}_1^* \subset \mathcal{P}_2^*$ implies that $\mathcal{P}_1 \backslash \mathcal{Q} \subset \mathcal{P}_2 \backslash \mathcal{Q}$, so that $\mathcal{P}_1 \subset \mathcal{P}_2$, and also that $U_1 \backslash X\langle \mathcal{Q} \rangle \subset U_2 \backslash X\langle \mathcal{Q} \rangle$. Since $U_1^* \subset U_2^*$ shows that $U_1 \cap X^{\mathcal{Q}} \subset U_2 \cap X^{\mathcal{Q}}$ and both U_1 and U_2 contain X, we must have $U_1 \subset U_2$. It follows that $[f_1, U_1, \mathcal{P}_1] \leqslant [f_2, U_2, \mathcal{P}_2]$. Finally, for a given class $[f, V, \mathcal{V}]$ in $L(W_Z, \mathcal{S})$, $\mathcal{V}$ determines a subfamily $\mathcal{V}_1$ of $\mathcal{R} \backslash \mathcal{Q}$ and a subset V_2 of $Y \backslash X\langle \mathcal{Q} \rangle$. If $\mathcal{P} = \mathcal{Q} \cup \mathcal{V}_1$ and $U = X \cup V \cup V_2$, we have $[f, U^*, \mathcal{P}^*] = [f, V, \mathcal{V}]$. ∎

Since Z is $\mathcal{S}$-minimal, the dimension of the largest element $[1, Z, \mathcal{S}]$ of $L(W_Z, \mathcal{S})$ is given by

$$\operatorname{card}(X^{\mathcal{Q}} \cap Y) + \operatorname{card}(\mathcal{R} \backslash \mathcal{Q}) + \operatorname{card}(Y \backslash X\langle \mathcal{Q} \rangle),$$

which is appropriately equal to $\dim[h, Y, \mathcal{R}] - \dim[g, X, \mathcal{Q}]$.

PROPOSITION 3.3. *Suppose that* $[g, X, \mathcal{Q}] \leqslant [h, Y, \mathcal{R}]$ *in* $L(W, \mathcal{T})$ *and*

$$\dim[h, Y, \mathcal{R}] \geqslant \dim[g, X, \mathcal{Q}] + 2. \qquad (*)$$

Then there exists an element $[f, U, \mathcal{P}]$ *in* $L(W, \mathcal{T})$ *such that* $[g, X, \mathcal{Q}] < [f, U, \mathcal{P}] < [h, Y, \mathcal{R}]$. *Furthermore, if equality holds in* $(*)$, *then there exist precisely two such elements* $[f, U, \mathcal{P}]$.

Proof. By Theorem 3.2, it suffices to consider the lattice $L(W_Z, \mathcal{S})$. Since by hypothesis $\operatorname{card}(Z) + \operatorname{card}(\mathcal{S}) \geqslant 2$ and $\mathcal{S}$ cannot be empty, $L(W_Z, \mathcal{S})$ always contains an element $[f, \varnothing, (T)]$ of dimension 1, which corresponds to the desired element $[f, U, \mathcal{P}]$.

In the case of equality in $(*)$, there are two possibilities. If $Z = \varnothing$ and $\mathcal{S} = (T_1, T_2)$, with $T_1 = T_2 = \varnothing$, there are two elements of dimension 1, namely $[1, \varnothing, (T_1)]$ and $[1, \varnothing, (T_2)]$. Second, if $Z = \{s\}$ and $\mathcal{S} = (T)$, with $T = \varnothing$ (as Z is $\mathcal{S}$-minimal), there are again two elements of dimension 1, namely $[1, \varnothing, (T)]$ and $[s, \varnothing, (T)]$. ∎

It follows that $L(W, \mathcal{T})$ is a pure lattice of dimension n, given by (3.3), and that the lattice dimension of a class $[g, X, Q]$ in $L(W, \mathcal{T})$ coincides with its dimension as defined above.

4. Cones

In this section, we record some facts about cones and establish notation which will be used later in the paper. The reader is referred to texts such as [2, 9] for further details.

A cone $\mathscr{C}$ in a real vector space V is a subset of V containing 0 and closed under addition and positive scalar multiplication. We call $\mathscr{C}$ *pointed* if $\mathscr{C} \cap (-\mathscr{C}) = \{0\}$; each subcone $\mathscr{D}$ of $\mathscr{C}$ is then also pointed. The *polar cone* $\mathscr{C}^\pi$ consists of all $\varphi \in V^*$ such that $\langle \mathscr{C}, \varphi \rangle \leqslant 0$; $\mathscr{C}^\pi$ is pointed whenever $\mathscr{C}$ spans V.

A *face* $\mathscr{D}$ of $\mathscr{C}$ is a subcone $\mathscr{D}$ with the property that if $u+v \in \mathscr{D}$ for $u, v \in \mathscr{C}$, then $u, v \in \mathscr{D}$. If a ray $\mathbb{R}^+u$ is a face of $\mathscr{C}$, it is called an *extreme ray* of $\mathscr{C}$; such rays exist only if $\mathscr{C}$ is pointed. The faces of $\mathscr{C}$ containing an extreme ray $\mathbb{R}^+u$ of $\mathscr{C}$ correspond to faces of the quotient cone $\mathscr{C}/\mathbb{R}u$. If $\mathscr{C}$ is generated by a subset X of $\mathscr{C}$, each face $\mathscr{D}$ of $\mathscr{C}$ is generated by those $u \in X$ which belong to $\mathscr{D}$. In particular, extreme rays of $\mathscr{C}$ correspond to certain elements of X.

A stronger concept is that of an *exposed face*; for this, there has to exist some $\varphi \in \mathscr{C}^\pi$ such that $\mathscr{D} = \mathscr{C} \cap \ker \varphi$. Each face $\mathscr{D}$ of $\mathscr{C}$ determines a *polar face* $\mathscr{D}^\Delta = \{\varphi \in \mathscr{C}^p \mid \langle \mathscr{D}, \varphi \rangle = 0\}$, which is an exposed face of $\mathscr{C}^\pi$.

If $u \in V$, the cone $\mathscr{C} + \mathbb{R}u$ is called the *support cone* of $\mathscr{C}$ at u. For lack of a suitable reference, we provide a proof of

PROPOSITION 4.1. *If $\mathscr{C}$ is a closed pointed cone of dimension $\geqslant 2$, then $\mathscr{C} = \cap \overline{(\mathscr{C} + \mathbb{R}u)}$, where the intersection is taken over all extreme rays $\mathbb{R}^+u$ of $\mathscr{C}$.*

Proof. We can assume that $\mathscr{C}$ spans V, so that $\mathscr{C}^\pi$ is also pointed and of dimension $\geqslant 2$. If $\mathbb{R}^+\psi$ is an extreme ray of C, its polar face $(\mathbb{R}^+\psi)^\Delta$ is nonzero and therefore contains some extreme ray $\mathbb{R}^+u$ of $\mathscr{C}$. Therefore $\mathbb{R}^+\psi$ is contained in the polar face $(\mathbb{R}^+u)^\Delta$. Since $\mathscr{C}^\pi$ is the sum of its extreme rays, it must also be a sum of its faces of the form $(\mathbb{R}^+u)^\Delta$, as $\mathbb{R}^+u$ varies over the extreme rays of $\mathscr{C}$. Taking polar cones, it follows that $\mathscr{C} = \cap \overline{(\mathscr{C} + \mathbb{R}u)}$, since the polar cone of $(\mathbb{R}^+u)^\Delta$ is equal to $\overline{\mathscr{C} + \mathbb{R}u}$. ∎

Let $\mathscr{C}^0$ denote the relative interior of $\mathscr{C}$ in the subspace of V spanned by $\mathscr{C}$. If $\mathscr{C}$ itself is not a subspace, one can easily show that an element $u \in V$ belongs to $\mathscr{C}^0$ if and only if one of the following equivalent statements is true:

(1) for all $v \in \mathscr{C}$, there exists some $a > 0$ such that $u - av \in \mathscr{C}$;

(2) $\mathscr{C}/\mathbb{R}u$ is a subspace and $-u \notin C$.

One can also prove that $u \in \mathscr{C}^0$ if and only if $\langle u, \varphi \rangle < 0$ for all $\varphi \in \mathscr{C}^\pi$ which do not vanish on $\mathscr{C}$. If $\mathscr{D}$ is a proper face of $\mathscr{C}$, then $\mathscr{D} \cap \mathscr{C}^0 = \varnothing$.

PROPOSITION 4.2. *Suppose that $\mathscr{C}$ is a closed cone generated by a set $X \subset \mathscr{C}^0$. Then $\mathscr{C}$ is either a ray or a subspace.*

Proof. If $\mathscr{D}$ is a nonzero face of $\mathscr{C}$, then $\mathscr{D}$ contains some element of X, which is a contradiction, unless $\mathscr{D} = \mathscr{C}$. Since a closed cone of dimension $\geqslant 2$ which is not a subspace has a nonzero proper face, the conclusion follows. ∎

If $\mathscr{C}$ is a finitely generated pointed cone, distinct extreme rays $\mathbb{R}^+ m$ and $\mathbb{R}^+ n$ of $\mathscr{C}$ are *adjacent* if $\mathbb{R}^+ m + \mathbb{R}^+ n$ is a face of $\mathscr{C}$. For every extreme ray $\mathbb{R}^+ k$ not adjacent (or equal) to $\mathbb{R}^+ m$, we have

$$k = -am + \sum b_n n \tag{4.1}$$

for some $a > 0$ and $b_n \geqslant 0$, where the summation is taken over rays $\mathbb{R}^+ u$ adjacent to $\mathbb{R}^+ m$. To see this, note that the extreme rays of $\mathscr{C}/\mathbb{R}^+ m$ correspond to 2-dimensional faces of $\mathscr{C}$ containing $\mathbb{R}^+ m$, i.e., to the neighbours of $\mathbb{R}^+ m$. Since these rays generate $\mathscr{C}/\mathbb{R}^+ m$, we can express k in the form (4.1) and observe that if $a \leqslant 0$, $\mathbb{R}^+ k$ is not an extreme ray of $\mathscr{C}$.

For the purpose of this paper, it is convenient to define a *tesselation* of a cone $\mathscr{C}$ to be a set $\mathscr{F}$ of proper subcones of $\mathscr{C}$ such that:

(1) Each $\mathscr{D} \in \mathscr{F}$ is closed and pointed.

(2) If $\mathscr{D} \in \mathscr{F}$ and $\mathscr{E}$ is a face of $\mathscr{D}$, then $\mathscr{E} \in \mathscr{F}$.

(3) If $\mathscr{D}, \mathscr{E} \in \mathscr{F}$ and $\mathscr{D} \subset \mathscr{E}$, then $\mathscr{D}$ is a face of $\mathscr{E}$.

(4) If $\mathscr{D}, \mathscr{E} \in \mathscr{F}$, then $\mathscr{D} \cap \mathscr{E} \in \mathscr{F}$.

(5) $\mathscr{C} = \bigcup \{\mathscr{D} \mid \mathscr{D} \in \mathscr{F}\}$.

(6) Every closed segment $[u, v]$ in $\mathscr{C}$ has a nonzero element in common only with finitely many $\mathscr{D} \in \mathscr{F}$.

It follows from (3) and (4) that $\mathscr{D} \cap \mathscr{E}$ is a face of both $\mathscr{D}$ and $\mathscr{E}$. Furthermore, $\mathscr{D}^0 \cap \mathscr{E}^0 = \varnothing$ whenever $\mathscr{D} \neq \mathscr{E}$. Indeed, if $u \in \mathscr{D}^0$, then u is not contained in any proper face of $\mathscr{D}$ and hence not in $\mathscr{D} \cap \mathscr{E}$, unless $\mathscr{D} \subset \mathscr{E}$. Each $v \in \mathscr{C}$ belongs to some $\mathscr{D}$ by (5) and therefore to the relative interior of some face $\mathscr{E}$ of $\mathscr{D}$, which must be in $\mathscr{F}$ by (2). Hence $\mathscr{C}$ is the disjoint union of the sets $\mathscr{D}^0$ for $\mathscr{D} \in \mathscr{F}$.

PROPOSITION 4.3. *Suppose that $\mathscr{C}$ is a finitely generated pointed cone and $u \in \mathscr{C}^0$. Then the set $\{\mathscr{D}/\mathbb{R}u \mid \mathscr{D}$ a proper face of $\mathscr{C}\}$ is a tesselation of the subspace $\mathscr{C}/\mathbb{R}u$.*

Proof. We only remark that, to establish (5), note that for a given $v \in \mathscr{C}$ there exists some $b \in \mathbb{R}$ such that $v - bu$ belongs to the boundary of $\mathscr{C}$ and hence to some proper face $\mathscr{D}$ of $\mathscr{C}$. It follows that $v + \mathbb{R}u \in \mathscr{D}/\mathbb{R}u$. ∎

5. The Lattice $L(W, P)$

Suppose that $W \to \mathrm{GL}(E)$ is the standard geometric representation of W [1, V.4] and $(\cdot, \cdot)$ the canonical symmetric bilinear form on E. The group W is called *euclidean* if $(\cdot, \cdot)$ is positive semidefinite and its graph Γ is connected. On the other hand, W is *hyperbolic* if $(\cdot, \cdot)$ is of signature $(n-1, 1)$, where $n = \mathrm{card}(S)$, while $W_{S \setminus s}$ is finite or euclidean for all $s \in S$.

Let V be the direct sum of E with a finite-dimensional real vector space E_1, with W acting trivially on E_1. Then W also acts in a contragradient manner on the dual space $V^* = E^* \oplus E_1^*$. In particular, the action of $s \in S$ on $p \in V^*$ is given by $s(p) = p - 2\langle e_s, p\rangle \varepsilon_s$, where $\varepsilon_s \in E^*$ is defined by $\langle v, \varepsilon_s\rangle = (v, e_s)$ for all $v \in E$. A relation $\sum_{s \in S} a_s \varepsilon_s = 0$ occurs whenever $\sum_{s \in S} a_s e_s$ belongs to the radical of $(\cdot, \cdot)$; hence the ε_s are linearly independent if and only if the form $(\cdot, \cdot)$ is nonsingular.

Let $\bar{C} = \{p \in V^* \mid \langle e_s, p\rangle \geqslant 0 \text{ for all } s \in S\}$ be the extension of the closed fundamental chamber in E^* to V^*. The following three results follow immediately from [1, V.4].

Proposition 5.1. *If $p \in \bar{C}$, $w \in W$, and $s \in S$ are such that $l(sw) > l(w)$, then $\langle e_s, w(p)\rangle \geqslant 0$.*

Proposition 5.2. *The stabiliser in W of any element $p \in \bar{C}$ is generated by those $s \in S$ for which $\langle e_s, p\rangle = 0$.*

Proposition 5.3. *If $p, q \in \bar{C}$ and $w, w' \in W$ are such that $w(p) = w'(q)$, then p and q have the same stabiliser W_T in W and $wW_T = w'W_T$.*

One also shows by induction on $l(w)$ that

Proposition 5.4. *If $p \in \bar{C}$ and $w \in W$, then $w(p) = p - \sum_{s \in S} a_s \varepsilon_s$ for some $a_s \geqslant 0$. Furthermore, if $\mathrm{Stab}_W(p) = W_T$ and w is $(\varnothing, T)$-minimal, then $a_s > 0$ precisely for $s \in S_w$.*

For each $X \subset S$, let E_X^* be the subspace of E^* spanned by $\{\varepsilon_s \mid s \in X\}$ and $U_X = \bigcup_{w \in W_X} w(C)$. The argument of [1, V.4.6] extends to show that each U_X is a (convex) cone. We call $U = U_S$ the "Tits cone" in V^*. The set X also determines a face

$$\bar{C}_X = \{p \in \bar{C} \mid \langle e_s, p\rangle = 0 \text{ for all } s \in X\}$$

of the cone $\bar{C}$.

PROPOSITION 5.5. *Suppose that X and Y are subsets of S and w is an (X, Y)-reduced element of W. Then*

$$U_X \cap wU_Y = \bigcup_{u \in W_Z} u(\bar{C}_{S_w}),$$

where $Z = X \cap Y \cap S_w^-$.

Proof. If $p \in \bar{C}_{S_w}$ and $u \in W_Z$, we have $w(p) = p$ and $uw = wu$, so that $u(p) = uw(p) = wu(p)$ belongs to $U_X \cap wU_Y$.

Conversely, an element of $U_X \cap wU_Y$ can be written as $x(p) = wy(q)$ for some $p, q \in \bar{C}$, $x \in W_X$, and $y \in W_Y$. It follows from Proposition 5.3 that p and q have the same stabiliser W_T in W and that $xW_T = wyW_T$. Therefore $w \in W_T$, since w is (X, Y)-reduced, and $x \in W_X W_T \cap wW_Y W_T = W_Z W_T$ by Theorem 1.3. We conclude that $S_w \subset T$, $p \in \bar{C}_{S_w}$, and $x(p) \in u(\bar{C}_{S_w})$ for some $u \in W_Z$. ∎

If P is a finite subset of $\bar{C}$, let $[W, P]$ be the W-invariant cone generated in V^* by all vectors $w(p)$, where $w \in W$ and $p \in P$. For each $g \in W$, $X \subset S$, and $Q \subset P$, let $[g, X, Q]$ be the subcone of $[W, P]$ generated by vectors of the form $gw(p)$, for $w \in W_X$ and $p \in Q$. If $L(W, P)$ denotes the set of all such cones, ordered by inclusion, W acts on $L(W, P)$ by the rule $w \cdot [g, X, Q] = [wg, X, Q]$. Note that a cone $[g, X, Q]$ is contained in gU_X.

The following formula will often be used in inductive arguments.

PROPOSITION 5.6. *Suppose that $p \in P$ and* $\mathrm{Stab}_W(p) = W_T$, *so that W_T acts on $V^*/\mathbb{R}p$. Then*

(a) $[W, P]/\mathbb{R}p = [W_T, \bar{P}]$, *where $\bar{P}$ is the image in $V^*/\mathbb{R}p$ of the set $(P \backslash p) \cup \{-\varepsilon_s \mid s \in S \backslash T\}$;*

(b) $\mathrm{card}(T) + \mathrm{card}(\bar{P}) < \mathrm{card}(S) + \mathrm{card}(P)$;

(c) *for every cone $[g, Y, \bar{R}]$ in $L(W_T, \bar{P})$, we have $\{g, Y, \bar{R}] = [g, X, Q]/\mathbb{R}p$, where $X = Y \cup \{s \in S \mid -\varepsilon_s + \mathbb{R}p \in \bar{R}\}$ and $Q = \{q \in P \mid q + \mathbb{R}p \in \bar{R}\} \cup \{p\}$.*

Proof. Since $\langle e_t, -\varepsilon_s \rangle \geq 0$ for $t \in T$ and $s \in S \backslash T$, $\bar{P}$ is contained in the closed fundamental chamber in $V^*/\mathbb{R}p$ for the group W_T.

Consider a generator $w(q)$ of $[W, P]$. Arguing by induction on $l(w)$, write $w = sw'$, with $l(w) > l(w')$, and express $w'(q) \bmod \mathbb{R}p$ in terms of the generators of $[W_T, \bar{P}]$. If $s \in T$, we multiply this expression by s to obtain a similar expression for $w(q)$. If $s \in S \backslash T$, then $w(q) = w'(q) + 2\langle e_s, w'(q) \rangle (-\varepsilon_s)$, while $\langle e_s, w'(q) \rangle \geq 0$ by Proposition 5.1. Conversely, a generator $-\varepsilon_s \in \bar{P}$ is the image of $s(p)/2\langle e_s, p \rangle$.

Statement (b) is obvious, while (c) can be obtained by applying (a) with $S = X$ and $P = Q$ and then multiplying by g. ∎

We also observe

PROPOSITION 5.7. *If g is $(\emptyset, X)$-reduced, then* $\bar{C} \cap [g, X, Q] = \bar{C} \cap [1, X \cap S_g^{\perp}, Q']$, *where* $Q' = \{p \in Q \mid g(p) = p\}$.

Proof. It is clear that an element of $[1, X \cap S_g^{\perp}, Q']$ is fixed by g and therefore belongs to $[g, X, Q]$.

Conversely, let v be a nonzero element of $\bar{C} \cap [g, X, Q]$. Then $v \in U_{\emptyset} \cap gU_X = \bar{C}_{S_g}$ by Proposition 5.5. In particular, v is fixed by g, so that $v \in [1, X, Q]$ and one can write $v = \sum a_u u(p)$, where all $a_u > 0$, $u \in W_X$, and $p \in Q$.

Let $X' \subset X$ be the union of the sets S_u, for those $u(p)$ which occur in this representation of v and $Q' \subset Q$ the set of all p. If $s \in S_g \backslash X'$, then $l(su) > l(u)$ for all u, so that $\langle e_s, u(p) \rangle \geqslant 0$ by Proposition 5.1. However, since v is fixed by g, we have $\langle e_s, v \rangle = 0$, so that in fact $\langle e_s, u(p) \rangle = 0$ and hence $su(p) = u(p)$. It follows that $su \in uW_T$, where $W_T = \mathrm{Stab}_W(p)$, which is possible only if $s \in T$ and $s \in S_u^{\perp}$ by Proposition 1.1. Therefore s commutes with elements of X' and g can be written as $g_1 g_2$, where g_1 is a product of $s \in S_g \backslash X'$, while g_2 is a product of $s \in S_g \cap X'$. Since g is assumed to be $(\emptyset, X)$-reduced, $g_2 = 1$. Therefore $g(p) = p$ for all $p \in Q'$ and $X' \subset X \subset S_g^{\perp}$, so that $v \in [1, X \cap S_g^{\perp}, Q']$. ∎

COROLLARY 5.8. *Suppose that* $\mathrm{Stab}_W(p)$ *is finite for all $p \in P$. Then if $[u, v]$ is a closed segment in U, we have $[u, v] \cap [g, X, Q] \neq \{0\}$ for only finitely many $[g, X, Q] \in L(W, P)$.*

Proof. It follows from [1, V.4.6] that $[u, v]$ is covered by finitely many sets of the form $w(\bar{C})$, for $w \in W$. On the other hand, $w(\bar{C}) \cap [g, X, Q] \neq \{0\}$ only if $w^{-1}g \in \mathrm{Stab}_W(p)$ for some $p \in Q$, by Proposition 5.7. ∎

PROPOSITION 5.9. *Suppose that* $\bar{C} \cap [1, X, Q] \cap [1, Y, R] = \bar{C} \cap [1, X \cap Y, Q \cap R]$ *for all elements of $L(W, P)$ of this form. Then for any $[g, X, Q]$ and $[h, Y, R]$ in $L(W, P)$, if $g^{-1}h = xwy$ with $x \in W_X$, $y \in W_Y$, and w (X, Y)-reduced, we have*

$$[g, X, Q] \cap [h, Y, R] = [gx, Z, S],$$

where $Z = X \cap Y \cap S_w^{\perp}$ *and* $S = \{p \in Q \cap R \mid w(p) = p\}$.

Proof. By multiplying with $(gx)^{-1}$, it suffices to consider $[1, X, Q] \cap [w, Y, R]$. If v belongs to this intersection, then $v \in U_X \cap wU_Y \subset U_Z$, so that

$v = u(v')$ for some $u \in W_Z$ and $v' \in C$. Since $uw = wu$, $v' \in [1, X, Q] \cap [w, Y, R]$. However, Proposition 5.7 shows that $v' \in [1, X, Q] \cap [1, Y \cap S_w^\perp, R']$, where $R' = \{p \in R \mid w(p) = p\}$, so that $v' \in [1, Z, S]$ and hence $v \in [1, Z, S]$. ▮

Let $\mathscr{T} = (T_p)_{p \in P}$, where $\mathrm{Stab}_W(p) = W_{T_p}$. Each subfamily $\mathscr{Q}$ of $\mathscr{T}$ corresponds to a subset Q of P and we may associate the cone $[g, X, Q]$ to the triple $[g, X, \mathscr{Q}]$.

PROPOSITION 5.10. *Suppose that a relation $\sum_{p \in P} a_p p \in E_S^*$ implies that all a_p are either $\geqslant 0$ or all $\leqslant 0$. Then $[g, X, Q] \subset [h, Y, R]$ if and only if $[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}]$.*

Proof. If $[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}]$, then $\mathscr{Q} \subset \mathscr{R}$ and $gW_X W_T \subset hW_Y W_T$ for all $T \in \mathscr{Q}$. Therefore $Q \subset R$ and each generator $gw(p)$ of $[g, X, Q]$ belongs to $[h, Y, R]$, so that $[g, X, Q] \subset [h, Y, R]$.

Conversely, if $[g, X, Q] \subset [h, Y, R]$, then for all $p \in Q$ and $w \in W_X$, $gw(p) \in [h, Y, R]$. Using Proposition 5.4, we deduce a relation $p = \sum_{r \in R} a_r r \bmod E_S^*$ for some $a_r \geqslant 0$, not all zero, which contradicts the hypothesis unless $p \in R$. Therefore $Q \subset R$ and hence $\mathscr{Q} \subset \mathscr{R}$. Second, since $[h, Y, R] \subset hU_Y$, there exist $u \in W_Y$ and $c \in \bar{C}$ such that $gw(p) = hu(c)$. Proposition 5.3 shows that $gw \in huW_T$, where $W_T = \mathrm{Stab}_W(p)$, so that $gW_X W_T \subset hW_Y W_T$ for all $p \in Q$ and hence $[g, X, \mathscr{Q}] \leqslant [h, Y, \mathscr{R}]$. ▮

It follows that under the hypothesis of Proposition 5.10, the map $[g, X, \mathscr{Q}] \to [g, X, Q]$ is then a bijection between the lattice $L(W, \mathscr{T})$ and the ordered set $L(W, P)$, which preserves order in both directions. Therefore $L(W, P)$ is a lattice isomorphic to $L(W, \mathscr{T})$. However, the greatest lower bound of two elements of $L(W, P)$ need not be their intersection, unless one knows that this intersection belongs to $L(W, P)$.

We shall consider the following four situations, in all of which the hypothesis of Proposition 5.10 is satisfied.

(F) The elements of $P \cup \{\varepsilon_s \mid s \in S\}$ are linearly independent.

(S) W is finite and there is a unique relation $\sum_{p \in P} a_p p = \sum_{s \in S} b_s \varepsilon_s$, in which all the a_p and b_s are >0.

(E) W is euclidean, $E_1 = \{0\}$, and $P = \{p\}$ for some nonzero $p \in \bar{C}$. We then have a unique relation $\sum_{s \in S} c_s \varepsilon_s = 0$ with all $c_s > 0$, while $p \notin E_S^*$.

(H) W is hyperbolic, $E_1 = \{0\}$, and $P = \{p\}$ for some nonzero $p \in \bar{C}$. Then $p = \sum_{s \in S} c_s \varepsilon_s$ with all $c_s \leqslant 0$.

The word "unique" in the above hypotheses means of course "unique up to a multiple."

THEOREM 5.11. *In case* (F), $L(W, P)$ *is the face lattice of the pointed cone* $[W, P]$. *Furthermore, every face of* $[W, P]$ *is exposed.*

Proof. If $[W, P]$ was not pointed, we would have a nontrivial relation $\sum a_{w,p} w(p) = 0$ among the generators of $[W, P]$, with all $a_{w,p} \geqslant 0$. By Proposition 5.4, this means that $\sum_p (\sum_w a_{w,p}) p \in E^*$, so that $\sum_w a_{w,p} = 0$ for all $p \in P$ and hence each $a_{w,p} = 0$, a contradiction.

We show next that every $[g, X, Q] \in L(W, P)$ is an exposed face of $[W, P]$, for which we may assume that $g = 1$. Since $P \cup \{\varepsilon_s \mid s \in S\}$ is linearly independent, there exists some $\varphi \in V$, regarded as the dual space of V^*, such that $\langle \varphi, p \rangle \leqslant 0$ for all $p \in R$, with equality precisely for $p \in Q$, and $\langle \varphi, \varepsilon_s \rangle \geqslant 0$ for all $s \in Y$, with equality precisely for $s \in X$. It is then clear from Proposition 5.4 that $\varphi \in [W, P]^\pi$, while $[W, P] \cap \operatorname{Ker} \varphi = [1, X, Q]$.

Conversely, we show by induction on $\operatorname{card}(S) + \operatorname{card}(P)$ that every face of $[W, P]$ is of the form $[g, X, Q]$. The zero face of $[W, P]$ is equal to any cone $[g, X, Q]$ with $Q = \varnothing$. If F is a nonzero face, choose an extreme ray $\mathbb{R}^+ w(p)$ of F; then $\mathbb{R}^+ p$ is an extreme ray of the face $w^{-1}(F)$. Consider the face $w^{-1}(F)/\mathbb{R}p$ of the quotient cone $[W, P]/\mathbb{R}p = [W_T, \bar{P}]$, in the notation of Proposition 5.6. Since the elements of $\bar{P} \cup \{\varepsilon_s + \mathbb{R}p \mid s \in T\}$ are linearly independent, the induction hypothesis implies that $w^{-1}(F)/\mathbb{R}p$ is equal to some face $[h, Y, \bar{R}]$ of $[W_T, \bar{P}]$. Since $[h, X, Q]$, in the same notation, is a face of $[W, P]$ containing $\mathbb{R}^+ p$, we must have $w^{-1}(F) = [h, X, Q]$ and $F = [g, X, Q]$ with $g = wh$. ∎

The intersection of two elements of $L(W, P)$ in this case is a face of $[W, P]$ and therefore belongs to $L(W, P)$. Since $L(W, P)$ is isomorphic to $L(W, \mathscr{T})$, Theorem 3.1 shows that the intersection formula of Proposition 5.9 is valid.

When W is finite, the cone $[W, P]$ is finitely generated and therefore closed. A bounded cross-section of $[W, P]$ is a polytope whose face lattice is isomorphic to $L(W, P)$.

For a given family $\mathscr{T} = (T_1, ..., T_k)$ of subsets of S, we can always find a family $P = (p_1, ..., p_k)$ of points in a suitably large space V^* such that $\operatorname{Stab}_W(p_i) = W_{T_i}$ and the elements of P are linearly independent mod E^*. If the form $(\cdot, \cdot)$ is nonsingular, Theorem 5.11 shows that the shadow lattice $L(W, \mathscr{T})$ is then realised as the lattice $L(W, P)$.

When W is finite, it follows that Euler's formula must be valid for $L(W, \mathscr{T})$. If $\mathscr{T}$ consists of a single subset T of S, this asserts that

$$\sum_{T\text{-minimal } X} (-1)^{\operatorname{card}(X)} [W : W_{X\langle T\rangle}] = 1.$$

When $T = \varnothing$, every subset of S is T-minimal, while $X\langle T\rangle = X$. The above equation can then be written as

$$\frac{1}{\operatorname{card}(W)} = \sum_{X \subset S} \frac{(-1)^{\operatorname{card}(X)}}{\operatorname{card}(W_X)},$$

which is the formula of [1, V.3, Exer. 5].

Let $L'(W, P)$ be the set obtained from $L(W, P)$ by omitting the cone $[W, P]$.

THEOREM 5.12. *In case* (S), $L'(W, P)$ *is a tesselation of the subspace* $[W, P]$. *Furthermore,*

$$\bar{C} \cap [W, P] \subset \bigcup_{(X, Q) < (S, P)} [1, X, Q].$$

Proof. We first show by induction on $\operatorname{card}(S) + \operatorname{card}(P)$ that $[W, P]$ is a subspace. For any $p \in P$, we have $[W, P]/\mathbb{R}p = [W_T, \bar{P}]$ in the notation of Proposition 5.6. Furthermore, there is a similar unique relation

$$\sum_{q \in P \setminus p} a_q q + \sum_{s \in S \setminus T} b_s(-\varepsilon_s) = \sum_{s \in T} b_s \varepsilon_s \bmod \mathbb{R}p$$

among the elements of $\bar{P} \cup \{\varepsilon_s + \mathbb{R}p \mid s \in T\}$. The inductive hypothesis shows that $[W_T, \bar{P}]$ is a subspace, so that p, and hence $w(p)$, belongs to $[W, P]^0$ for all $p \in Q$ and $w \in W$. Since W is finite, $[W, P]$ is closed and therefore must be either a subspace or a ray by Proposition 4.2. The latter case is ruled out because P then consists of a single element $p \neq 0$ fixed by W, so that p cannot belong to E^*.

Now let $V' = V \oplus \mathbb{R}$, with W acting trivially on $\mathbb{R}$ and P' the set of all points $p' = (p, 1)$ in V' corresponding to $p \in P$. A relation $\sum_{p' \in P'} a_p p' \in E^*$ means that $\sum_{p \in P} a_p p \in E^*$ and $\sum_{p \in P} a_p = 0$; since all the a_p must be of the same sign, all are zero. Therefore the elements of $P' \cup \{\varepsilon_s \mid s \in S\}$ are linearly independent.

It follows from Theorem 5.11 that $[W, P']$ is a pointed cone in V' whose face lattice is $L(W, P')$. Furthermore, $(0, 1) \in [W, P']^0$, since $[W, P']/\mathbb{R}(0, 1)$ is the subspace $[W, P]$. Using Proposition 4.3, we conclude that $L'(W, P)$ is a tesselation of $[W, P]$.

An element of $\bar{C} \cap [W, P]$ therefore belongs to some cone $[g, X, Q]$ with $(X, Q) < (S, P)$, and hence to $[1, X \cap S_g^{\perp}, Q']$ by Proposition 5.7. ∎

It again follows that the intersection formula of Proposition 5.9 is valid for $L(W, P)$. By taking the intersections of elements of $L'(W, P)$ with a sphere centred on 0 in $[W, P]$, we obtain a spherical tesselation isomorphic to $L'(W, P)$.

The following observation will be useful in the remaining cases.

PROPOSITION 5.13. *If $v \in \bar{C}$ belongs to $[1, X, Q] + \mathbb{R}p$ for some $p \in Q$, then v also belongs to $[1, X, Q] + \mathbb{R}w(p)$ for all $w \in W_X$.*

Proof. Since $[1, X, Q]$ is invariant under W_X, it is equivalent to show that $w(v) \in [1, X, Q] + \mathbb{R}p$ for all $w \in W_X$. We argue by induction on $l(w)$. Write $w = sw'$ with $l(w) > l(w')$ and assume that $w'(p) \in [1, X, Q] + \mathbb{R}p$. If $s(p) = p$, the assertion is trivial. When $s(p) \neq p$, i.e., $\langle e_s, p\rangle > 0$, we have $w(v) = w'(v) - 2\langle e_s, w'(v)\rangle \varepsilon_s$ with $\langle e_s, w'(v)\rangle \geqslant 0$ by Proposition 5.1 and $s(p) = p - 2\langle e_s, p\rangle \varepsilon_s$, so that

$$w(v) = w'(v) + \frac{\langle e_s, w'(v)\rangle}{\langle e_s, p\rangle}(s(p) - p)$$

belongs to $[1, X, Q] + \mathbb{R}p$. ∎

THEOREM 5.14. *In case* (E), *the elements of $L'(W, P)$ form a tesselation of $[W, P] = U$. Furthermore, $\bar{C} \subset \bigcup_{X \neq S}[1, X, p]$.*

Proof. For $X \neq S$, W_X is finite and the elements of $\{p\} \cup \{\varepsilon_s \mid s \in X\}$ are linearly independent. By Theorem 5.11, each cone $[g, X, p]$ is closed and pointed, while every face of $[g, X, p]$ belongs to $L'(W, P)$.

If $[g, X, p] \subset [h, Y, p]$, then $X \subset Y$ by (3.2), assuming that X is taken to be T-minimal, where $W_T = \mathrm{Stab}_W(p)$. Furthermore, if $g^{-1}h = xwy$ in the same notation, then $S_w \subset X^T$ and $h^{-1}g \in W_Y W_{X\langle T\rangle}$, so that we may assume that $h^{-1}g \in W_Y$. If $Y \neq S$, Theorem 5.11 shows that $[h^{-1}g, X, p]$ is a face of $[1, Y, p]$ and hence that $[g, X, p]$ is a face of $[h, Y, p]$.

Consider the quotient $[W, P]/\mathbb{R}p = [W_T, \bar{P}]$, in the notation of Proposition 5.6. Then W_T is finite and we have a unique relation

$$\sum_{s \in S \setminus T} c_s(-\varepsilon_s) = \sum_{s \in T} c_s \varepsilon_s \bmod \mathbb{R}p \tag{5.1}$$

among elements of $P \cup \{\varepsilon_s + \mathbb{R}p \mid s \in T\}$, with all $c_s > 0$. Therefore the pair $(W_T, \bar{P})$ is of type (S) and $L'(W_T, \bar{P})$ is a tesselation of the subspace $[W_T, \bar{P}] = E^*/\mathbb{R}p$ by Theorem 5.12.

We have $[1, X, p]/\mathbb{R}p \cap [1, Y, p]/\mathbb{R}p = [1, X \cap Y, p]/\mathbb{R}p$, by the intersection formula of Proposition 5.9, applied for $L'(W_T, P)$, as well as Proposition 5.6. Therefore an element $v \in [1, X, p] \cap [1, Y, p]$ belongs to $[1, X \cap Y, p]$. If $v \in \bar{C}$ as well, Proposition 5.13 shows that v belongs to the support cone of $[1, X \cap Y, p]$ at $w(p)$ for all $w \in W_{X \cap Y}$ and hence to $[1, X \cap Y, p]$ itself by Proposition 4.1, since $W_{X \cap Y}$ is finite. It now follows from Proposition 5.9 that the intersection formula given there is valid for $L'(W, P)$.

If $v \in \bar{C}$, its image $\tilde{v}$ in $E^*/\mathbb{R}p$ is in the closed fundamental chamber for the group W_T. Therefore $\tilde{v}$ belongs to some cone $[1, Y, \bar{R}]$ with

$(Y, \bar{R}) < (T, \bar{P})$ by Theorem 5.12, so that $v \in [1, X, p] + \mathbb{R}p$ for some $X \neq S$. Using Proposition 5.13, it follows as above that $v \in [1, X, p]$ since W_X is finite. Therefore $\bar{C} \subset \bigcup_{X \neq S} [1, X, p]$ and $U \subset \bigcup_{g \in W, X \neq S} [g, X, p] \subset [W, P] \subset U$, which proves property (5), as well as showing that $[W, P] = U$.

Finally, Property (6) follows from Corollary 5.8. ∎

The intersection of elements of $L'(W, P)$ with the affine hyperplane $A = \{v \in E^* \mid \langle c, v \rangle = 1\}$ in E^*, where $c = \sum_{s \in S} c_s e_s$, is then a tesselation of A isomorphic to $L'(W, P)$.

It remains to consider the case (H). We shall identify E^* with E and ε_s with e_s by using the nonsingular form $(\cdot, \cdot)$. If $\{\omega_s\}_{s \in S}$ is the dual basis of $\{e_s\}_{s \in S}$ in E, we have $(\omega_s, \omega_{s'}) \leqslant 0$ for all $s, s' \in S$, with equality only if $s = s'$ and $W_{S \setminus s}$ is euclidean. Therefore, if $p \in \bar{C}$, then $p = \sum_{s \in S} (p, \omega_s) e_s$, with all $(p, \omega_s) \leqslant 0$; furthermore, $(p, \omega_s) = 0$ for some $s \in S$ only if $p \in \mathbb{R}^+ \omega_s$ and $(\omega_s, \omega_s) = 0$. The Tits cone U is equal to a connected component of the set $\{v \in E \mid (v, v) < 0\}$, together with 0 and the rays $\mathbb{R}^+ w(\omega_s)$ for all $w \in W$ and ω_s satisfying $(\omega_s, \omega_s) = 0$. See, for example, [1] or [7].

If $X = S \setminus s$ is such that W_X is euclidean, a cone $[g, X, p]$, for $g \in W$, need not be closed and we shall first describe its closure $\overline{[g, X, p]}$.

Proposition 5.15. *Suppose that $X = S \setminus s$ is such that W_X is euclidean. Then for any $p \neq 0$ in $\bar{C}$, $\omega_s \in \overline{[1, X, p]}$.*

Proof. Since $(\omega_s, \omega_s) = 0$, $\omega_s = \sum_{i \neq s} (\omega_i, \omega_s) e_i \in E_X$, so that $\mathbb{R}\omega_s$ is the radical of E_X with respect to the form $(\cdot, \cdot)$. The group W_X acts on $E_X/\mathbb{R}\omega_s$, resulting in a homomorphism $W_X \to \mathrm{GL}(E_X/\mathbb{R}\omega_s)$. Let W'_X be the kernel of this homomorphism. For any $w \in W'_X$, we have $(w - 1)(e_i) \in \mathbb{R}\omega_s$ for all $i \in X$, so that $(w - 1)^2$ vanishes on E_X. Furthermore, since $s \notin S_w$, $(w - 1)(e_s) = a_w \in E_X$, and hence $(w - 1)^3 = 0$ on all of E. In fact, w is then a "translation" in W_X with respect to the class of a_w in $E_X/\mathbb{R}\omega_s$ [1]. Since such translations form a lattice, we know in particular that $W'_X \neq \{1\}$.

The proposition is trivial if p is a positive multiple of ω_s. Otherwise, let $w \neq 1$ be an element of W'_X and suppose $k \in X$ is such that $w(e_k) \neq e_k$. Write $w(e_k) = e_k + a\omega_s$ for some $a \neq 0$ and form the commutator $w_1 = s_k w s_k^{-1} w^{-1} \in W'_X$. Then $(w_1 - 1)^2 (p) = b\omega_s$, where $b = -4a^2(\omega_s, p)$. Since p is a nonnegative combination of the ω_i and $(\omega_s, \omega_i) < 0$ for all $i \neq s$, we have $(\omega_s, p) < 0$ and hence $b > 0$.

Using the fact that $(w_1 - 1)^3 = 0$, it follows that

$$w_1^N(p) = p + \binom{N}{1} (w_1 - 1)(p) + \binom{N}{2} b\omega_s$$

for all integers $N \geqslant 1$, so that the sequence $w_1^N(p)/\binom{N}{2} b$ tends to ω_s as $N \to \infty$. ∎

Proposition 5.16. *Suppose that $X = S\backslash s$ is such that W_X is euclidean. Then for any nonzero $p \in \bar{C}$ and $g \in W$,*

(a) $\overline{[g, X, p]} = [g, X, p] \cup \mathbb{R}^+ g(\omega_s)$;

(b) *the faces of $\overline{[g, X, p]}$ are precisely the faces of $[g, X, p]$, together with the ray $\mathbb{R}^+ g(\omega_s)$.*

Proof. The statement is trivial if $p \in \mathbb{R}^+ \omega_s$. Otherwise, we may assume that $g = 1$ and regard $E/\mathbb{R}\omega_s$ as the dual space of E_X, with the class $\tilde{e}_i$ of e_i playing the role of ε_i for $i \neq s$. The class $\tilde{p}$ of p is nonzero and belongs to the closed fundamental chamber in $E/\mathbb{R}\omega_s$ for the group W_X. We already know from Proposition 5.15 that $\omega_s \in \overline{[1, X, p]}$.

It follows from Theorem 5.14 that $L'(W_X, \{\tilde{p}\})$ is a tesselation of the Tits cone U' in $E/\mathbb{R}\omega_s$, with $[1, X, \tilde{p}] = U'$. The closure $\overline{U'}$ of U' is the set of all $\tilde{v} \in E/\mathbb{R}\omega_s$ such that $\langle \omega_s, \tilde{v} \rangle \leqslant 0$, while $\langle \omega_s, \tilde{v} \rangle < 0$ for nonzero elements $\tilde{v} \in U'$.

If $v \in \overline{[1, X, p]}$, then $\tilde{v} \in \overline{U'}$, so that either $\langle \omega_s, \tilde{v} \rangle = 0$, or $\tilde{v} \in [h, Y, \tilde{p}]$ for some $h \in W_X$ and $Y \subsetneq X$. In the first case, $v \in E_X$, which is only possible if $v \in \mathbb{R}^+ \omega_s$, since $(v, v) \leqslant 0$. In the second, write $v = u + a\omega_s$ for some $u \in [h, Y, p]$ and $a \in \mathbb{R}$.

Since the elements of $\{p\} \cup \{e_s \mid s \in X\}$ are linearly independent, Theorem 5.11 applies to the cone $[1, X, p]$. For every $\varphi \in \overline{[1, X, p]}^\pi$, we have $(\omega_s, \varphi) \leqslant 0$ since $\omega_s \in \overline{[1, X, p]}$. Furthermore, if $(\omega_s, \varphi) = 0$, then φ belongs to the polar cone of $[1, X, \tilde{p}] = U'$ and is therefore a positive multiple of ω_s, since $\overline{U'}$ is the halfspace determined by ω_s. Thus $\overline{[1, X, p]}^\pi \cap (\mathbb{R}\omega_s)^\perp = \mathbb{R}^+ \omega_s$ and hence, by taking polar cones,

$$\overline{[1, X, p] + \mathbb{R}\omega_s} = \overline{U'}. \tag{5.2}$$

In particular, if φ is such that $[h, Y, p] = [1, X, p] \cap \ker \varphi$, we have $(\omega_s, \varphi) < 0$ and $0 \geqslant (v, \varphi) = 0 + a(\omega_s, \varphi)$, so that $a \geqslant 0$. If $a = 0$, $v = u$ belongs to $[h, Y, p]$. On the other hand, if $a > 0$, then $(v, \varphi) < 0$ for all nonzero $\varphi \in \overline{[1, X, p]}^\pi$, so that $v \in \overline{[1, X, p]}^0 = [1, X, p]^0$. In either case, we have $v \in [1, X, p]$, so that $\overline{[1, X, p]} = [1, X, p] \cup \mathbb{R}^+ \omega_s$.

If F is a face of $\overline{[1, X, p]}$ not containing ω_s, then F is a face of $[1, X, p]$. On the other hand, if $\omega_s \in F$ and $u \in F \backslash \mathbb{R}^+ \omega_s$, the preceding argument shows that $u + \omega_s \in F \cap \overline{[1, X, p]}^0$, which is a contradiction unless $F = \overline{[1, X, p]}$. ∎

Let $L''(W, P)$ be the set derived from $L'(W, P)$ by replacing each cone $[g, X, p]$ by $\overline{[g, X, p]}$ whenever W_X is euclidean and adding the ray $\mathbb{R}^+ g(\omega_s)$, where $X = S\backslash s$.

Theorem 5.17. *In case* (H), *the elements of $L''(W, P)$ form a tesselation* *of U, except that for property (6) to hold, p should not be a multiple of any ω_s with $(\omega_s, \omega_s) = 0$. Furthermore, $\bar{C} \subset \bigcup_{X \neq S} \overline{[1, X, p]}$.*

Proof. Properties (1–4) are established as in the proof of Theorem 5.14, with minor modifications based on Proposition 5.16, while the statement about property (6) follows from Corollary 5.8.

If $v \in \bar{C}$, one shows in the same way that $v \in [1, X, p] + \mathbb{R}w(p)$ for some $X \subsetneq S$ and all $w \in W_X$, except that $W_T = \mathrm{Stab}_W(p)$ may be euclidean if p is a multiple of some ω_j with $(\omega_j, \omega_j) = 0$. In that case, (5.1) is replaced by the relation

$$\sum_{i \neq j} -(p, \omega_i) e_i = 0 \qquad \mathrm{mod}\ \mathbb{R}p$$

with all $-(p, \omega_i) > 0$ and Theorem 5.14 is used instead of Theorem 5.12. Furthermore, if $X = S\backslash s$ is such that W_X is euclidean, then v also belongs to $\overline{[1, X, p] + \mathbb{R}\omega_s}$ by (5.2), so that Proposition 4.1 again shows that $v \in \overline{[1, X, p]}$. ∎

6. Hyperbolic Groups of Level $\geqslant 2$

Suppose that $W = W_S$ is a Coxeter group such that the form $(\cdot, \cdot)$ is of signature $(n-1, 1)$, where $n = \mathrm{card}(S)$, but that W is not "hyperbolic" in the sense of [1]. Then W is "hyperbolic of level $\geqslant 2$" in the sense of [7]. If $\{\omega_s\}_{s \in S}$ is the dual basis of $\{e_s\}_{s \in S}$ in E, we have $(\omega_s, \omega_s) > 0$ for at least one $s \in S$. Let U_r be the subcone of U generated by all elements $w(\omega_s)$, with $w \in W$ and $(\omega_s, \omega_s) > 0$.

THEOREM 6.1. *If W is hyperbolic of level $\geqslant 2$ with a connected graph Γ, then $\bar{U}_r = \bar{U}$.*

Proof. Since U is generated by all elements $w(\omega_s)$, with $w \in W$ and $s \in S$, while $\bar{U}_r$ is invariant under W, it will suffice to show that $\omega_k \in \bar{U}_r$ whenever $(\omega_k, \omega_k) \leqslant 0$.

If $(\omega_k, \omega_k) < 0$, the group $W_{S\backslash k}$ is finite. Choose any ω_s such that $(\omega_s, \omega_s) > 0$ and consider the element $v = \sum_{w \in W_{S\backslash k}} w(\omega_s)$ in U_r. Since $s(v) = v$ for all $s \neq k$, $v = a\omega_k$, where $a = \sum_{w \in W_{S\backslash k}} (e_k, w(\omega_s))$ is >0 since Γ is connected.

Now suppose that $(\omega_k, \omega_k) = 0$ and choose some ω_s for which $(\omega_s, \omega_s) > 0$. Then $\Gamma\backslash k$ is a union of connected components of finite or euclidean type. In fact, precisely one of the components is euclidean, since otherwise there exist two orthogonal isotropic vectors in E. If s belongs to a component Y of finite type, one sees as above that $v = \sum_{w \in W_Y} w(\omega_s)$ is a positive multiple of ω_k. On the other hand, suppose that s belongs to the euclidean component X.

Let $\omega'_k = \sum_{i \in X} (\omega_k, \omega_i) e_i$ and $\omega''_k = \omega_k - \omega'_k$; then ω'_k and ω''_k are orthogonal and $(\omega'_k, \omega'_k) + (\omega''_k, \omega''_k) = (\omega_k, \omega_k) = 0$. Therefore both (ω'_k, ω'_k) and (ω''_k, ω''_k) are zero and hence $\omega''_k = 0$, since $\Gamma \backslash (X \cup \{k\})$ is of finite type, so that $\omega_k = \omega'_k$ spans the radical of E_X. Therefore all the numbers (ω_k, ω_i) for $i \in X$ are either >0 or <0. In fact, the latter alternative holds since $1 = (e_k, \omega_k) = \sum_{i \in X} (\omega_k, \omega_i)(e_k, e_i)$ and $(e_k, e_i) \leqslant 0$ for all $i \in X$.

In particular, $(\omega_k, \omega_s) < 0$ and one can now imitate the proof of Proposition 5.15 with ω_k in place of ω_s and $p = \omega_s$ to conclude that $\omega_k \in \overline{[1, X, \omega_s]} \subset \bar{U}_r$. ∎

Suppose now that W is of level 2 and Ω_r is the sphere packing associated to W (see [7]). For each $s \in S$ such that $W_{S \backslash s}$ is hyperbolic of level 1 (i.e., hyperbolic in the sense of [1]), let $T_s = S \backslash s$ and $\mathscr{T} = (T_s)$. The map

$$[g, X, \mathscr{Q}] \rightarrow \{ g w(\omega_s) \mid w \in W_X, T_s \in \mathscr{Q} \}$$

then establishes an isomorphism between $L(W, \mathscr{T})$ and a certain lattice of subsets of the packing Ω_r. In particular, the "points" of $L(W, \mathscr{T})$ correspond to the elements of Ω_r.

7. The Lattice $L(W^+, P)$

In this section, we retain the notation of Section 5 and discuss the analogous theory for the rotation subgroup W^+ of W, in the case when P consists of a single element $p \in \bar{C}$ for which $\mathrm{Stab}_W(p) = \{1\}$.

Let $L(W^+)$ be the set of all subsets of W^+ of the form

(a) gW_X^+, for $g \in W^+$ and $X \subset S$;

(b) wX, for $w \in W^- = W \backslash W^+$ and $X \subset S$.

It is not difficult to show that, when ordered by inclusion, $L(W^+)$ is a W^+-invariant pure lattice of dimension $\mathrm{card}(S) + 1$, unless $\mathrm{card}(S) \leqslant 1$ or $S = \{s_1, s_2\}$ with $s_1 s_2 = s_2 s_1$, in which case the dimension is $\mathrm{card}(S)$. One only needs to consider sets of type (a) for $\mathrm{card}(X) \geqslant 3$, or $X = \{s_1, s_2\}$ with $s_1 s_2 \neq s_2 s_1$, since otherwise they also occur as sets of type (b). The dimension of such a set gW_X^+ in $L(W^+)$ is $\mathrm{card}(X) + 1$, while the dimension of a set wX is equal to $\mathrm{card}(X)$.

For each $A \in L(W^+)$, let $[A, p]$ be the cone generated in V^* by all vectors $w(p)$, with $w \in A$, and let $L(W^+, P)$ denote the set of all such cones, ordered by inclusion. We also use the notation $[A, p]$ for other subsets A of W.

Let $C = \{v \in V^* \mid \langle e_s, v\rangle > 0 \text{ for all } s \in S\}$. It follows from [1, V.4] that, for $v \in C$, $w \in W$, and $s \in S$, we have $\langle e_s, w(v)\rangle > 0$ whenever $l(sw) > l(w)$. The element p belongs to C since $\mathrm{Stab}_W(p) = \{1\}$.

PROPOSITION 7.1. *Suppose that $X \subset S$ and $v \in [X, p]$, $v \neq 0$. Then one of the following possibilities holds*:

(a) $v \in \bar{C}$, *with $\langle e_i, v\rangle = 0$ for at most one $s_i \in S$, which belongs to X*;

(b) $v = a(s_i(p) + s_j(p))$ *for some $a > 0$ and $s_i, s_j \in X$ such that $s_i s_j = s_j s_i$; in this case $\langle e_i, v\rangle = \langle e_j, v\rangle = 0$ and $\langle e_k, v\rangle > 0$ for $s_k \neq s_i, s_j$*;

(c) $v \in s_i(C)$ *for some $s_i \in X$.*

Proof. Let $v = \sum_{s \in S} a_s s(p)$, with all $a_s \geqslant 0$ and $a_s = 0$ for $s \notin X$. Since $\langle e_i, s(p)\rangle > 0$ for all $s \neq s_i$, $\langle e_i, v\rangle \leqslant 0$ implies that $s_i \in X$ and, for all $s_j \neq s_i$,

$$a_j \langle e_i, s_j(p)\rangle \leqslant \sum_{s \neq s_i} a_s \langle e_i, s(p)\rangle \leqslant a_i \langle e_i, p\rangle.$$

However, $\langle e_i, s_j(p)\rangle = \langle e_i, p\rangle - 2(e_i, e_j)\langle e_j, v\rangle \geqslant \langle e_i, p\rangle$, so that $\langle e_i, p\rangle / \langle e_i, s_j(p)\rangle \leqslant 1$ and hence $a_j \leqslant a_i$. Furthermore, $a_j = a_i$ only if $\langle e_i, v\rangle = 0$, $a_s = 0$ for $s \neq s_i, s_j$, and $s_i s_j = s_j s_i$.

It follows that if $\langle e_i, v\rangle = \langle e_j, v\rangle = 0$ for distinct $s_i, s_j \in X$, then $a_j = a_i$, so that v is as described in (b). On the other hand, if $\langle e_i, v\rangle < 0$ for some $s_i \in X$, then $a_j < a_i$ for all $s_j \neq s_i$ and hence $\langle e_j, v\rangle > 0$ for all $s_j \neq s_i$ in S. Consider the element $s_i(v)$, for which $\langle e_i, s_i(v)\rangle = -\langle e_i, v\rangle > 0$. If $j \neq i$ and $s_i s_j = s_j s_i$, then $\langle e_j, s_i(v)\rangle = \langle e_j, v\rangle > 0$, whereas if $s_i s_j \neq s_j s_i$, then $\langle e_j, s_i s(p)\rangle > 0$ for all $s \in X$, so that $\langle e_j, s_i(v)\rangle > 0$. Therefore $s_i(v) \in C$ and $v \in s_i(C)$. ∎

PROPOSITION 7.2. *For all $A, B \in L(W^+)$, we have $[A, p] \subset [B, p]$ if and only if $A \subset B$.*

Proof. Suppose that $h \in A$. If $B = gW_X^+$ is of type (a), then $h(p) \in gU_X$, and hence $h(p) = gu(c)$ for some $u \in W_X$ and $c \in \bar{C}$, which implies by Proposition 5.3 that $h = gu$, so that $u \in W_X^+$ and $h \in B$. On the other hand, if $B = wX$ is of type (b), then $h(p)$ belongs to one of the cones $w(\bar{C})$, $ws(\bar{C})$ for $s \in X$, so that $h = ws$ for some $s \in S$ since h is even. ∎

It follows that $L(W^+, P)$ is a lattice isomorphic to $L(W^+)$.

THEOREM 7.3. *Suppose that W is finite and $p \in C \backslash E^*$. Then $L(W^+, P)$ is the face lattice of the pointed cone $[W^+, P]$.*

Proof. We first show that every face F of $[W^+, P]$ is of the form

$[A, p]$ for some $A \in L(W^+)$. If $F = \{0\}$, let $A = \varnothing$. Otherwise, since $p \in g^{-1}(F)$ for some $g \in W^+$, we may assume that $p \in F$. Since the cone $[W^+, P]$ is finitely generated, F is exposed, so that there exists some $\varphi \in V$, regarded as the dual space of V^*, with the property that $\varphi \in [W^+, P]^\pi$ and $F = [W^+, P] \cap \ker \varphi$.

If $\varphi \in [W, P]^\pi$, then $[W, P] \cap \ker \varphi$ is a face $[1, X, p]$ of $[W, P]$ for some $X \subset S$. Furthermore, $\langle \varphi, p \rangle = 0$ and $\langle \varphi, s(p) \rangle \leqslant 0$ for all $s \in S$; i.e., $\langle \varphi, \varepsilon_s \rangle \geqslant 0$, with equality precisely for $s \in X$. This implies that $F = [W_X^+, p]$.

On the other hand, suppose that $\langle \varphi, w(p) \rangle > 0$ for some $w \in W \backslash W^+$. Since $\langle \varphi, ws(p) \rangle \leqslant 0$ for every ray $\mathbb{R}^+ ws(p)$ adjacent to $\mathbb{R}^+ w(p)$, it follows from (4.1) that $\langle \varphi, k \rangle < 0$ for every other extreme ray $\mathbb{R}^+ k$ of $[W^+, p]$. Therefore the extreme rays of F must all be adjacent to $\mathbb{R}^+ w(p)$. In particular, we have $w \in S$ since $\mathbb{R}^+ p$ is an extreme ray of F, while the remaining extreme rays of F are of the form $ws(p)$, for s belonging to some subset X of S; i.e., $F = [wX, p]$. Furthermore, $\varphi = w(\psi)$, where $\langle \psi, p \rangle > 0$ and $\langle \psi, s(p) \rangle \leqslant 0$; i.e., $\langle \psi, \varepsilon_s \rangle \geqslant \langle \psi, p \rangle / 2 \langle e_s, p \rangle$, for all $s \in S$, with equality precisely for $s \in X$.

Since the ε_s are linearly independent and $p \in E^*$, the elements $\varphi \in V$ discussed above actually exist for all $X \subset S$, which shows that $[A, p]$ is a face of $[W^+, p]$ for all $A \in L(W^+)$. ∎

Suppose now that W is euclidean, $E_1 = \{0\}$, and $p \in C$. Let $\sum_{s \in S} c_s \varepsilon_s = 0$ be the unique (up to a multiple) relation among the ε_s, with all $c_s > 0$. For each $t \in S$, the set $\{p\} \cup \{\varepsilon_s \mid s \in S \backslash t\}$ is linearly independent, so that there exists an element $\psi_t \in E$ for which $\langle \psi_t, p \rangle = 1$ and $\langle \psi_t, \varepsilon_s \rangle = \frac{1}{2} \langle e_s, p \rangle$ for $s \neq t$. It is easy to see that $\langle \psi_t, t(p) \rangle > 0$ and $\langle \psi_t, w(p) \rangle \leqslant 0$ for all $w \neq 1$ in $W_{S \backslash t}$, with equality precisely for $w \in S \backslash t$.

Let $K(t)$ be the cone generated by p and all elements $s(p)$, for $s \neq t$. We have

$$[1, S \backslash t, p] \cup \{v \in E^* \mid \langle w\psi_t, v \rangle \geqslant 0\} = wK(t) \tag{7.1}$$

for all $w \in W_{S \backslash t}$. Indeed, by using (4.1), an element $v \in [1, S \backslash t, p]$ can be written as $v = aw(p) + \sum_{s \neq t} b_s ws(p)$, with all $b_s \geqslant 0$, so that $\langle w\psi_t, v \rangle \geqslant 0$ if and only if $a \geqslant 0$.

Applying Theorem 7.3 to the finite group $W_{S \backslash t}$, we see that if $v \in [1, S \backslash t, p]$ does not belong to $[W_{S \backslash t}^+, p]$, then $\langle w\psi_t, v \rangle \geqslant 0$ for some $w \in W_{S \backslash t}^-$, so that $v \in wK(t)$ by (7.1). Therefore

$$[1, S \backslash t, p] = [W_{S \backslash t}^+, p] \cup \Bigl(\bigcup_{w \in W_{S \backslash t}^-} wK(t) \Bigr). \tag{7.2}$$

Similarly, for any $s \in S\backslash t$, we have

$$[1, S\backslash t, p] = s[W^+_{S\backslash t}, p] \cup \left(\bigcup_{w \in W^+_{S\backslash t}} wK(t) \right). \tag{7.3}$$

The formula

$$\left(\sum_{s \in S} c_s/\langle e_s, p\rangle \right) p = \sum_{s \in S} c_s s(p)/\langle e_s, p\rangle \tag{7.4}$$

implies that $[S, p]$ is the union of the cones $K(t)$, for $t \in S$, so that for all $w \in W$,

$$[wS, p] = \bigcup_{t \in S} wK(t), \tag{7.5}$$

Let $L'(W^+, p)$ be the set obtained from $L(W^+, p)$ by omitting $[W^+, p]$.

THEOREM 7.4. *Suppose that W is euclidean, $E_1 = \{0\}$, and $p \in C$. Then $L'(W^+, p)$ is a tesselation of $[W^+, p] = U$.*

Proof. Since W^+_X is finite for $X \subset S$ and $p \notin E^*_X$, each cone in $L'(W^+, p)$ is closed and pointed. By Theorem 7.3, if B is of type (a) and $B \neq W^+$, every face of $[B, p]$ is of the form $[A, p]$ for some $A \subset B$. On the other hand, since $\{s(p) \mid s \in S\}$ is linearly independent, $[B, p]$ is a simplicial cone when B is of type (b), so that its faces correspond to all subsets A of B. If $[A, p] \subset [B, p]$ and $B \neq W^+$, we have $A \subset B$ by Proposition 7.2 and the preceding argument shows that $[A, p]$ is a face of $[B, p]$. The verification of Property (4) is rather tedious and will be sketched below.

For $g \in W^+$, Eqs. (7.2) and (7.5) show that $[g, S\backslash t, p]$ is covered by elements of $L'(W^+, p)$, whereas for $g \in W^-$, the same conclusion is reached by using (7.3) in place fo (7.2). Therefore the union of elements of $L'(W^+, p)$ is equal to U by Theorem 5.14, so that $[W^+, p] = U$.

Finally, note that if a segment $[u, v]$ in U has a nonzero element r in common with a cone of the form $[wX, p]$, then $r \in wK(t)$ for some $t \in S$ by (7.5) and therefore $r \in [w, S\backslash t, p]$ by (7.1). Property (6) now follows from Corollary 5.8. ∎

We still need to show that in the euclidean case,

PROPOSITION 7.5. *For all $A, B \in L(W^+)$, we have $[A, p] \cap [B, p] = [A \cap B, p]$.*

Proof. First consider a nonzero intersection $[gW^+_X, p] \cap [hW^+_Y, p]$ of two tiles of type (a). Let $g^{-1}h = xwy$, where $x \in W_X$, $y \in W_Y$, and w is (X, Y)-reduced. Since $[g, X, p] \cap [h, Y, p] = \{0\}$ unless $w(p) = p$, we have

$w=1$ and $g^{-1}h=xy$. By multiplying with a suitable element of W^+, it suffices to consider the following two cases:

(i) $[W_X^+, p] \cap [W_Y^+, p]$;

(ii) $[W_X^+, p] \cap [s_i s_j W_Y^+, p]$ with $s_i \in X$ and $s_j \in Y$.

In case (i), an element $m \in [W_X^+, p] \cap [W_Y^+, p]$ can be written as $m = \sum a_u u(p) = \sum b_v v(p)$, with all $a_u, b_v > 0$, $u \in W_X^+$, and $v \in W_Y^+$. Let $X' \subset X$ be the union of all sets S_u for each $u(p)$ that occurs in m, with a similar meaning for $Y' \subset Y$. Using Proposition 5.4, we deduce a relation,

$$m = ap - \sum_{s \in X'} a_s \varepsilon_s = bp - \sum_{s \in Y'} b_s \varepsilon_s,$$

with all $a_s, b_s > 0$, which is only possible if $X' = Y'$, so that $m \in [W_{X \cap Y}^+, p]$. In case (ii), one proves in a similar manner that $X \cap Y \neq \varnothing$ and $m \in [ss' W_{X \cap Y}^+, p]$ for any $s' \in X \cap Y$.

For two tiles of type (b), it suffices to determine the intersection $[X, p] \cap [wY, p]$, where $X, Y \subset S$ and $w \in W^+$. If $w=1$, $[X, p] \cap [Y, p] = [X \cap Y, p]$, since the set $\{s(p) \mid s \in S\}$ is linearly independent. On the other hand, if $l(w) > 2$, an element v in $[X, p] \cap [wY, p]$ belongs to $\bar{C} \cap w\bar{C}$ by Proposition 7.1, so that $\langle e_i, v \rangle = 0$ for all $s_i \in S_w$ and hence $v=0$, again by Proposition 7.1. Finally, suppose that $w = s_i s_j$ for some $s_i \neq s_j$. Let $v = \sum_{s \in S} a_s s(p)$, with all $a_s \geqslant 0$ and $a_s = 0$ for $s \notin X$. Then

$$\langle e_j, s_i(v) \rangle / \langle e_j, p \rangle \geqslant a_i + a_j \langle e_j, s_i s_j(p) \rangle / \langle e_j, p \rangle,$$

while

$$-\langle e_i, v \rangle / \langle e_i, p \rangle \leqslant a_i - a_j \langle e_i, s_j(p) \rangle / \langle e_i, p \rangle,$$

so that

$$\langle e_j, s_i(v) \rangle / \langle e_j, p \rangle \geqslant -\langle e_i, v \rangle / \langle e_i, p \rangle. \tag{7.6}$$

Similarly, we have $\langle e_i, s_j(v) \rangle / \langle e_i, p \rangle \geqslant -\langle e_j, v \rangle / \langle e_j, p \rangle$. Applying this inequality to $s_j s_i(v) \in [S, p]$, it follows that $-\langle e_i, v \rangle / \langle e_i, p \rangle \geqslant \langle e_j, s_i(v) \rangle / \langle e_j, p \rangle$, so that equality holds in (7.6). If $s_i s_j \neq s_j s_i$, this is only possible if $v \in \mathbb{R}^+ s_i(p)$, whereas if $s_i s_j = s_j s_i$, v can be an element of $\mathbb{R}^+ s_i(p) + \mathbb{R}^+ s_j(p)$ so long as $s_i, s_j \in X \cap Y$.

For an intersection of a tile of type (a) with a tile of type (b), it suffices to consider the case $[W_X^+, p] \cap [sY, p]$ for some $s \in S$. Note that $[sY, p] \subset \bigcup_{t \in S} sK(t)$ by (7.5) and look at each intersection $[W_X^+, p] \cap sK(t)$. For instance, if $s \neq t$ and $s \notin X$, an element $v \in [W_X^+, p] \cap sK(t)$ belongs to $[1, X, p] \cap [1, S \backslash t, p] = \{1, X \backslash t, p\}$. Writing $v = \sum_{u \in W_X^+} a_u u(p)$, we have $\langle s\psi_t, v \rangle \geqslant 0$ since $v \in sK(t)$, and also

$\langle s\psi_t, u(p)\rangle \leqslant 0$ for all $u \in W_{X\setminus t}$, with equality only for $u = 1$, which shows that $v \in \mathbb{R}^+ p$. ∎

The hyperbolic case will be left to the patient reader. One needs to find an argument which determines the faces of a cone $[gW_X^+, p]$ of type (a) when W_X is euclidean.

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