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REU 2018 Day 7 Vic Reiner

Weight polytopes & Wythoff's construction

1. Reflection & Coxeter groups
2. Weight polytopes (= Wythoff's construction)
3. Simple polytopes & f-vectors
4. REU Problem 7

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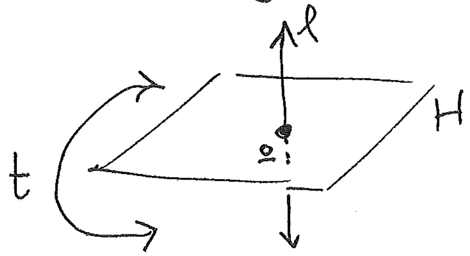
1. Ref'n groups

DEF'N: A *reflection* t acting on $V=\mathbb{R}^n$ is an element $t\in GL_n(\mathbb{R})$ that fixes a *hyperplane* H (called its *reflecting hyperplane*)

a codimension 1 linear subspace through $\underline{0}$ $(=(n-1)$-dimensional$)$

and negates the line $\ell = H^\perp$ orthogonal to H



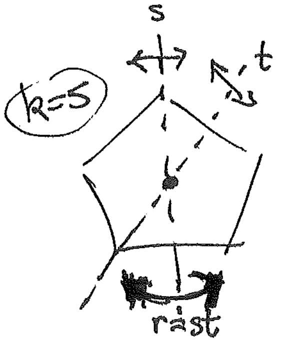
A *finite reflection group* W is a finite subgroup of $GL_n(\mathbb{R})$ generated by reflections.

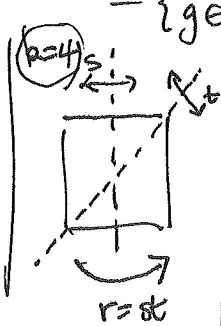
EXAMPLE: $W = I_2(k)$ $(k\geq 3)$

$=$ dihedral group of order $2k$

$:=$ linear symmetries of a regular k-sided polygon P

$= \{ g\in GL_2(\mathbb{R}) : g(P) = P \}$





REU Exercise 17:

(a) Prove $I_2(k) = \underbrace{\{e,r,r^2,\ldots,r^{k-1}\}}_{\text{rotations}} \sqcup \underbrace{\{s,sr,sr^2,\ldots,sr^{k-1}\}}_{\text{reflections}}$

(b) Prove the abstract presentation
$I_2(k) \cong \langle s,r \mid s^2=r^k=e,\ srs=r^{-1}\rangle$

(c) Prove the Coxeter presentation
$I_2(k) \cong \langle s,t \mid s^2=t^2=e,\ (st)^k=e\rangle$

Recall from Day 6...

DEF'N: A Coxeter presentation for a group W is one of the form

$$W \cong \Big\langle \underbrace{\{s_1, s_2, \ldots, s_n\}}_{S :=} \;\Big|\; \begin{array}{l} s_i^2 = e \;\; \forall i = 1, \ldots, n \\ (s_i s_j)^{m_{ij}} = e \;\text{ for some } m_{ij} \in \{2,3,\ldots\} \cup \{\infty\} \end{array} \Big\rangle$$

and it can be encapsulated in the Coxeter graph for (W,S)

having vertices $:= S$

edges: $(s_i) \xrightarrow{m_{ij}} (s_j)$ with the edge omitted if $m_{ij} = 2$ (s_i, s_j commute)

and the labels $m_{ij} = 3$ omitted

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e.g. $\Big(\overset{W}{\overset{\shortparallel}{I_2(k)}}, \overset{S}{\overset{\shortparallel}{\{s,t\}}}\Big)$ has Coxeter graph $(s) \overset{k}{\text{———}} (t)$

Which finite groups W have a Coxeter presentation?

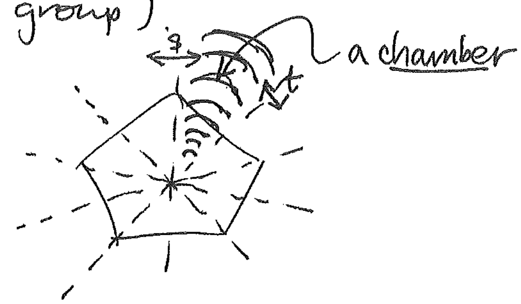
THM (Coxeter) Finite reflection groups W always have one,

specifically by choosing $S = \{$ reflections s_i through hyperplanes bounding a particular chamber cut out by the refl'n hyperplanes $\}$

any connected component of $V - \bigcup_{\substack{\text{refl'n} \\ \text{hyperplanes } H}} H$

(Conversely, if (W,S) gives a Coxeter presentation & W is finite, then W is a finite refl'n group)

EXAMPLES: ① $W = I_2(m)$
$S = \{s,t\}$



② $W = S_n$ = symmetric group on n letters

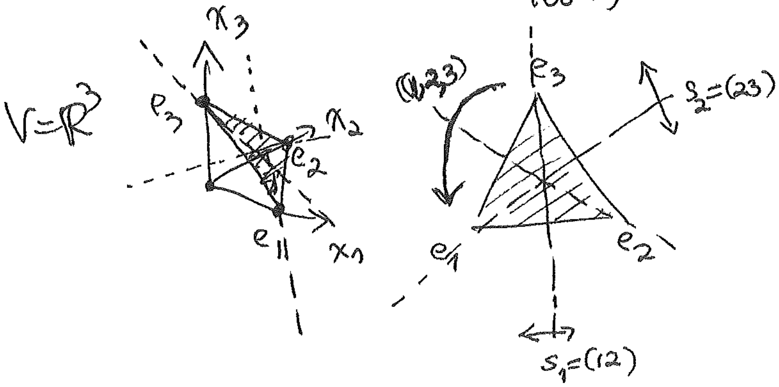
$\subset GL_n(\mathbb{R})$ as permutation matrices (permuting coordinates)

is actually a refl'n group...

(3) e.g. $n=3$ $\quad \mathfrak{S}_3 = \{ e, (12), (13), (23), (123), (132) \}$

$(12) \mapsto \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}$ $\qquad (132) \mapsto \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix}$

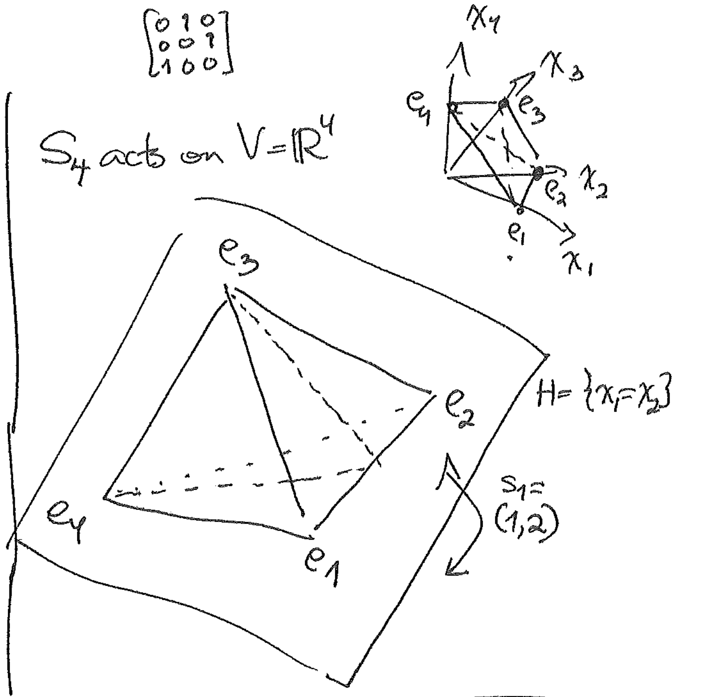
$V = \mathbb{R}^3$



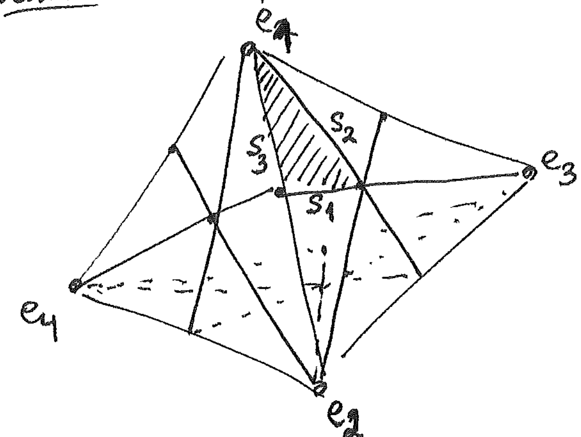
$S_3 = I_2(3)$
(type A_2)

(s_1) — (s_2)

S_4 acts on $V=\mathbb{R}^4$



The ~~chambers~~ hyperplanes & chambers for S_4 cut out a (barycentric) subdivision of tetrahedron boundary:

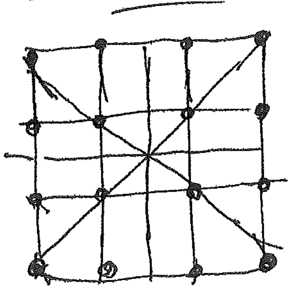


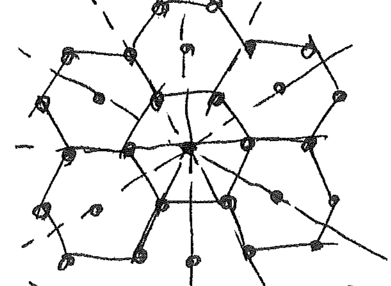
(s_1) — (s_2) — (s_3)
$(1,2)$ $\quad (2,3)$ $\quad (3,4)$

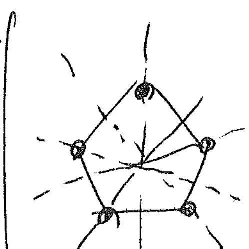
Generally, $S_n = A_{n-1}$ $\quad (s_1)$ — (s_2) — (s_3) — $\cdots$ — (s_{n-1}) (12) $\quad (23)$ $\quad (34)$ $\quad \cdots \quad (n-1,n)$	$s_i^2 = e$ $(s_i s_j)^2 = e$ if $\lvert i-j \rvert \geq 2$ $(s_i s_{i+1})^3 = e$

Some finite refl'n groups W acting on $V = \mathbb{R}^n$ stabilize a lattice of full rank $\Lambda \cong \mathbb{Z}^n$ inside V, and are called crystallographic refl'n groups or Weyl groups

EXAMPLES: $I_2(k)$ for $k=3,4,6$ are Weyl groups ; $I_2(5), I_2(7), I_2(8), \ldots$ are not







$I_2(4) (= B_2$ or $C_2)$ / $I_2(6)$ $(= G_2)$ $\supset I_2(3)$

REP. Exercise 18: Prove that if W is a Weyl group, then all labels m_{ij} in $(s_i s_j)^{m_{ij}} = e$ must have $m_{ij} \in \{2, 3, 4, 6\}$.

(Conversely, a finite refln group (W, S) with all $m_{ij} \in \{2, 3, 4, 6\}$ turns out to always be a Weyl group).

REMARK: Weyl groups W always have associated algebraic groups G (like $G = GL_n(F)$ for $W = S_n$) with Borel subgroups $B \subset G$ (like $B = \{\text{upper triangulars}\}$) and Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$, and more...

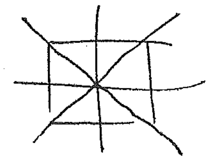
Roughly, W controls the structure and rep'n theory of G.

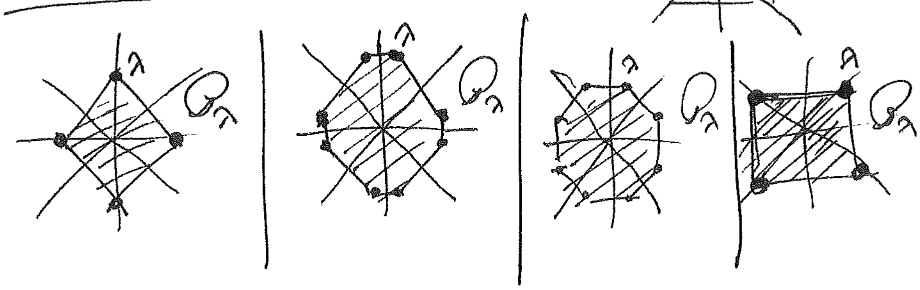
2. Weight polytopes/Wythoff's construction

DEFN: Given a finite refl'n group W acting on $V = \mathbb{R}^n$, and $\lambda \in V$, the weight polytope P_λ (or Wythoff's construction (of a semiregular solid) from λ) is the convex hull of the W-orbit of $\lambda := \{w(\lambda) : w \in W\}$

smallest convex set containing it

EXAMPLES: $W = I_2(4) = B_2/C_2$





Every $\lambda \in V$ has a unique W-orbit rep $\omega(\lambda)$ lying in the closure of your favorite chamber C, whose walls give Coxeter generators $S = \{s_1, \ldots, s_n\}$.

If we let $J(\lambda) = \{s \in S : s(\lambda) = \lambda$, i.e. λ lies on the refl'n hyperplane for $s\}$

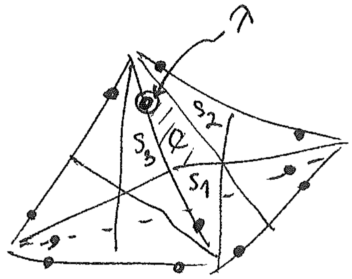
then $J(\lambda)$ controls the facial structure of P_λ ...

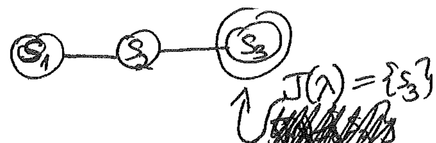
(5) <u>THM</u> (see e.g. [Renner 2009, Cor 1.3] for <u>Weyl</u> groups W; Maxwell in <u>all</u> cases (?))

- $\mathcal{P}_\lambda$ has exactly one W-orbit of faces for each $I \subseteq S$ such that no connected component of I is contained in $J(\lambda)$. — call these I the set $\mathcal{S}(\lambda)$
- This W-orbit is represented by a face F_I whose relative interior intersects $\bar{C}$ and has the parabolic subgroup $W_I := \langle s_i \rangle_{s_i \in I}$ stabilizing F_I, but acting <u>nontrivially</u> restricted to F_I.
- The W-stabilizer of F_I is W_{I^*} where $I^* = I \cup \{s \in J_\lambda : st = ts \ \forall t \in I\}$

EXAMPLES:

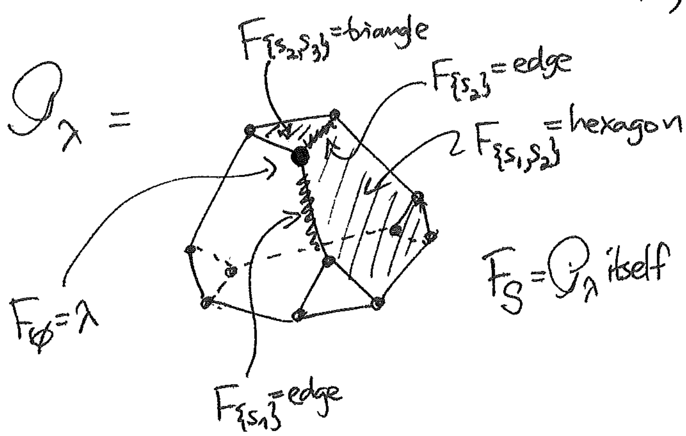
① $W = S_4$





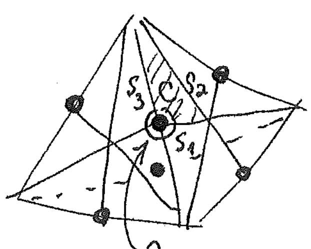
$$I \in \{\emptyset, \{s_1\}, \{s_2\}, \{s_1,s_2\}, \{s_2,s_3\}, \overset{S}{\overset{\parallel}{\{s_1,s_2,s_3\}}}\} =: \mathcal{S}(\lambda)$$

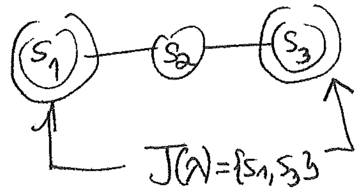
$\mathcal{P}_\lambda =$



$F_S = \mathcal{P}_\lambda$ itself

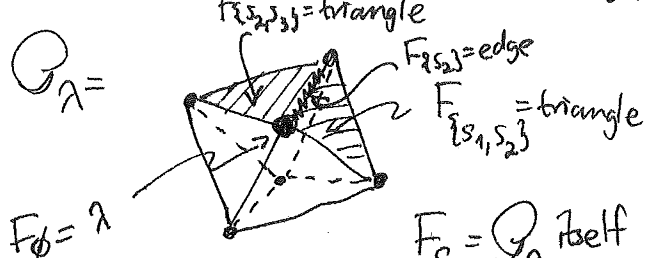
② $W = S_4$





$$I \in \{\emptyset, \{s_2\}, \{s_1,s_2\}, \{s_2,s_3\}, S\} =: \mathcal{S}(\lambda)$$

$\mathcal{P}_\lambda =$



$F_S = \mathcal{P}_\lambda$ itself

(6)

3. Simple polytopes & f-vectors

DEFN: For a convex n-dimensional polytope P,

its <u>f-vector</u> is $f(P) = (\underset{1}{\underset{\parallel}{f_{-1}}}, f_0, f_1, \ldots, f_{n-1}, \underset{1}{\underset{\parallel}{f_n}})$

where $f_i :=$ # of i-<u>dimensional</u> <u>faces</u> of P

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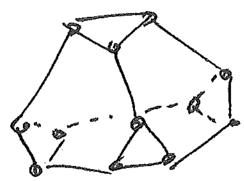
e.g. $P_\lambda =$  has $f(P_\lambda) = (\cancel{f_{-1}}, f_0, f_1, f_2, f_3)$

$= (\cancel{1}, 12, 18, 8, 1)$

vertices, edges, polygon 2-faces, P_λ itself

Since the W-orbit of F_I in P_λ looks like cosets $W/\underbrace{W_{I^*}}_{\text{stabilizer of } F_I}$,

it has size $[W : W_{I^*}] = \dfrac{|W|}{|W_{I^*}|}$.

Also F_I has dimension $|I|$.

$\Rightarrow$ <u>COROLLARY</u>: $f_i(P_\lambda) = \sum\limits_{I \in \mathcal{S}_\lambda}' \dfrac{|W|}{|W_{I^*}|}$

SAGE/COCALC know polytopes & f-vectors! Try this in COCALC:

```
def weight_polytope(lam):
  P=polyhedron(vertices=
              Arrangements(
                lam, len(lam))
  return(P)

P=weight_polytope([1,2,2,3])
P.f_vector()
P.show()
```

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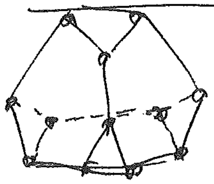
e.g. example above

s_1 — s_2 — s_3

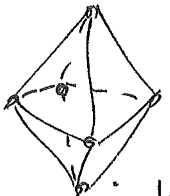
J_λ

I	I^*	$\frac{\lvert W \rvert}{\lvert W_{I^*} \rvert}$	
$\emptyset$	s_3	$24/2 = 12$	$\} f_0 = 12$ ✓
s_1	s_1, s_3	$24/4 = 6$	$\} f_1 = 6+12 = 18$ ✓
s_2	s_2	$24/2 = 12$	
s_1, s_2	s_1, s_2	$24/6 = 4$	$\} f_2 = 4+4 = 8$ ✓
s_2, s_3	s_2, s_3	$24/6 = 4$	
s_1, s_2, s_3	s_1, s_2, s_3	$24/24 = 1$	$\} f_3 = 1$ ✓

(7) When P is a simple n-dimensional polytope
every vertex touches exactly n edges (smallest possible),



simple



not simple

there is a particularly pleasant encoding of

$$f(P) = (f_0, f_1, \ldots, f_n)$$

called the h-vector $h(P) := (h_0, h_1, \ldots, h_n)$

defined by $\underbrace{h_0 + h_1 t + \cdots + h_n t^n}_{h(P_\lambda, t) :=} = f_0 + f_1(t-1) + f_2(t-1)^2 + \cdots + f_n(t-1)^n$

e.g. $f(P_\lambda) = \overset{f_0\ \ f_1\ \ f_2\ f_3}{(12, 18, 8, 1)} \rightsquigarrow 12 + 18(t-1) + 8(t-1)^2 + 1\cdot(t-1)^3 = 1 + 5t + 5t^2 + t^3$

has $h(P_\lambda) = \overset{h_0\ h_1\ h_2\ h_3}{(1, 5, 5, 1)}$

One always has $h(P_\lambda)$ symmetric: $h_i = h_{n-i}$

and $h(P_\lambda) \in \mathbb{N}$, with many interpretations

S_3 EXAMPLE: When $W = S_n$ and λ is generic,

$f = (6, 6, 1)$

$h = (1, 4, 1)$

w	$\#\{i : w(i) > w(i+1)\}$
123	0
132	1
213	1
231	1
312	1
321	2

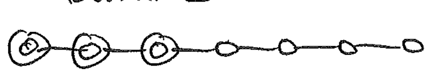
$h(P_\lambda, t) =$ Eulerian polynomial $:= \sum_{w \in S_n} t^{\#\{i : w(i) > w(i+1)\}}$

Called the permutohedron $\qquad E_n(t)$

which have a nice (exponential) generating function $\displaystyle\sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!} = \frac{t-1}{t - e^{(t-1)x}}$

REU Problem 7: (a) Use Renner's easy classification of the simple P_λ in all types [Renner 2009, Thm. 3.2], and continue the work of his student Golubitsky 2014, by computing f/h-vectors and generating functions for them as families

e.g. Type A_{n-1}:



and

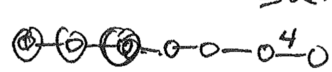
were done by Golubitsky

but

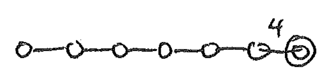
was not.

≥ 2 nodes

Type B_n:



and



were not done.

⋮

(b) Check that Renner's ~~[illegible]~~ [2009, Cor 1.3] was known to Maxwell (or others) for all W, not just Weyl groups.