AN INTRODUCTION TO REDUCTIVE MONOIDS

LOUIS SOLOMON*
Department of Mathematics
University of Wisconsin
Madison, WI 53706
U. S. A.

For Elsbeth Solomon-Aellig

1. Introduction to the Introduction

The aim of this paper is to give an introduction to the theory of reductive monoids, a well developed branch of semigroup theory with good prospects. It has highly structured problems which, like those in the theory of reductive groups, are rooted in the combinatorics of the Weyl group. This theory is the work of Mohan Putcha and Lex Renner, who originated it independently around 1980. There are now at least 50 papers on the subject - most of them by Putcha and/or Renner, although others, notably Okniński, have been involved. Their work has been more or less ignored by those who might enjoy it and profit from it. These include workers in (the order is alphabetic) algebraic groups and related finite groups, algebraic combinatorics, semigroups, and possibly other parts of mathematics. This subject has a marketing problem. My estimate, based on some minimal evidence, is that those who do algebraic groups are sympathetic but uninterested, those who do algebraic combinatorics do not know that the subject exists, and those who do semigroups are put off by prerequisites which seem formidable.

A reductive monoid M is pieced together from its group of units $G = G(M)$ and its set of idempotents $E = E(M)$. The group G is a reductive group. The idempotents are intimately connected to the group structure. Although one can see this connection in the simplest example, where M is the monoid of all matrices over a field and G is the general linear group, it is remarkable that natural questions about idempotents lead to all the standard constructs in semisimple Lie theory: roots, parabolic subgroups, Tits building, and so forth.

My intent in this paper is to give an introduction to the theory of reductive monoids from scratch, for a reader with general background and interest in algebra but no special knowledge of semigroup theory or reductive algebraic groups. The theory of reductive groups is itself a big subject with various prerequisites. To overcome the prerequisites in minimal space we concentrate on examples in the context of both groups and monoids. Statements of the main theorems are accompanied by references but not by proofs. Comments written as footnotes provide some further

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references as well as bits of argument and hints concerning the depth of unsupported statements. The footnotes may be ignored at first reading without loss of continuity. A willing beginner should be able to check most of the details in the examples. In the first steps, and perhaps at any time, the examples are more important than the theorems.

The best course is to begin with an example which is elementary but sufficiently interesting. Example 2.3 in Section 2 should do the job. Once this is done there should be some incentive for you to accept the basic definitions in the theory of reductive groups: weights, roots, Coxeter-Dynkin diagram, parabolic subgroup, BN-pair and so forth. Weights are introduced in Section 2 and the rest of the apparatus is introduced in Section 4. The exposition here has been stripped to a minimum; we do what is necessary for an understanding of the monoid problems and no more. This will allow us to describe in Section 5 by example and statement of theorem, some of the main results of Putcha and Renner, in particular Putcha's cross section lattice and Renner's analogue of the Bruhat decomposition via the Renner monoid.

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2. Algebraic Monoids

A monoid is a semigroup with 1. What is an algebraic monoid? Let K be an algebraically closed field. Let $\mathbf{M}_n = \mathbf{M}_n(K)$ denote the set of all $n \times n$ matrices over K. We may think of $\mathbf{M}_n$ as an affine space of dimension n^2 with coordinate functions X_{ij} defined by

$$X_{ij}(\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}) = c_{ij} \qquad c_{ij} \in K.$$

It is also a monoid with the general linear group $\mathbf{GL}_n = \mathbf{GL}_n(K)$ as its group of units. If V is a vector space over K of dimension n, then choice of a basis allows us to identify $\mathbf{M}_n$ with $\mathbf{End}_K(V)$ and $\mathbf{GL}_n$ with $\mathbf{GL}(V)$. In these notes V is usually the space K^n of column vectors and $\{v_1, \dots, v_n\}$ is the standard basis for K^n.

A *linear algebraic monoid*, for short an *algebraic monoid*, is a submonoid M of $\mathbf{M}_n$ which is a Zariski closed set. [1] This means that M is the set of common zeros of a family of polynomials in the X_{ij}. For example $\mathbf{M}_n$ itself is algebraic, defined by the empty family of polynomials. The monoid $\mathbf{D}_n = \mathbf{D}_n(K)$ of diagonal matrices is defined by vanishing of the X_{ij} with $i \neq j$ and is thus algebraic. Similarly, the monoid of upper triangular matrices is defined by vanishing of the X_{ij} with $i > j$ and is thus algebraic.

If $K = \overline{\mathbf{F}}_q$ is the algebraic closure of a finite field $\mathbf{F}_q$ with q elements, there are closely related finite monoids. Let $\sigma : \mathbf{M}_n \to \mathbf{M}_n$ be the Frobenius map defined

[1] We use Putcha [31] as a reference on algebraic monoids; we have tried, whenever possible, to give a reference to original source.

by $[c_{ij}] \mapsto [c_{ij}^q]$. The set of fixed points of σ is the monoid $\mathbf{M}_n(\mathbf{F}_q)$ of all matrices over $\mathbf{F}_q$. If $M \subseteq \mathbf{M}_n$ is any algebraic monoid which is the zero set of a family of polynomials with coefficients in $\mathbf{F}_q$ then $\sigma M = M$. Let $M_\sigma = \{a \in M \mid \sigma a = a\}$ be the set of fixed points. Then $M_\sigma \subseteq \mathbf{M}_n(\mathbf{F}_q)$ is a finite monoid. For example if $M = \mathbf{M}_n$ then $M_\sigma = \mathbf{M}_n(\mathbf{F}_q)$; if $M = \mathbf{D}_n$ then M_σ is the monoid of diagonal matrices with coefficients in $\mathbf{F}_q$. [2]

An algebraic monoid M is *connected* if it is connected in the Zariski topology. An algebraic monoid is *irreducible* if it is irreducible as affine algebraic set - it cannot be written as a union of proper Zariski closed subsets. Irreducible implies connected but not conversely. For example the monoid $M = \{\operatorname{diag}(c_1, c_2) \in \mathbf{D}_2 \mid c_1c_2 = 0\}$ is connected but not irreducible. Since every finite monoid M is an algebraic monoid, we assume throughout that M is irreducible. Every algebraic monoid M has a *dimension*, dim M, which we will not attempt to define [15, 3.1]. On the intuitive level, dim $\mathbf{M}_n = n^2$ and dim $\mathbf{D}_n = n$.

A *linear algebraic group*, for short an *algebraic group*, is a subgroup of $\mathbf{GL}_n$ which is the intersection of $\mathbf{GL}_n$ with a Zariski closed subset of $\mathbf{M}_n$. [3] For example $G = \mathbf{GL}_n$ is an algebraic group, as is the group $G = \mathbf{T}_n$ of invertible diagonal matrices. The *coordinate ring* or *affine ring* of G is the K-algebra of functions on G generated by the restrictions to G of the coordinate functions X_{ij} as well as the reciprocal of the determinant function det : $\mathbf{GL}_n \to K^*$, where $K^* \simeq \mathbf{GL}_1$ denotes the multiplicative group of K. Thus, for example, if $G = \mathbf{T}_n$ and X_i denotes the restriction of X_{ii} to G then $K[G] = K[X_1, \ldots, X_n, (X_1 \cdots X_n)^{-1}] = K[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$ is a ring of Laurent polynomials in n indeterminates with coefficients in K. A subgroup of G is, by definition, closed if it is the intersection of G with a Zariski closed subset of $\mathbf{M}_n$; thus a closed subgroup of an algebraic group G is itself an algebraic group. A direct product of algebraic groups is an algebraic group. An algebraic group G is connected if and only if it is irreducible as algebraic set [15, 7.3]. [4] If $G = \langle Y_1, \ldots, Y_r \rangle$ where the Y_i are closed connected subgroups which generate G as an abstract group, then G is connected [15, Corollary 7.5]. This is a useful fact which we will apply several times.

It is very easy to construct algebraic monoids. If X is any subset of $\mathbf{M}_n$ let $\overline{X}$ denote the Zariski closure of X, the intersection of all Zariski closed sets which include X. The Zariski closure of any submonoid of $\mathbf{M}_n$ is a monoid [5] and hence an algebraic monoid. In particular, the Zariski closure $\overline{G}$ of any subgroup $G \subseteq \mathbf{GL}_n$ is

[2] The classical groups over finite fields were introduced by Jordan and were placed in a Lie theoretic context by Dickson and Chevalley. Steinberg [51, §10 – §15] studied the Chevalley groups, and certain variations on them which include the finite unitary groups, as fixed point sets $G_\sigma = \{g \in G \mid \sigma g = g\}$ where G is a semisimple algebraic group defined over $\mathbf{F}_q$. Renner introduced the analogous construction $M \rightsquigarrow M_\sigma$ for algebraic monoids [44],[46] and studied the M_σ when M is reductive; see Section 5 of this paper.

[3] We use Borel [3] and Humphreys [15] as references on algebraic groups; with occasional reference to Springer [50] and Séminaire Chevalley [6]. The boldface notation used here differs from the boldface notation in [3] and [15]. It is "dictated" if we let $\mathbf{D}_n$ denote the monoid of diagonal matrices in $\mathbf{M}_n$. Thus, here, $\mathbf{T}_n$ is the standard torus of invertible diagonal matrices and $\mathbf{B}_n$ is the Borel subgroup of invertible upper triangular matrices in $\mathbf{GL}_n$.

[4] In the theory of algebraic groups the term "irreducible" is usually suppressed in this context and is reserved for representation theory. By convention, this is also done for monoids in [31, Definition 1], so that "connected" in [31] means "irreducible" here.

[5] To prove this, argue as in Lemma 2.1 of [25].

an algebraic monoid. If $G \subseteq \mathbf{GL}_n(K)$ is an algebraic group then $\overline{G}$ is an algebraic monoid which has G as its group of units. In the opposite direction, if $M \subseteq \mathbf{M}_n$ is an algebraic monoid then $M \cap \mathbf{GL}_n$ is an algebraic group which is in fact the group of units of M, the set of elements of M with an inverse in M [10, Corollary II, §2, 3.5], [39, Corollary 2.2.2]. [6] Thus if $G \subseteq \mathbf{GL}_n(K)$ is an algebraic group, then $\overline{G}$ is an algebraic monoid which has G as its group of units. For example, if $G = \mathbf{GL}_n$ then $\overline{G} = \mathbf{M}_n$ and $\mathbf{GL}_n$ is the group of units of $\mathbf{M}_n$. If $G = \mathbf{T}_n$ then $\overline{G} = \mathbf{D}_n$ and $\mathbf{T}_n$ is the group of units of $\mathbf{D}_n$. Thus we have a method for constructing algebraic monoids with a prescribed algebraic group G of units. If G is connected then $\overline{G}$ is irreducible [31, p.48].

To construct interesting algebraic monoids we choose an algebraic group $G_0 \subseteq \mathbf{GL}_m$ and let $G = \rho(G_0)$ where where ρ is a *rational representation*. Recall that a representation of a group G_0 is a homomorphism $\rho : G_0 \to \mathbf{GL}_n$ for some n. A representation ρ of an algebraic group G_0 is a rational representation if, for each coordinate function X_{ij} on $\mathbf{M}_n$, the function $g \mapsto X_{ij}(\rho(g))$, $g \in G_0$, lies in the affine ring $K[G_0]$. For example, if $A = [a_{ij}]$ is any $m \times n$ matrix of integers then we may define a rational representation $\rho : \mathbf{T}_m \to \mathbf{GL}_n$ by

$$\rho(g) = \operatorname{diag}(t_1^{a_{11}} \cdots t_m^{a_{m1}}, \dots, t_1^{a_{1n}} \cdots t_m^{a_{mn}}) \in \mathbf{T}_n \subseteq \mathbf{GL}_n \tag{1}$$

for $g = \operatorname{diag}(t_1, \dots, t_m) \in \mathbf{T}_m$. The representation is rational because $X_{ij}(\rho(g)) = 0$ for $i \neq j$ and $X_{jj}(\rho(g)) = t_1^{a_{1j}} \cdots t_m^{a_{mj}}$ is a Laurent polynomial in the matrix entries of g. In fact every rational representation of $\mathbf{T}_m$ is equivalent to a representation of the form (1). [7] If $G_0 = \mathbf{GL}_m$ and $^\top$ means transpose then $\rho : G_0 \to \mathbf{GL}_m$ defined by $g \to (g^{-1})^\top$ is a rational representation because each matrix entry of g^{-1} is the product of $(\det g)^{-1}$ and a cofactor of g; the cofactor is a polynomial in the matrix entries of g. If G_0 is any algebraic group and ρ is a rational representation then $\rho(G_0)$ is an algebraic group [3, 1.4],[15, Proposition 7.4.B(b)]. Unfortunately, the proof of this theorem does not tell you how to construct polynomials in n^2 indeterminates X_{ij} which generate the ideal of polynomials which vanish on $\rho(G_0)$. It is hard to give an explicit description of M in terms of concretely given G_0 and ρ.

Let's look at an example. Let $G_0 = \mathbf{SL}_m = \mathbf{SL}_m(K)$ be the *special linear group*, which consists of all elements of $\mathbf{GL}_m$ with determinant 1. Let $\rho : G_0 \to \mathbf{GL}_n$ be a rational representation. Since $G = \rho(G_0)$ is an algebraic group, there is a Zariski closed subset X of $\mathbf{M}_n$ such that $G = X \cap \mathbf{GL}_n$. Since every $g \in G_0$ is a product of commutators $xyx^{-1}y^{-1}$ it follows that $\rho(g)$ is a product of commutators $\rho(x)\rho(y)\rho(x)^{-1}\rho(y)^{-1}$. Thus $\det \rho(g) = 1$ so $G \subseteq \mathbf{SL}_n$. Thus $G = X \cap \mathbf{SL}_n$. But $\mathbf{SL}_n$ is Zariski closed because it is the zero set of the polynomial $\det[X_{ij}] - 1$. Thus G is Zariski closed in $\mathbf{M}_n$. So the monoid $\overline{G} = G$ is a group and we have accomplished nothing.

[6] It is not clear without argument that if $a \in M$ is an invertible matrix then $a^{-1} \in M$. For example, if $a \in K^*$ has infinite order and $M = \{1, a, a^2, \dots\}$ then M is a monoid, not an algebraic monoid, in which every element is invertible in K, but M has no units except 1.

[7] The key point here is the case $m = n = 1$ where one must check that every rational homomorphism $K^* \to K^*$ has the form $t \mapsto t^a$ for some $a \in \mathbf{Z}$. Then use the fact [3, Proposition 8.4] that $\rho(\mathbf{T}_m)$ is conjugate to a subgroup of $\mathbf{T}_n$.

This is just setting you up. We have learned something. To construct monoids which are not groups, G must contain matrices with determinant different from 1. [8] This suggests that we adjoin the scalar matrices to $\rho(G_0)$ and choose

$$G = \{c\rho(g) \mid c \in K^* \text{ and } g \in G_0\} = K^*\rho(G_0) \supseteq \rho(G_0). \tag{2}$$

We have taken our first attempt $\rho(G_0)$ and modified it by adjoining the nonzero scalar matrices. The group G is an algebraic group but it is *not* a closed subset of $\mathbf{M}_n$. [9] Let M be its Zariski closure. Then, as we have remarked, M is an algebraic monoid with G as its group of units. Write $M = M(\rho)$ to denote the dependence on ρ. Since G_0 and K^* are connected, so is $G = K^*\rho(G_0)$. Thus $M(\rho)$ is an irreducible algebraic monoid. Since Ker ρ is a normal subgroup of $G_0 = \mathbf{SL}_m$, it is either equal to G_0, a case we agree to ignore, or it is a finite central subgroup of G_0 of order equal to a divisor of m. Thus for any nontrivial representation ρ, the group $G = K^*\rho(G_0)$ of units of $M(\rho)$ is a close relative of $\mathbf{GL}_m$. [10] Any significant structural difference between $M(\rho)$ and $\mathbf{M}_m$ must lie in its set of idempotents. [11]

If M is a monoid, let $E(M)$ denote its set of idempotents. We give $E(M)$ the partial order

$$f \leq e \Leftrightarrow ef = f = fe\,. \tag{3}$$

We agree that the partial order on any subset of $E(M)$ is inherited from this one.

We give three examples of the construction $(G_0, \rho) \rightsquigarrow M(\rho)$ with $G_0 = \mathbf{SL}_m$ and various ρ. They presuppose no special knowledge about the representation theory of $\mathbf{SL}_m$. Even so, the monoid in the third example is quite different from familiar objects. Let K^m be the space of column vectors and let $v_1, \ldots, v_m$ be the standard basis for K^m. Let $T_0 = G_0 \cap \mathbf{T}_m$ be the group of diagonal matrices in G_0 and let $T = K^*\rho(T_0) \subseteq K^*\rho(G_0) = G$. Thus $\overline{T} \subseteq \overline{G} = M(\rho)$.

Example 2.1 Let $V = K^m$. Thus $n = \dim V = m$. Define $\rho : G_0 \to \mathbf{GL}(V)$ by $\rho(g) = g$. Then $G = \{cg \mid c \in K^*, g \in \mathbf{SL}_m\} = \mathbf{GL}_n$, $T = \mathbf{T}_n$, and $M(\rho) = \mathbf{M}_n$. We are on very familiar territory. This example is the prototype. Let $E_{ij} \in \mathbf{M}_n$ be the matrix units. The idempotents in $E(\overline{T})$ are of the form $e_I = \sum_{i \in I} E_{ii}$ where I ranges over the subsets of $\{1, \ldots, n\}$. The poset $E(\overline{T})$ is isomorphic to the Boolean

[8] Renner [39, Theorem 3.3.6] and Waterhouse [54] gave precise conditions under which an algebraic group G may be imbedded as the group of units of an algebraic monoid M which is not a group.

[9] Imbed $K^* \times G_0 \to \mathbf{GL}_{n+1}$ in the natural way. Then G is the image of the algebraic group $K^* \times G_0$ under the representation $(c, g) \mapsto c\rho(g)$ and hence is an algebraic group. Since the closure of $K^*\rho(1)$ contains the zero matrix, so does $\overline{G}$.

[10] To be precise, note that the homomorphism $K^* \times G_0 \to G$ is surjective with finite kernel. So is the homomorphism $K^* \times G_0 \to \mathbf{GL}_m$ given by $(c, g) \mapsto cg$.

[11] One should not conclude from this statement that the distinction between $K^* \times \mathbf{SL}_m$ and $\mathbf{GL}_m$ may be ignored in the context of monoids. For example, if $m = 2$ it enters the proof of Renner's classification theorem for semisimple algebraic monoids [40], [41] and into Renner's construction of an analogue for M of the Bruhat decomposition for G [43]. However, in the present context, the difference between groups is subtle while the difference between idempotent sets can be spectacular. Thus it seems best, with first examples, to concentrate on the idempotents.

lattice of subsets of $\{1,\dots,n\}$. In view of Example 5.4, it is equally true, and closer to the general Theorem 5.4 which governs $E(\overline{T})$, to say that $E(\overline{T})$ is isomorphic to the lattice of faces of a simplicial cone in $\mathbf{R}^n$ generated by n rays through the origin.

Example 2.2 Let $V = K^m \otimes K^m$. Then V has a basis $\{v_i \otimes v_j \mid 1 \le i < j \le m\}$ so $n = \dim V = m^2$. Define $\rho : G_0 \to \mathbf{GL}(V)$ by $\rho(g)(v \otimes v') = gv \otimes gv'$. Then $G = \{c\rho(g) \mid c \in K^*, g \in \mathbf{SL}_m\} = \{g \otimes g \mid g \in \mathbf{GL}_m\}$. Note that ρ is a rational representation: if $g = [c_{ij}]$ then the matrix entries of $\rho(g)$ have the form $c_{ij}c_{kl}$ and are thus polynomials in the matrix entries of g. We may extend ρ to a representation of $\mathbf{M}_m$ by defining $\rho(a)(v \otimes v') = av \otimes av'$ for $a \in \mathbf{M}_m$. Then $\rho(\mathbf{M}_m) = \{a \otimes a \mid a \in \mathbf{M}_m\}$ is Zariski closed[12] and has G as its group of units, so $\overline{G} = \rho(\mathbf{M}_m)$ and thus $M(\rho) = \rho(\mathbf{M}_m)$. In particular, $E(M(\rho)) \simeq E(\mathbf{M}_m)$, isomorphism of posets, via $e \otimes e \leftrightarrow e$. Thus we get nothing new.

Example 2.3 Let $V = K^m \otimes K^m$ be as in Example 2.2 so again $n = \dim V = m^2$. Define $\rho : G_0 \to \mathbf{GL}(V)$ by $\rho(g)(v \otimes v') = gv \otimes (g^{-1})^{\top} v'$. Note that ρ is a rational representation: if $g = (c_{ij})$ then the matrix entries of $\rho(g)$ have the form $c_{ij}c'_{kl}$ where the c'_{kl} are cofactors of g and hence polynomials in the matrix entries of g. We will construct some idempotents in $E(\overline{T})$ which show that $E(M(\rho))$ is radically different from $E(\mathbf{M}_m)$. Our calculation here contains the germ of a general argument whose ancestry may be traced to Hilbert; see Theorem 5.4 for a formulation in the context of reductive monoids. If $g \in T_0$ write $g = \operatorname{diag}(t_1,\dots,t_m)$ where $t_i \in K^*$ and $t_1 \cdots t_m = \det g = 1$. Then $\rho(g)(v_i \otimes v_j) = t_i t_j^{-1}(v_i \otimes v_j)$. Thus, using the basis of elements $v_i \otimes v_j$, the matrix $\rho(g)$ is diagonal:

$$\rho(g) = \operatorname{diag}(\ \dots\ , t_i t_j^{-1},\ \dots\ )\ . \tag{4}$$

Let's try to produce idempotents in $\overline{T}$. Here is the idea. Suppose we can find non-negative integers $a_1,\dots,a_n$ such that

$$\operatorname{diag}(t^{a_1},\dots,t^{a_n}) \in T \text{ for all } t \in K^*\ . \tag{5}$$

Then

$$\operatorname{diag}(t^{a_1},\dots,t^{a_n}) \in \overline{T} \text{ for all } t \in K\ . \tag{6}$$

We may set $t = 0$ in (6). Under this substitution, $t^{a_i} \mapsto 1$ if $a_i = 0$ and $t^{a_i} \mapsto 0$ if $a_i > 0$. Thus the element in (6) is a diagonal matrix with entries in $\{0,1\}$ and hence is an idempotent $e \in \overline{T}$. It is tempting to write $e = \lim_{t\to 0} \operatorname{diag}(t^{a_1},\dots,t^{a_n})$. See (77) for an elaboration of this limit argument. Set $t_i = t^{b_i}$ and $t_j = t^{b_j}$ in

[12] This must be checked. If $M \subseteq \mathbf{M}_m$ is an algebraic monoid and $\rho : M \to \mathbf{M}_n$ is a homomorphism of monoids such that the matrix entries of $\rho(a)$ are polynomials in the matrix entries of a, it does not follow that $\rho(M)$ is a closed submonoid of $\mathbf{M}_n$. This is quite different from the situation with algebraic groups [15, Proposition 7.4.B]. For example take $M = \mathbf{M}_m$ and let $\rho(a) = \det(a)a$.

(4) where $b_1, \ldots, b_m \in \mathbf{Z}$ are to be found. The condition $t_1 \cdots t_m = 1$ demands $b_1 + \cdots + b_m = 0$. Thus

$$\text{(7)} \qquad \operatorname{diag}(\ldots, t^{b_i - b_j}, \ldots) \in \rho(T_0) \subset T$$

for all $t \in K^*$ and all integers $b_1, \ldots, b_m$ such that $b_1 + \cdots + b_m = 0$. There is still some freedom left since $T = K^* \rho(T_0)$ contains the scalar matrices $t^{b_0} \rho(1)$ for $t \in K^*$ and $b_0 \in \mathbf{Z}$. Thus

$$\text{(8)} \qquad \operatorname{diag}(\ldots, t^{b_0 + b_i - b_j}, \ldots) \in T\ .$$

We want to choose integers $b_0, b_1, \ldots, b_m$ so that $b_1 + \cdots + b_m = 0$, all $b_0 + b_i - b_j$ are non-negative, and some $b_0 + b_i - b_j$ is positive. This is a problem in linear programming over $\mathbf{Z}$. Let's exhibit some solutions. Fix a pair of indices i, j with $1 \le i \neq j \le m$. Choose $b_0 = 2, b_i = -1, b_j = 1$, and let $b_k = 0$ for $k \neq i, j$. Then $b_0 + b_i - b_j = 0$ and all other $b_0 + b_p - b_q$ are positive. This gives us an idempotent, say $e_{ij} \in \overline{T}$.

Let $M = M(\rho)$. Since $e_{ij} \in \mathbf{D}_m \otimes \mathbf{D}_m \simeq \mathbf{D}_n$ is a diagonal matrix with just one nonzero entry, it has rank 1 and hence is a minimal element of the poset $E(M)$. Thus $E(M)$ has a set of $m(m-1)$ minimal elements which mutually commute. On the other hand, in $E(\mathbf{M}_m)$ any set of commuting minimal idempotents may be simultaneously diagonalized and thus has cardinality at most m. Thus $E(M)$ and $E(\mathbf{M}_m)$ are not isomorphic. The monoids M and $\mathbf{M}_m$ have closely related groups of units but they are radically different from one another. The poset $E(\overline{T})$ is in fact isomorphic to the lattice of faces of the polytope $\mathcal{P}$ in $\mathbf{R}^m$ which has its vertices at the points $\mathbf{e}_{ij} = \mathbf{e}_i - \mathbf{e}_j$, where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the standard basis for the space $\mathbf{R}^m$ of column vectors. This fact may be deduced from (103). By contrast, the corresponding poset $E(\mathbf{T}_m)$ is isomorphic to the lattice of faces of a simplex.

Let's describe the polytope $\mathcal{P}$ explicitly in case $m = 4$. The calculation begun here which leads to this description will be completed in Example 5.6. We can draw the polytope $\mathcal{P}$ in $\mathbf{R}^3$ because its vertices $\mathbf{e}_{ij}$ lie in the hyperplane $H = \{\sum_{i=1}^4 x_i \mathbf{e}_i \mid \sum_{i=1}^4 x_i = 0\}$ of $\mathbf{R}^4$. Define an $\mathbf{R}$-linear map $\phi : H \to \mathbf{R}^3 \subset \mathbf{R}^4$ by $\phi(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$, $\phi(\mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_1 - \mathbf{e}_2$, $\phi(\mathbf{e}_3 - \mathbf{e}_4) = \mathbf{e}_2 + \mathbf{e}_3$. The $\phi(\mathbf{e}_{ij})$ are the 12 points $\sum_{i=1}^3 x_i \mathbf{e}_i$ such that $\{x_1, x_2, x_3\} \subseteq \{0, \pm 1\}$ and precisely one of x_1, x_2, x_3 is 0. These points are the midpoints of the edges of a cube with vertices at $\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3$. Thus the polytope $\mathcal{P}$ is the convex hull of the midpoints of a cube. This is a cuboctahedron. In Figure 2.1, $\phi(\mathbf{e}_{ij})$ is labeled ij and the "invisible" vertices 41, 42, 43 are omitted.

We may perform the construction $(G_0, \rho) \rightsquigarrow M(\rho)$ with any algebraic group $G_0 \subseteq \mathbf{GL}_m$ and rational representation ρ. If we want interesting monoids we start with interesting groups. By common agreement (?/!) these are the reductive groups. We postpone the definition of "reductive" to Section 3 and simply remark here that the family of reductive groups includes the special linear, general linear, symplectic and orthogonal groups, and that reductive groups have an extraordinary structure governed by finite group W called the Weyl group. One might hope that if we apply

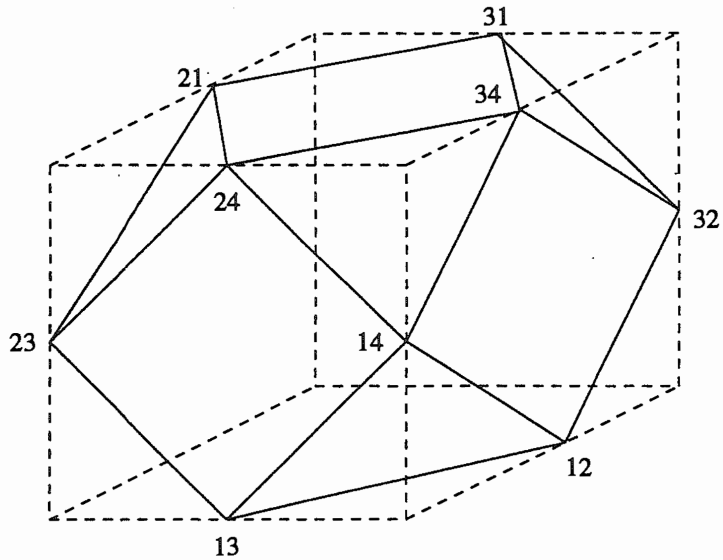

Figure 2.1. Cuboctahedron

the construction $(G_0, \rho) \rightsquigarrow M(\rho)$ to a reductive group, then the resulting monoid will also have extraordinary structure. This turns out to be the case; see Section 5 where we apply this construction in case G_0 is *semisimple* and ρ has finite kernel. Example 2.3 exhibits some of the appeal of the theory of reductive groups. At the core of an enormous edifice lie the regular polyhedra and their close relatives. In any event, since $G = \mathbf{GL}_n$ is reductive with Zariski closure $M = \mathbf{M}_n$, the hypothesis that a monoid is reductive can't lead us too far astray - any general theory must include the multiplicative aspects of matrix algebra over a field. [13]

An *algebraic torus*, or simply a *torus*, is an algebraic group T isomorphic to $\mathbf{T}_m$ for some integer m. The dimension of T is m. We often identify $\mathbf{T}_m$ with $(K^*)^m$. A closed connected subgroup of a torus T is itself a torus [3, Corollary 8.5]. For example $\mathbf{T}_m \cap \mathbf{SL}_m$ is a closed subgroup of $\mathbf{T}_m$ of dimension $m-1$, which is isomorphic to $(K^*)^{m-1}$ via $\mathrm{diag}(t_1, \ldots, t_m) \mapsto (t_1, \ldots, t_{m-1})$ because $t_1 \cdots t_m = 1$. A *rational character* of a torus T is a rational representation $\chi : T \to K^*$. For example if $g = \mathrm{diag}(t_1, \ldots, t_m) \in \mathbf{T}_m$ then

$$\varepsilon_i(g) = t_i \ , \ 1 \leq i \leq m \tag{9}$$

[13] There is no reason except for the infancy of the subject, to assume that K is a field; one might envision $K = \mathbf{Z}$ or a more general coefficient ring as in the theory of algebraic groups. In case K is a field Renner, [43] has given a monoid analogue of the row echelon form for square matrices.

defines a rational character ε_i of $\mathbf{T}_m$. The rational characters of a torus T form a group $X(T)$ under multiplication, which is free abelian of rank equal to dim T [3, 8.5]. For example $X(\mathbf{T}_m)$ has a basis $\{\varepsilon_1, \dots, \varepsilon_m\}$; if $\chi \in X(\mathbf{T}_m)$ is a rational character of $\mathbf{T}_m$ then there exist unique $a_1, \dots, a_m \in \mathbf{Z}$ such that $\chi = \varepsilon_1^{a_1} \cdots \varepsilon_m^{a_m}$. Thus, if $g = \operatorname{diag}(t_1, \dots, t_m)$ then

$$\chi(g) = t_1^{a_1} \cdots t_m^{a_m} \ . \tag{10}$$

This is just (1) when $n = 1$.

Let $T_0 \subseteq \mathbf{T}_m$ be a torus and let $\rho : T_0 \to \mathbf{GL}(V) \simeq \mathbf{GL}_n$ be a rational representation. [14] A nonzero vector $v \in V$ is a *weight vector* for ρ if $\rho(T_0)v = Kv$. Thus v is an eigenvector for all $\rho(g)$ with $g \in T_0$. Define a function $\chi : T_0 \to K^*$ by

$$\rho(g)v = \chi(g)v \ g \in T_0 \ . \tag{11}$$

Since ρ is a representation, χ is a character of T_0. The characters χ which arise in this way are called *weights* of the representation ρ. Let $\Phi(\rho) \subset X(T_0)$ denote the set of weights of ρ. The set $\Phi(\rho)$ is finite, of cardinality at most n. If $\chi \in \Phi(\rho)$ then the space $V_\chi = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in T_0\}$ is called the *weight space* corresponding to χ. The dimension dim V_χ is called the *multiplicity* of the weight χ. It is a fact [3, 8.17] that there exists a basis $\mathcal{B}$ for V which consists of weight vectors for ρ. We usually use $\mathcal{B}$ to identify $\mathbf{GL}(V)$ with $\mathbf{GL}_n$ so that $\rho(T_0) \subseteq \mathbf{T}_n$. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is the standard basis for K^n and $\rho(g)v_j = \chi_j(g)v_j$ for $g \in T_0$ then $\Phi(\rho) = \{\chi_1, \dots, \chi_n\}$. It may happen that $\chi_j = \chi_k$ for $j \neq k$.

For example if $\rho : \mathbf{T}_m \to \mathbf{T}_n$ is the representation in (1), define $\chi_j \in X(\mathbf{T}_m)$ for $1 \le j \le n$ by $\chi_j(\operatorname{diag}(t_1, \dots, t_m)) = t_1^{a_{1j}} \cdots t_m^{a_{mj}}$. Then $\Phi(\rho) = \{\chi_1, \dots, \chi_n\}$. Let's describe $\Phi(\rho)$ for the representations ρ of $\mathbf{SL}_m$ in Examples 2.1-2.3. We view ρ as a representation of the torus $T_0 = \mathbf{T}_m \cap \mathbf{SL}_m$. Define $\varepsilon_i : T_0 \to K^*$ by

$$\varepsilon_i(g) = t_i \ , \ 1 \le i \le m \tag{12}$$

for $g = \operatorname{diag}(t_1, \dots, t_m) \in T_0$. Although the formula in (12) is exactly as in (9), one must remember that in $X(T_0)$ we have $\varepsilon_1 \cdots \varepsilon_m = 1$ because $\det g = 1$ for $g \in T_0$. Thus the ε_i in (12) are the restrictions to T_0 of the ε_i in (9). The bases $\mathcal{B}$ of weight vectors and the corresponding sets $\Phi(\rho)$ are given in the following table. Here ε_i is defined as in (12) and $v_1, \dots, v_m$ is the standard basis for K^m.

Example	$\mathcal{B}$	$\Phi(\rho)$
(2.1)	$\{v_i \mid 1 \le i \le m\}$	$\{\varepsilon_i \mid 1 \le i \le m\}$
(2.2)	$\{v_i \otimes v_j \mid 1 \le i, j \le m\}$	$\{\varepsilon_i \varepsilon_j \mid 1 \le i, j \le m\}$
(2.3)	$\{v_i \otimes v_j \mid 1 \le i, j \le m\}$	$\{\varepsilon_i \varepsilon_j^{-1} \mid 1 \le i, j \le m\}$.

In Example 2.3 the weights $\varepsilon_i \varepsilon_j^{-1}$ with $i \neq j$ have multiplicity 1 while the weight $1 = \varepsilon_1 \varepsilon_1^{-1} = \cdots = \varepsilon_m \varepsilon_m^{-1}$ has multiplicity m.

[14] The change in notation from T to T_0 will be useful in Section 5 where $T_0 \subseteq \mathbf{T}_m$ and $\rho(T_0) \subseteq T \subseteq \mathbf{T}_n$. In Section 4 a torus will usually be written T.

3. The $\mathcal{J}$-class structure of an Algebraic Monoid

Let M be an algebraic monoid. How shall we begin a structure theory of M? Let G be the group of units of M. Since $GM \subseteq M$ and $MG \subseteq M$, the group $G \times G$ acts on M by $(g,h)\cdot a = gah^{-1}$ for $g,h \in G$ and $a \in M$. Let $G\backslash M/G$ denote the set of orbits $\mathcal{O} = GaG$ for this action. From the group theoretic point of view, the first main problem is to describe the set $G\backslash M/G$.

From the point of view of semigroup theory, the analogous problem is to describe the equivalence classes for Green's $\mathcal{J}$-relation, which is defined by $a\mathcal{J}b$ if and only if $MaM = MbM$ [31, 1.1]. Clearly $GaG = GbG \Rightarrow MaM = MbM$. Recall that we have agreed to assume throughout that M is irreducible. If M is irreducible then $MaM = MbM \Rightarrow GaG = GbG$ [31, 6.1], [24, Theorem 13]. Thus a, b lie in the same $\mathcal{J}$-class if and only if they lie in the same $G \times G$ orbit, so that the natural group theoretic and semigroup theoretic equivalence relations are the same. The $\mathcal{J}$-class of a is thus the orbit GaG. We will often write $G \times G$ orbit rather than $\mathcal{J}$-class because it suggests a connection with other parts of mathematics. In general, the set of $\mathcal{J}$-classes in a monoid M carries a natural partial order given by inclusion: $MaM \le MbM \Leftrightarrow MaM \subseteq MbM$ [31, 1.1].

Example 3.1 Let $M = \mathbf{M}_n$. Then $G = \mathbf{GL}_n$. If $a, b \in M$ then $GaG = GbG$ if and only if rank a = rank b. Thus there is a bijection $G\backslash M/G \leftrightarrow \{0, 1, \ldots, n\}$ given by $GaG \to$ rank a. In particular, the number of $G \times G$ orbits is finite. The partial order is the natural linear order on $\{0, 1, \ldots, n\}$.

Example 3.2 ([24, Example 15]) Let $M \subset \mathbf{M}_{n+1}$ consist of all matrices

$$a = \begin{bmatrix} x & x_1 & x_2 & \ldots & x_n \\ 0 & x & 0 & \ldots & 0 \\ 0 & 0 & x & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x \end{bmatrix}$$

where $x, x_1, \ldots, x_n \in K$. For simplicity of notation write $a = (x, x_1, \ldots, x_n)$. The group G consists of those a with $x \neq 0$. The $G \times G$ orbits are G, $\{0 = (0, \ldots, 0)\}$ and orbits which contain matrices $(0, x_1, \ldots, x_n)$ with x_i not all 0. For the latter we have $(0, x_1, \ldots, x_n)$ and $(0, y_1, \ldots, y_n)$ in the same orbit if and only if there exists $c \in K^{\bullet}$ with $y_i = cx_i$ for all i. Thus these orbits are in one to one correspondence with points in the projective space $\mathbf{P}^{n-1}(K)$. There is a bijection $G\backslash M/G \leftrightarrow G \cup \{0\} \cup \mathbf{P}^{n-1}(K)$. This example has some undesired features. The number of $G \times G$ orbits is infinite (unless $n = 1$) and M has no idempotents except $0, 1$. Thus we are led to the following definition [31, 1.5].

Definition 1 Let $\mathcal{U}(M) = \{\mathcal{O} \in G\backslash M/G \mid \mathcal{O} \cap E(M) \neq \emptyset\}$.

Thus $\mathcal{U}(M)$ is the set of $\mathcal{J}$-classes which contain an idempotent. It inherits a partial order from the set of all $\mathcal{J}$-classes. In Example 3.2 the poset $\mathcal{U}(M)$ is isomorphic to the poset $\{0,1\}$. In general, if M is an irreducible algebraic monoid then $\mathcal{U}(M)$ is a finite lattice [31, Theorem 5.10], [21, Theorem 1.7], [22, Theorem 2.7] and any two maximal chains in $\mathcal{U}(M)$ have the same length [23, Theorem 1.9]. The following conjugacy theorem will be useful in Section 5.

Theorem 3.1 **([24, Theorem 9])** *Let M be an irreducible algebraic monoid. Suppose $e, f \in E(M)$. Then $GeG = GfG$ if and only if e and f are conjugate under G.*

Example 3.3 Suppose $M = \mathbf{M}_n$. Then $G = \mathbf{GL}_n$. If $e, f \in E(M)$ and $GeG = GfG$, then rank $e =$ rank f so e, f are conjugate under G.

We will impose conditions on M which exclude the pathology in Example 3.2 and insure that $G\backslash M/G = \mathcal{U}(M)$. A monoid M is *unit regular* if $M = GE(M)$. A monoid M is *regular* if given $a \in M$ there exists $b \in M$ such that $a = aba$.

Proposition 3.2 *Let M be an irreducible algebraic monoid. Then the conditions (1) M is unit regular, (2) M is regular and (3) $G\backslash M/G = \mathcal{U}(M)$, are equivalent.*

For (1) $\Rightarrow$ (2) note that if $a = eg$ with $e \in E(M)$ and $g \in G$ then $ag^{-1}a = ea = a$. For (2) $\Rightarrow$ (3) note that, by our earlier remark, the $\mathcal{J}$-classes are the $G \times G$ orbits, and that if $aba = a$ then $ab = abab$ is an idempotent in the $\mathcal{J}$-class of a. To prove that (3) $\Rightarrow$ (1), given $a \in M$ choose $g, h \in G$ and $e \in E(M)$ with $a = geh$. Then $a = gh \cdot h^{-1}eh \in GE(M)$.

Thus, if M is regular then the set $G\backslash M/G$ is finite. If $a \in M$ then MaM is stable under the $G \times G$ action and is thus a finite union of orbits GbG. The closure $\overline{MaM}$ is thus a finite union of sets $\overline{GbG}$. On the other hand, if M regular then MaM is closed and irreducible as algebraic set. This is proved in [22, Corollary 2.5] under the hypothesis that a is idempotent, but this is no restriction since $G\backslash M/G = \mathcal{U}(M)$. Thus $MaM = \overline{GaG}$. It follows that if M is regular then for $a, b \in M$ we have $MaM \leq MbM \Leftrightarrow GaG \subseteq \overline{GbG}$. Thus the partial order on $\mathcal{U}(M)$ which is defined naturally in terms of the semigroup structure has an algebro-geometric interpretation in terms of the closures of group orbits. In Example 3.1, if rank $a = r$ then $\overline{GaG}$ consists of the matrices of rank at most r.

We may also exclude the undesired behavior in Example 3.2 by imposing the requirement that G be a *reductive group*. What is a reductive group? To define it we need some preliminaries. A matrix $u \in \mathbf{GL}_n$ is *unipotent* if all its eigenvalues are 1 [15, 15.1]. A subgroup of $\mathbf{GL}_n$ is unipotent if all its elements are unipotent [15, 17.5]. For example the group

$$\mathbf{U}_n = \{I + \sum_{1 \leq i < j \leq n} c_{ij}E_{ij} \mid c_{ij} \in K\} \tag{13}$$

of all upper unitriangular matrices is unipotent. In fact every unipotent subgroup of $\mathbf{GL}_n$ is conjugate to a subgroup of $\mathbf{U}_n$ [3, Theorem 4.8], [15, Corollary 17.5]. The set of all connected unipotent normal subgroups of a connected algebraic group G

has a unique maximal element $\mathcal{R}_u(G)$ called the *unipotent radical* [3, 11.21], [15, 19.5] of G. The group G is said to be *reductive* [3, 11.21], [15, 19.5] if $\mathcal{R}_u(G) = \{1\}$. Let $\mathbf{B}_n \subseteq \mathbf{GL}_n$ denote the group of all invertible upper triangular matrices.

Example 3.4 If $G = \mathbf{B}_n$ then $\mathcal{R}_u(G) = \mathbf{U}_n$. Thus $\mathbf{B}_n$ is not reductive for $n \geq 2$.

Example 3.5 Let $G = \mathbf{GL}_n$. Since any normal subgroup of G either contains $\mathbf{SL}_n$ or is included in the center $Z(G) = \{tI \mid t \in K^*\}$, G is reductive. Similarly $\mathbf{SL}_n$ is reductive.

If $e \in E(M)$ is idempotent, define the *right centralizer* $P(e)$ and *left centralizer* $P^-(e)$ of e in G to be the subgroups [31, p.47] [15]

$$(14) \qquad P(e) = \{g \in G \mid ge = ege\}\ , \quad P^-(e) = \{g \in G \mid eg = ege\}\ .$$

Let $C_G(e) = \{g \in G \mid ge = eg\}$ be the centralizer of e in G. Then

$$(15) \qquad C_G(e) = P(e) \cap P^-(e)\ .$$

Sometimes we write $L(e) = C_G(e)$ since it turns out that $L(e)$ is a common Levi factor of $P(e)$ and $P^-(e)$; see (64) for the definition of Levi factor and see the case $\Gamma = \{e\}$ of Theorem 5.7.

Example 3.6 Let $G = \mathbf{GL}_n$. For $1 \leq r \leq n$ let $e = e_r = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathbf{D}_n$ where there are r entries equal to 1. Then $P(e) = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$ and $P^-(e) = \left[\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right]$ where the $*$'s are arbitrary matrices of the appropriate size such that the diagonal $*$'s are invertible. These groups have unipotent radicals $U(e) = \left[\begin{smallmatrix} I & * \\ 0 & I \end{smallmatrix}\right]$ and $U^-(e) = \left[\begin{smallmatrix} I & 0 \\ * & I \end{smallmatrix}\right]$ respectively, where the $I's$ are identity matrices of the appropriate size. Note that the intersection $L(e) = \left[\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right]$ is the *centralizer* of e in $\mathbf{GL}_n$. It is isomorphic to $\mathbf{GL}_r \times \mathbf{GL}_{n-r}$ and hence is a reductive group. Note also that $P(e)$ and $P^-(e)$ factor as semidirect products $P(e) = L(e)U(e)$ and $P^-(e) = L(e)U^-(e)$. Since $L(e)$ is reductive and $U(e)$ is a unipotent normal subgroup of $P(e)$ we have $U(e) = \mathcal{R}_u(P(e))$. Similarly $U^-(e) = \mathcal{R}_u(P^-(e))$. This is an important example in the general theory; see (64) and Example 4.30.

Definition 2 A *reductive monoid* is an algebraic monoid which is irreducible as algebraic set and has a connected reductive group of units. [16]

Thus, for example, $\mathbf{M}_n$ is reductive because $\mathbf{GL}_n$ is reductive. The monoid in Example 3.2 is not reductive. An element $g = (x, x_1, \ldots, x_n) \in G$ is unipotent if and only if $x = 1$. The set $U = \{(1, x_1, \ldots, x_n)\}$ of all unipotent elements in G is a normal subgroup of G which is Zariski closed because it is isomorphic to a product of n copies of K. Thus $U = \mathcal{R}_u(G)$ and G is not reductive.

It is truly remarkable, an unexpected confluence of group theory and semigroup theory, that M is reductive if and only if M is regular.

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[15] These are sometimes written as $C^r_G(e)$ and $C^l_G(e)$. The notation $P(e), P^-(e)$ is more suggestive in the context of reductive monoids, since $P(e), P^-(e)$ are parabolic subgroups of G; see the remarks which precede Example 4.29 for the definition of parabolic subgroup.

[16] The connectivity of G is in fact a consequence of the irreducibility of M [25, p. 695].

Theorem 3.3 *Let M be an irreducible algebraic monoid with zero. Then M is reductive if and only if M is regular.* [17]

There is a clear signal here. Regular semigroups are a distinguished class in semigroup theory. Reductive groups form the centerpiece of the theory of algebraic groups. Two notions which, on the face of it, have nothing to do with one another are in fact closely related. From the point of view of semigroup theory, it might be a good idea to learn something about reductive groups. We will do this in the next section. From Theorem 3.3 and Proposition 3.2 we conclude:

Theorem 3.4 *Let M be a reductive monoid with zero. Let G be the group of units of M. Then $G\backslash M/G$ is finite, and every $G \times G$ orbit contains an idempotent.*

4. Reductive Groups

This is a big subject. Our aim is to maximize understanding in minimal time by study of some of the main examples. Thus the sequence of definitions and theorems given here is not the sequence one would ordinarily use to develop the theory. We have not in general used the formal notice **Theorem**. There are two exceptions. It would not be possible to prove the main theorems on reductive groups in the order given here. The Lie algebra has been suppressed, the roots are defined in terms of their corresponding root groups and introduction of the Borel subgroup is long postponed. We will make the necessary definitions, state some of the main theorems, and spend most of our time on examples. The examples will be $\mathbf{GL}_n$, $\mathbf{SL}_n$, and the symplectic group $\mathbf{Sp}_n$. To avoid trivial cases we assume throughout that $n \geq 2$. Note though that $\mathbf{GL}_1 \simeq K^*$.

This is a long section, which I have tried to make "user-friendly." A reader who cares about semigroups, but knows nothing about reductive groups, might turn to the last paragraph of this section. Here, in the correct general context, we find the right and left centralizers $P(e), P^-(e)$ of an idempotent $e \in \mathbf{M}_n$ which were defined in (14) and computed in Example 3.6.

Our first aim is to introduce the *roots* of a reductive group, relative to a chosen *maximal torus*. Let's begin in comfort with $\mathbf{GL}_n$. For $1 \leq i \neq j \leq n$ let

$$U_{ij} = \{I + cE_{ij} \mid c \in K\} \tag{16}$$

where E_{ij} is a matrix unit. Then U_{ij} is a unipotent subgroup of G and the map $x_{ij} : K^+ \to U_{ij}$ defined by

[17] The implication "regular $\Rightarrow$ reductive" was proved by Putcha in [26, Theorem 2.11]. The converse was proved by Putcha [28, Theorem 2.4] if K has characteristic zero, using complete reducibility of the rational representations of G. The converse was proved in arbitrary characteristic by Renner in his Thesis [39, Theorem 4.4.15], [42, Theorem 3.1]. Renner's proof uses a fair amount of algebraic geometry. Putcha [29, Theorem 1.1] gave a different proof, valid in arbitrary characteristic, which uses the structure of reductive groups and some geometry. This argument also appears in [31, Theorem 7.3]. The hypothesis concerning zero is "necessary". If M is a regular algebraic monoid and H is a connected algebraic group which is not reductive, then the monoid of all matrices $\mathrm{diag}(a, h)$ with $a \in M$ and $h \in H$ is regular but not reductive.

$$x_{ij}(c) = I + cE_{ij} \ , \ c \in K \tag{17}$$

is an isomorphism. Here K^+ denotes the additive group of K. It has been known since Jordan in the case of the finite field $\mathbf{F}_p$, and since Dickson in general, that the subgroups U_{ij} generate $\mathbf{SL}_n$ and hence, together with the invertible diagonal matrices, generate $\mathbf{GL}_n$. If $t = \mathrm{diag}(t_1, \dots, t_n) \in \mathbf{T}_n$ then

$$tx_{ij}(c)t^{-1} = x_{ij}(t_i t_j^{-1} c) \ , \ t \in \mathbf{T}_n, \ c \in K \ . \tag{18}$$

For $i \neq j$ define $\alpha_{ij} \in X(\mathbf{T}_n)$ by $\alpha_{ij} = \varepsilon_i \varepsilon_j^{-1}$ where ε_i is as in (9). Then we may rewrite (18) as

$$tx_{ij}(c)t^{-1} = x_{ij}(\alpha_{ij}(t)c) \ , \ t \in \mathbf{T}_n, \ c \in K \ . \tag{19}$$

This formula suggests a definition, (20) below, in any connected reductive group G. First we define the analogue in G of the subgroup $\mathbf{T}_n$ of $\mathbf{GL}_n$. A *maximal torus* of G is a torus $T \subseteq G$ of maximal dimension. Since a closed connected subgroup of a torus is itself a torus [3, Corollary 8.5], a maximal torus is maximal in the set theoretic sense. Choose, once and for all, a maximal torus T. Since T is uniquely determined up to conjugacy in G [3, Corollary 11.3], [15, Corollary 21.3.A], our later constructions do not depend on the choice of T. A character $\alpha \in X(T)$ is a *root* of G relative to T if there exists a monomorphism $\varphi : K^+ \to G$ such that [18]

$$t\varphi(c)t^{-1} = \varphi(\alpha(t)c) \ , \ t \in T, \ c \in K \ . \tag{20}$$

Define

$$U_\alpha = \varphi(K) \ . \tag{21}$$

In view of the preceding footnote, the group $\varphi(K)$ is uniquely determined by α and our usage of U_α is consistent with that in [3, 15]. The subgroups U_α are called *root*

[18] This is not the usual definition of "root" as given for example in [3, 8.17] or [15, 16.4] but is equivalent to it. The point of the definition given here is that it allows us to enter the subject with minimal prerequisites. Since roots are the essence of reductive groups we show the reader where to find a proof that the two definitions are equivalent. The group G and hence the maximal torus T has a representation on the Lie algebra of G - not defined in this paper - called the ***adjoint representation*** [15, 9.1], [3, 3.13] and written Ad. The set of roots of G relative to T is, according to the usual definition, the set $\Phi(Ad)$ of weights of the adjoint representation. If $\alpha \in \Phi(Ad)$ then [15, Theorem 26.3(c)] proves the existence of a monomorphism φ which satisfies (20). Conversely suppose $\alpha \in X(T)$ satisfies (20). Then T normalizes $\varphi(K)$. It follows from [3, Proposition 13.20] with $H = \varphi(K)$ that there exists $\beta \in \Phi(Ad)$ such that $\varphi(K) = U_\beta$ where the group U_β is defined in [3, Theorem 13.18(4)(d)] or [15, Theorem 26.3(a)]. It follows from [15, Theorem 26.3(c)] that there exists a monomorphism $\psi : K^+ \to G$ such that $t\psi(c)t^{-1} = \psi(\beta(t)c)$ for $t \in T$ and $c \in K$. Since $\psi^{-1}\varphi$ is an automorphism of K^+ it follows from [3, 10.10] that there exists $k \in K^*$ with $\psi(c) = \varphi(kc)$. Then $\varphi(k\alpha(t)c) = \varphi(k\beta(t)c)$ so $\alpha = \beta$. Thus $\alpha \in \Phi(Ad)$. This argument also shows that the group $\varphi(K)$ in (19) is uniquely determined by the root α as $\varphi(K) = U_\alpha$.

groups. The map φ is not uniquely determined by α. However, as remarked in the preceding footnote, the automorphisms of K^+ have the form $c \mapsto kc$, so the map φ is determined up to replacement of φ by $c \mapsto \varphi(kc)$ for some $k \in K^*$. For given α we make a fixed choice of φ.

The term "root" may puzzle a beginner. The Lie algebra of $\mathbf{GL}_n$ is $\mathbf{M}_n$ with the bracket operation $[a,b] = ab - ba$ for $a,b \in \mathbf{M}_n$. If $a = \mathrm{diag}(a_1, \cdots, a_n) \in \mathbf{D}_n \subseteq \mathbf{M}_n$ is a diagonal matrix, then the formula $[a, E_{ij}] = (a_i - a_j)E_{ij}$ shows that E_{ij} is an eigenvector for the K-linear map denoted ad $a : \mathbf{M}_n \to \mathbf{M}_n$ which is defined by $b \mapsto [a,b]$, for $b \in \mathbf{M}_n$ and that the corresponding eigenvalue, alias characteristic *root*, is $a_i - a_j$. This is the Lie algebraic version of (18) and is the source of the terminology.

The set $\Phi(G,T)$ of roots is finite and [19]

$$|\Phi(G,T)| = \dim G - \dim T \; . \tag{22}$$

Example 4.1 Let $G = \mathbf{GL}_n$. Then G is a connected reductive group and $T = \mathbf{T}_n$ is a maximal torus. [20] Let ε_i be as in (9). Then $\Phi(G,T) = \{\varepsilon_i\varepsilon_j^{-1} \mid 1 \le i \ne j \le n\}$ because (18) shows that $\varepsilon_i\varepsilon_j^{-1} \in \Phi(G,T)$ and $|\Phi(G,T)| = \dim G - \dim T = n^2 - n$ is the number of $\varepsilon_i\varepsilon_j^{-1}$ with $i \ne j$. If $\alpha = \varepsilon_i\varepsilon_j^{-1}$ then $U_\alpha = U_{ij}$ is as in (16).

Example 4.2 Let $G = \mathbf{SL}_n$. Then G is a connected reductive group and $T = G \cap \mathbf{T}_n$ is a maximal torus. Let ε_i be as in (12). Then $\Phi(G,T) = \{\varepsilon_i\varepsilon_j^{-1} \mid 1 \le i \ne j \le n\}$. The homomorphisms x_{ij} are exactly as in $\mathbf{GL}_n$ and $|\Phi(G,T)| = \dim G - \dim T = (n^2-1) - (n-1) = n^2 - n$ as in $\mathbf{GL}_n$. Again, if $\alpha = \varepsilon_i\varepsilon_j^{-1}$ then U_α is as in (16).

Example 4.3 Let $n = 2l$ be even and let $\{v_1, \dots, v_n\}$ be the standard basis for K^n. Let $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right] \in \mathbf{M}_n$ where I is the identity matrix of size l. Thus J is nonsingular and skew symmetric. The *symplectic group* is by definition

$$G = \mathbf{Sp}_n = \{g \in \mathbf{GL}_n \mid g^\top J g = J\} \; . \tag{23}$$

It is connected and reductive. [21] Let $T = G \cap \mathbf{T}_n$. Elements of T have the shape

$$t = \mathrm{diag}(t_1, \dots, t_l, t_1^{-1}, \dots, t_l^{-1}) \tag{24}$$

[19] This follows from [3, Proposition 13.20], [15, Corollary 26.2.B] and the fact that the dimension of G is equal to the dimension of its Lie algebra [3, 3.5], [15, 9.1].

[20] Since the torus T is its own centralizer in G it is a maximal torus. To prove that G is connected, note that $G = \langle T, U_{ij} \mid 1 \le i \ne j \le n \rangle$. Since $T \simeq (K^*)^n$ and the $U_{ij} \simeq K^+$ are connected we may apply [15, Corollary 7.5]. We have already remarked that G is reductive.

[21] The group G is generated by transvections [1, Theorem 3.21], each of which may be imbedded in a subgroup isomorphic to the connected group K^+ [50, 6.6]. Thus we may apply [15, Corollary 7.5] to prove that G is connected. To prove that G is reductive one may use the fact that the only proper normal subgroup of G, as abstract group, is its center of order at most 2 [1, Theorem 5.2]. Alternatively, one may exploit the fact that the natural representation of G on K^n is irreducible [6, Exposé 22, p.1].

where $t_1, \dots, t_l$ are arbitrary in K^*. Thus T is a torus of dimension l. As in the case of $\mathbf{GL}_n$, it is a maximal torus because it is its own centralizer in G. Define $\varepsilon_i : T \to K^*$ for $1 \le i \le l$ by

$$\varepsilon_i(\mathrm{diag}(t_1, \dots, t_l, t_1^{-1}, \dots, t_l^{-1})) = t_i. \tag{25}$$

Since $\dim G = l(2l+1)$ [6, Exposé 22, p.3] we have $|\Phi(G,T)| = \dim G - \dim T = l(2l+1) - l = 2l^2$. [22] We will exhibit $2l^2$ homomorphisms $\varphi : K^+ \to G$ which satisfy (20) for suitable characters $\alpha \in X(T)$ [6, Exposé 22, p.2]. For $1 \le i \ne j \le l$ define $y_{ij} : K^+ \to \mathbf{SL}_l$ by $y_{ij}(c) = I + cE_{ij}$ for $c \in K$. These are precisely the maps used in (17) with a change of notation because the letter "x" is now reserved for $\mathbf{Sp}_n$. Define $x_{ij} : K^+ \to \mathbf{Sp}_n$ by

$$x_{ij}(c) = \begin{bmatrix} y_{ij}(c) & 0 \\ 0 & y_{ji}(-c) \end{bmatrix}. \tag{26}$$

To check that $x_{ij}(c) \in \mathbf{Sp}_n$ use $y_{ji}(c) = y_{ij}(c)^\top$. For $1 \le i \ne j \le l$ and $c \in K$, define x'_{ij}, x''_{ij} by

$$x'_{ij}(c) = \begin{bmatrix} I & c(E_{ij}+E_{ji}) \\ 0 & I \end{bmatrix}, \quad x''_{ij}(c) = \begin{bmatrix} I & 0 \\ c(E_{ij}+E_{ji}) & I \end{bmatrix}. \tag{27}$$

For $1 \le i \le l$ and $c \in K$, define x_i, x'_i by

$$x_i(c) = \begin{bmatrix} I & cE_{ii} \\ 0 & I \end{bmatrix}, \quad x'_i(c) = \begin{bmatrix} I & 0 \\ cE_{ii} & I \end{bmatrix}. \tag{28}$$

Here the $E_{ij} \in \mathbf{M}_l$ are matrix units and I is the identity matrix of size l. These maps $\varphi = x_{ij}, x'_{ij}, x''_{ij}, x_i, x'_i$ are all monomorphisms from $K^+ \to \mathbf{Sp}_n$. [23] We must compute $t\varphi(c)t^{-1}$ for these monomorphisms φ and check that there exist characters $\alpha \in X(T)$ which satisfy (20). This calculation may be done easily with block matrices of size l and we simply tabulate the results below. We have

(29)

φ	$t\varphi(c)t^{-1}$	α	
x_{ij}	$x_{ij}(t_it_j^{-1}c)$	$\varepsilon_i\varepsilon_j^{-1}$	$1 \le i \ne j \le l$
x'_{ij}	$x'_{ij}(t_it_jc)$	$\varepsilon_i\varepsilon_j$	$1 \le i < j \le l$
x''_{ij}	$x'_{ij}(t_i^{-1}t_j^{-1}c)$	$\varepsilon_i^{-1}\varepsilon_j^{-1}$	$1 \le i < j \le l$
x_i	$x_i(t_i^2c)$	ε_i^2	$1 \le i \le l$
x'_i	$x'_i(t_i^{-2}c)$	ε_i^{-2}	$1 \le i \le l$

[22] The dimension of G is computed in [6, Exposé 22] by algebro-geometric argument; one may also compute it as the dimension of the Lie algebra $\mathrm{Lie}(G) = \{a \in \mathbf{M}_n \mid a^\top J + Ja = 0\}$.

[23] They may be found, with slightly different notation, in [6, Exposé 22]

where the ε_i are as in (25) and t and the t_i are as in (24). Thus the set $\Phi(G,T)$ of roots of $G = \mathbf{Sp}_n$ relative to the chosen maximal torus T is the set of characters α in the right hand column of the table, namely

(30) $$\Phi(G,T) = \{\varepsilon_i\varepsilon_j^{-1},\ \varepsilon_i\varepsilon_j,\ \varepsilon_i^{-1}\varepsilon_j^{-1},\ \varepsilon_i^2, \varepsilon_i^{-2}\}$$

with the restrictions on i,j given above. If $\alpha \in \Phi$ then U_α is the set of matrices which appear in the appropriate line (26)-(28).

Let G be a connected reductive group. Then

(31) $$G = < T, U_\alpha \mid \alpha \in \Phi(G,T) >$$

is generated by the maximal torus T and the root groups U_α [3, Corollary 14.8.1], [15, Theorem 26.3(d)]. Thus the structure of G is determined in large part by the manner in which the various U_α are glued together. This is controlled by the *Weyl group*, a finite group with extraordinary properties. The setup is complex enough to be interesting but structured enough to be manageable. Our next objective is to study the Weyl group. The Weyl group of G relative to T is by definition

(32) $$W(G,T) = N_G(T)/T$$

where $N_G(T)$ is the normalizer of T in G. [24] Let's simplify notation. Henceforth we fix a maximal torus T and write $\Phi = \Phi(G,T)$ for the set of roots, $N = N_G(T)$ for the normalizer and $W = W(G,T)$ for the Weyl group. The Weyl group is finite [3, 11.19], [15, Proposition 24.1.A].

Example 4.4 Let $G = \mathbf{GL}_n$. Let S_n be the symmetric group on $\{1, \dots, n\}$. Define a subgroup $\mathbf{N}_n$ of $\mathbf{GL}_n$ by

(33) $$\mathbf{N}_n = \{\omega = \textstyle\sum_{j=1}^{m} t_j E_{\pi j,j} \mid t_j \in K^* \text{ and } \pi \in S_n\}\ .$$

The group $\mathbf{N}_n$ consists of all matrices which have just one nonzero entry in each row and column. These are sometimes called *monomial matrices*. The notation $\mathbf{N}_n$ is not standard but will be useful. Let's choose $T = \mathbf{T}_n$ as our maximal torus. Then $N = \mathbf{N}_n$. The map $\omega = \sum_{j=1}^{n} t_j E_{\pi j,j} \mapsto \pi$ is a homomorphism from N onto S_n with kernel T. Thus $W = N/T \simeq S_n$ so the Weyl group is isomorphic to the symmetric group on n letters. Let $P_n \subset G$ be the group of *permutation matrices*. By definition these are matrices of the form $p = \sum_{j=1}^{n} E_{\pi j,j}$ where $\pi \in S_n$. Then $N = TP_n$ splits as a semidirect product and we may realize W as a subgroup of G. This simplifies calculations in case $G = \mathbf{GL}_n$ but it is misleading. In general there is no semidirect product decomposition $N = TP$ with P isomorphic to W.

[24] If G is a connected algebraic group and $T \subseteq G$ is a torus, then the Weyl group is defined [3, 11.19], [15, 24.1] by $W(G,T) = N_G(T)/Z_G(T)$ where $Z_G(T)$ is the centralizer of T in G. If G is reductive and T is a maximal torus then $Z_G(T) = T$ [3, Corollary 13.17(2)], [15, Corollary 26.2.A] so $W(G,T)$ is as in (32).

Example 4.5 Let $G = \mathbf{SL}_n$ and $T = G \cap \mathbf{T}_n$. Then $N = G \cap \mathbf{N}_n$. The map $N \to S_n$ defined by $p = \sum_{j=1}^{n} t_j E_{\pi j,j} \mapsto \pi$ is surjective, as in the case of $\mathbf{GL}_n$ and has kernel T. Thus again $W = N/T \simeq S_n$. Note that permutation matrices may have determinant -1 and thus need not be in G. For example, if $n = 2$ and $\pi = (12)$, the corresponding coset in N may be represented by $\omega = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ but not by $p = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Example 4.6 Let $G = \mathbf{Sp}_n$ and let T be as in Example 4.3. Let's compute W. The description of N is complicated by the fact that $\mathbf{Sp}_n \subseteq \mathbf{SL}_n$ so that $\mathbf{Sp}_n$ does not contain the group P_n of permutation matrices; see the last sentence in Example 4.5. Let's ignore this difficulty for the moment. If $\pi \in S_n$ let $p_\pi = \sum_{i=1}^{n} E_{\pi i,i} \in P_n$ be the corresponding permutation matrix. Define an involutory permutation $i \mapsto \bar{i}$ of $\{1, \ldots, n\}$ by $\bar{i} = i + l$ if $1 \le i \le l$ and $\bar{i} = i - l$ if $l + 1 \le i \le n$. Let C denote the centralizer of this involution in S_n. Then p_π normalizes T if and only if $\pi \in C$. The group C factors as a semidirect product $C = C_1 C_2$ where C_1 is a normal abelian subgroup of order 2^l generated by the transpositions $(1\bar{1}), \ldots, (l\bar{l})$ and $C_2 \simeq S_l$ consists of all permutations $\pi \in S_n$ which stabilize $\{1, \ldots, l\}$ and act on the complement $\{l+1, \ldots, n\}$ in the unique manner consistent with the assertion that $\pi \in C$. For example, with $l = 5$, if the restriction of π to $\{1, \ldots, 5\}$ is $(135)(24)$ in cycle notation, then $\pi = (135)(24)(\bar{1}\bar{3}\bar{5})(\bar{2}\bar{4})$. Note that $|C| = |C_1||C_2| = 2^l l!$. Now return to the difficulty mentioned at the beginning. It is possible to multiply p_π by a suitable diagonal matrix d_π with entries ± 1 such that $\omega_\pi = d_\pi p_\pi \in \mathbf{Sp}_n$. Since p_π normalizes T so does ω_π. Thus $\omega_\pi \in N$. This shows that $N \supseteq \sqcup_{\pi \in C} \omega_\pi T$, where $\sqcup$ means disjoint union. In fact $N = \sqcup_{\pi \in C} \omega_\pi T$. Thus $W = N/T \simeq C \subseteq S_n$. For $i = 1, 2$ let $N_i = \sqcup_{\pi \in C_i} \omega_\pi T$ and let $W_i = N_i/T$. Then $W = W_1 W_2 \simeq C_1 C_2 = C \subseteq S_n$ where the products are semidirect. There exists an isomorphism $\theta : W \to \mathbf{GL}_l(\mathbf{R})$ such that $\theta(W_1)$ is the group of diagonal matrices with entries in $\{\pm 1\}$ and $\theta(W_2) = P_l$ is the group of permutation matrices. The group $\theta(W)$ is the symmetry group of the *hyperoctahedron* or *cross polytope* in $\mathbf{R}^l$ with vertices at the $2l$ points $\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_l$ where $\mathbf{e}_1, \ldots, \mathbf{e}_l$ is the standard basis for $\mathbf{R}^l$. Thus W is sometimes called the *hyperoctahedral group*.

Let G be a connected reductive group and let $T \subseteq G$ be a maximal torus. If $\chi \in X(T)$ and $\omega \in N$, define a function $\omega\chi : T \to K$ by $(\omega\chi)(t) = \chi(\omega^{-1} t \omega)$ for $t \in T$. Then $\omega\chi \in X(T)$. Since $\omega^{-1} t \omega$ depends only on $w = \omega T$ we write $w\chi = \omega\chi$. Thus W acts on $X(T)$. If $\chi, \chi' \in X(T)$ then $\omega(\chi\chi') = (\omega\chi)(\omega\chi')$, so $w(\chi\chi') = (w\chi)(w\chi')$. Thus W acts as a group of automorphisms of the abelian group $X(T)$. This action is faithful: if $w\chi = \chi$ for all $\chi \in X(T)$ then $w = 1$.[25] Let $\rho : G \to \mathbf{GL}(V)$ be a rational representation. View ρ as a representation of T and, as in Section 2, let $\Phi(\rho) \subset X(T)$ be the set of weights of ρ. Suppose $\chi \in \Phi(\rho)$ and $v \in V$ is a vector of weight χ. Thus $\rho(t)v = \chi(t)v$ for $t \in T$. If $\omega \in N$ then $\rho(t)\rho(\omega)v = \rho(\omega)\rho(\omega^{-1} t \omega)v = \rho(\omega)\chi(\omega^{-1} t \omega)v = (\omega\chi)(t)\rho(\omega)v$. Thus $\omega\chi \in \Phi(\rho)$. Thus

$$W\Phi(\rho) = \Phi(\rho)\,. \tag{34}$$

[25] Suppose $\omega \in N$ and $\omega\chi = \chi$ for all $\chi \in X(T)$. If $t \in T$ then $\chi(\omega^{-1} t \omega t^{-1}) = \chi(\omega^{-1} t \omega)\chi(t)^{-1} = 1$. Thus $\omega t \omega^{-1} t^{-1} = 1$ so ω centralizes T and thus $\omega \in T$; see the footnote which follows (32).

If we had defined roots using the adjoint representation - see the footnote to (20) - then it would follow in particular from (34) that $W\Phi = \Phi$. Let's check this directly using our definition (20). Note that N acts on $\mathrm{Hom}(K^+, G)$ by $(\omega\varphi)(c) = \omega\varphi(c)\omega^{-1}$ for $\omega \in N$, $\varphi \in \mathrm{Hom}(K^+, G)$ and $c \in K$. If $\alpha \in \Phi$ and $\varphi \in \mathrm{Hom}(K^+, G)$ satisfy (20) then $\beta = \omega\alpha$ and $\psi = \omega\varphi$ satisfy $t\psi(c)t^{-1} = \psi(\beta(t)c)$ for $t \in T$ and $c \in K$ so $\beta = \omega\alpha \in \Phi$. Thus indeed

(35) $$W\Phi = \Phi\ .$$

If $\alpha \in \Phi$ then, since T normalizes U_α, the group $\omega U_\alpha \omega^{-1}$ depends only on $w = \omega T \in W$. We abuse notation and write $wU_\alpha w^{-1} = \omega U_\alpha \omega^{-1}$. Thus [15, Theorem 26.3(b)]

(36) $$wU_\alpha w^{-1} = U_{w\alpha} \text{ for } w \in W\ .$$

Example 4.7 Let $G = \mathbf{GL}_n$ and let T, N be as in Example 4.4. If $\pi \in S_n$ let $p = \sum_{j=1}^n E_{\pi j,j} \in P_n \subseteq N$ be the corresponding permutation matrix. Let $\varepsilon_i \in X(T)$ be as in (9). Then $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for the abelian group $X(T)$. Since W acts as a group of automorphisms of $X(T)$, the action of w on $X(T)$ is determined by the $w\varepsilon_i$. Let $w = pT$. Then $w\varepsilon_i = \varepsilon_{\pi i}$. Thus, as one might expect, $W \simeq S_n$ permutes $\varepsilon_1, \dots, \varepsilon_n$ in the same way that S_n permutes $\{1, \dots, n\}$. From Example 4.1 we have $\Phi = \{\varepsilon_i\varepsilon_j^{-1} \mid 1 \leq i \neq j \leq n\}$. Then $w(\varepsilon_i\varepsilon_j^{-1}) = (w\varepsilon_i)(w\varepsilon_j)^{-1} = \varepsilon_{\pi i}\varepsilon_{\pi j}^{-1} \in \Phi$. If U_{ij} is the root group corresponding to $\varepsilon_i\varepsilon_j^{-1}$ and $\pi \in S_n$ is the permutation corresponding to $w \in W$ then $wU_{ij}w^{-1} = U_{\pi i, \pi j}$.

Example 4.8 Let $G = \mathbf{SL}_n$. Let T, N be as in Example 4.5 and let $\varepsilon_1, \dots, \varepsilon_n$ be as in (12). Remember in this case that $\varepsilon_1 \cdots \varepsilon_n = 1$. Here we may choose $\varepsilon_1, \dots, \varepsilon_{n-1}$ as a basis for $X(T)$. In spite of this difference, the bottom line for $\mathbf{SL}_n$ is the same as for $\mathbf{GL}_n$. By Example 4.2 we have $\Phi = \{\varepsilon_i\varepsilon_j^{-1} \mid 1 \leq i \neq j \leq n\}$. If $w \in W$ corresponds to $\pi \in S_n$ then $w\varepsilon_i = \varepsilon_{\pi i}$ and $w(\varepsilon_i\varepsilon_j^{-1}) = \varepsilon_{\pi i}\varepsilon_{\pi j}^{-1} \in \Phi$ and $wU_{ij}w^{-1} = U_{\pi i, \pi j}$.

Example 4.9 Let $G = \mathbf{Sp}_n$ where $n = 2l$. Let T, N be as in Example 4.6 and let $\varepsilon_1, \dots, \varepsilon_l$ be as in (25). Recall from Example 4.6 that $W = W_1W_2 \simeq C_1C_2 = C \subseteq S_n$. To describe the action of $w \in W$ on $X(T)$ it suffices to know the action of $w \in W_1$ and $w \in W_2$. If $w \in W_1$ and $\pi \in S_n$ is the corresponding permutation then there exists a subset I of $\{1, \dots, l\}$ such that $\pi = \prod_{i \in I}(i\bar{i})$. Then for $1 \leq i \leq l$ we have $w\varepsilon_i = \varepsilon_i^{-1}$ if $i \in I$ and $w\varepsilon_i = \varepsilon_i$ if $i \notin I$. If $w \in W_2$ and $\pi \in S_n$ is the corresponding permutation then $\pi\{1, \dots, l\} \subseteq \{1, \dots, l\}$ and the action of w on $X(T)$ is given by $w\varepsilon_i = \varepsilon_{\pi i}$ for $1 \leq i \leq l$. It follows that the set Φ listed in (30) is stable under both W_1 and W_2 and is thus stable under W. To check that $wU_\alpha w^{-1} = U_{w\alpha}$ in $\mathbf{Sp}_n$ one must separate the cases (26), (27), (28) and one may choose $w \in W_1$ or $w \in W_2$.

Let G be a connected reductive group and let $T \subseteq G$ be a maximal torus. Since $X(T)$ is a free abelian group, we may use additive notation and write $\chi + \chi'$ rather than $\chi\chi'$. If $w \in W$ and $\chi, \chi' \in X(T)$ then $w(\chi + \chi') = w\chi + w\chi'$. Thus W acts faithfully as a group of $\mathbf{Z}$-linear mappings of $X(T)$. Identify $X(T)$ with a $\mathbf{Z}$-submodule of the real vector space $X = X(T) \otimes \mathbf{R}$ via $\chi = \chi \otimes 1$ [3, 14.8],[15, 26.1][26].The dimension $\dim_{\mathbf{R}} X$ of X as vector space over $\mathbf{R}$ is equal to the rank of $X(T)$ as free abelian group which is dim T. If $\{\chi_1, \ldots, \chi_r\}$ is a basis for $X(T)$ as free abelian group, then $\{\chi_1, \ldots, \chi_r\}$ is also a basis for X as vector space over $\mathbf{R}$. The Weyl group W may now be viewed as a group of $\mathbf{R}$-linear transformations of X, as well as a group of $\mathbf{Z}$-linear transformations of $X(T)$.

The roots Φ need not span the real vector space X. To describe the difference between $\dim_{\mathbf{R}} X$ and $\dim_{\mathbf{R}} \sum_{\alpha\in\Phi} \mathbf{R}\alpha$ we introduce the notions of rank and semisimple rank. The integer $r = \dim T = \dim_{\mathbf{R}} X$ is called the *rank* [15, 21.3] of G. [27] To define the semisimple rank of G we introduce the notions of radical and semisimple group. Any connected algebraic group G has a unique maximal connected solvable normal subgroup $\mathcal{R}(G)$ called the *radical* [3, 11.21], [15, 19.5] of G. The radical is automatically a closed subgroup of G. The group G is by definition *semisimple* [3, 11.21], [15, 19.5] if $G \neq \{1\}$ and $\mathcal{R}(G) = \{1\}$. If G is semisimple then the connected component $Z(G)^\circ$ of the center is $\{1\}$ so $Z(G)$ is finite. For example $G = \mathbf{SL}_n$ and $G = \mathbf{Sp}_n$ are semisimple; in either case a normal subgroup different from G is a subgroup of $Z(G)$. If G is semisimple then G is reductive. In the opposite direction, if G is reductive then $\mathcal{R}(G) = Z(G)^\circ$ is a torus [3, Proposition 11.21], [15, Lemma 19.5]. If G is reductive then the commutator subgroup (G, G) is semisimple [3, 14.2]. Furthermore $G = (G, G)\, Z(G)^\circ$ and $(G, G) \cap Z(G)$ is finite [3, Proposition 14.2], [15, 27.5]. For example, if $G = \mathbf{GL}_n$ then $\mathcal{R}(G) = Z(G) = \{tI \mid t \in K^*\}$ and $(G, G) = \mathbf{SL}_n$ so that $(G, G) \cap Z(G) = \{tI \mid t^n = 1\}$. If G is a connected reductive group and T is a maximal torus of G, the *semisimple rank* [3, 13.13], [15, 25.3] of G is by definition $l = \dim T - \dim\, (T \cap \mathcal{R}(G))$. Then $l = \dim T - \dim Z(G) = \dim_{\mathbf{R}} \sum_{\alpha\in\Phi} \mathbf{R}\alpha$ by [3, Theorem 14.8]. Thus the semisimple rank of G is equal to the dimension of the real vector space spanned by the roots. Let's translate the statements in Examples 4.7-4.9 into the additive language.

Example 4.10 Let $G = \mathbf{GL}_n$. Here $X(T) = \mathbf{Z}\varepsilon_1 \oplus \cdots \oplus \mathbf{Z}\varepsilon_n$ and $X = \mathbf{R}\varepsilon_1 \oplus \cdots \oplus \mathbf{R}\varepsilon_n$. Then $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$. If $w \in W$ corresponds to the permutation $\pi \in S_n$ then $w\varepsilon_i = \varepsilon_{\pi i}$ and thus $w(\varepsilon_i - \varepsilon_j) = \varepsilon_{\pi i} - \varepsilon_{\pi j}$. The rank of G is $r = n$; the semisimple rank is $l = n - 1$.

Example 4.11 Let $G = \mathbf{SL}_n$. Here $X(T) = \mathbf{Z}\varepsilon_1 + \cdots + \mathbf{Z}\varepsilon_n$ and $X = \mathbf{R}\varepsilon_1 + \cdots + \mathbf{R}\varepsilon_n$. The sums are not direct since $\varepsilon_1 + \cdots + \varepsilon_n = 0$. We have $X(T) = \mathbf{Z}\varepsilon_1 \oplus \cdots \oplus \mathbf{Z}\varepsilon_{n-1}$ and $X = \mathbf{R}\varepsilon_1 \oplus \cdots \oplus \mathbf{R}\varepsilon_{n-1}$. The bottom line for $\mathbf{SL}_n$ is the same as for $\mathbf{GL}_n$. We have $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$ and $w\varepsilon_i = \varepsilon_{\pi i}$ so $w(\varepsilon_i - \varepsilon_j) = \varepsilon_{\pi i} - \varepsilon_{\pi j} \in \Phi$. The rank of G and the semisimple rank of G are given by $r = l = n - 1$.

Example 4.12 Let $G = \mathbf{Sp}_n$. Here $X(T) = \mathbf{Z}\varepsilon_1 \oplus \cdots \oplus \mathbf{Z}\varepsilon_l$ and $X = \mathbf{R}\varepsilon_1 \oplus \cdots \oplus \mathbf{R}\varepsilon_l$ where $n = 2l$. From (30) we get

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le l\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \le i < j \le l\} \cup \{\pm 2\varepsilon_i \mid 1 \le i \le l\}.$$

[26] In the rest of this section it would be possible, in most statements, to replace the real field $\mathbf{R}$ by the rational field $\mathbf{Q}$ and work in $X(T) \otimes \mathbf{Q}$ as in [3, 15].

[27] This differs from the definition of rank given in [3, 12.2] but the two definitions are equivalent when G is reductive, by [3, Corollary 13.17.2(c)].

In the notation of Example 4.9 the action of W is given as follows. If $w \in W_1$ corresponds to $\pi = \prod_{i \in I}(i\bar{i}) \in S_n$ then $w\varepsilon_i = -\varepsilon_i$ when $i \in I$ and $w\varepsilon_i = \varepsilon_i$ when $i \notin I$. If $w \in W_2$ corresponds to $\pi \in S_n$ then $w\varepsilon_i = \varepsilon_{\pi i}$ for $1 \leq i \leq l$. Thus W, in its action on X, permutes the set $\{\pm\varepsilon_1, \ldots, \pm\varepsilon_l\}$. If we use the basis $\varepsilon_1, \ldots, \varepsilon_l$ to identify X with Euclidean space $\mathbf{R}^l$ then the points $\pm\varepsilon_i$ are the vertices of the cross polytope, and this representation of W by linear transformations is the one mentioned in Example 4.6. The rank of G and the semisimple rank of G are given by $r = l = \frac{1}{2}n$.

Let X be a real vector space of finite dimension. An element $s \in \mathbf{GL}(X)$ is a *reflection* [3, 14.7], [15, p.229] if $s \neq 1$ and s fixes a subspace of codimension 1. Then s has just one eigenvalue different from 1 which, since the characteristic polynomial of s has real coefficients, must be -1. Thus $s^2 = 1$ and there is a nonzero vector α, determined uniquely up to a scalar multiple, such that $s\alpha = -\alpha$.

Theorem 4.1 *Let G be a connected reductive group and let T be a maximal torus. Let $\Phi = \Phi(G,T)$ be the set of roots and let $W = W(G,T) \hookrightarrow \mathbf{GL}(X)$ be the Weyl group. If $\alpha \in \Phi$ then $-\alpha \in \Phi$ and there exists a reflection $s_\alpha \in W$ such that $s_\alpha\alpha = -\alpha$. The group*

$$W = < s_\alpha \mid \alpha \in \Delta > \tag{37}$$

is generated by the reflections s_α corresponding to the simple roots $\alpha \in \Delta$. Furthermore, if $\alpha, \beta \in \Phi$ then there exists $n_{\beta,\alpha} \in \mathbf{Z}$ such that

$$s_\alpha\beta = \beta - n_{\beta,\alpha}\alpha \,. \tag{38}$$

This theorem is the most amazing and important fact in the structure theory of reductive groups. It makes the subject. The proof is long and difficult. [28] Note that $n_{\alpha,\alpha} = 2$ since $s_\alpha\alpha = -\alpha$.

For the moment, let X be any real vector space of finite dimension and let $W \subset \mathbf{GL}(X)$ be a finite group of linear transformations. There exists a positive definite inner product (symmetric bilinear form) on X which is W-invariant: [29] $(wx, wx') = (x, x')$ for $w \in W$ and $x, x' \in X$. Let $\mathbf{O}(X) \subset \mathbf{GL}(X)$ be the orthogonal group, the group of all linear transformations $g \in \mathbf{GL}(X)$ such that $(gx, gx') = (x, x')$ for all $x, x' \in X$. Then $W \subset \mathbf{O}(X)$. If $s \in \mathbf{O}(X)$ is a reflection and $\alpha \in V$ is a nonzero vector with $s\alpha = -\alpha$ then

$$sx = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\,\alpha \,. \tag{39}$$

[28] The main effort is to show that Φ is a *root system* in the real vector space $\sum_{\alpha\in\Phi}\mathbf{R}\alpha$. See [3, 14.7], [9, 64.4], [15, 27.1], for the definition of root system, and [3, Theorem 14.8] for the last lines in the proof. There are important preliminary steps in the proof, which concern reductive groups of semisimple rank 1; these are used to prove the existence of the groups U_α. See [3, Theorem 13.18(4)], [15, 27.1] for the existence of a reflection s_α with $s_\alpha\alpha = -\alpha$. See [3, Corollary 14.6], [15, 27.2] for the proof of (38). Once one knows that Φ is indeed a root system, the assertion (37) follows from properties of "abstract root systems" in a real vector space [3, 14.7], [15, p.229].

[29] If $\langle x, x' \rangle$ is any positive definite inner product on X then $(x, x') = \sum_{w\in W}\langle wx, wx' \rangle$ is a W-invariant positive definite inner product.

Return to the case of a reductive group where W is the Weyl group. It is convenient in the examples, but not necesary, to choose a W-invariant inner product on $X = X(T) \otimes \mathbf{R}$. This inner product is not in general unique. It is unique up to scalar multiple in case W is the Weyl group of $\mathbf{SL}_n$, or $\mathbf{Sp}_n$ because W acts irreducibly on X in these cases.

Example 4.13 Suppose that $G = \mathbf{GL}_n$, $\mathbf{SL}_n$, or $\mathbf{Sp}_n$ and that the ε_i are as in Examples 4.10-4.12. If $G = \mathbf{GL}_n$ or $\mathbf{Sp}_n$ then the ε_i are an $\mathbf{R}$-basis for X and we may define an inner product on X by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, Kronecker delta. If $G = \mathbf{SL}_n$ we must be careful because $\varepsilon_1 + \cdots + \varepsilon_n = 0$. In this case there is a well defined inner product on X such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij} - \frac{1}{n}$. In all three cases, it is clear from the formulas for $w \in W$ in Examples 4.4-4.6 that the inner product is W-invariant.

If we compare (38) with (39) we see that

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = n_{\beta,\alpha} \in \mathbf{Z} \; . \tag{40}$$

Example 4.14 Let $G = \mathbf{GL}_n$. Use the notation of Example 4.7 and the inner product in Example 4.13. If $\alpha = \varepsilon_i - \varepsilon_j$, it follows from (39) that $s_\alpha \varepsilon_i = \varepsilon_j$, $s_\alpha \varepsilon_j = \varepsilon_i$, and $s_\alpha \varepsilon_k = \varepsilon_k$ for $k \neq i, j$. Thus the matrix for $s_\alpha \in \mathbf{O}(X)$ with respect to the basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the permutation matrix corresponding to the transposition (ij). The assertion (37) amounts in this case to the fact that the transpositions (ij) generate the symmetric group. If $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_p - \varepsilon_q$ then $(\alpha, \alpha) = 2$ and the $n_{\beta,\alpha}$ in (40) are given by

$$n_{\beta,\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \delta_{ip} - \delta_{iq} - \delta_{jp} + \delta_{jq} \in \mathbf{Z} \; . \tag{41}$$

Example 4.15 Let $G = \mathbf{SL}_n$. Use the notation of Example 4.8 and the inner product in Example 4.13. Although the inner product here is not the same as in $\mathbf{GL}_n$ the bottom line is the same. If $\alpha = \varepsilon_i - \varepsilon_j$ then again $s_\alpha \varepsilon_i = \varepsilon_j$, $s_\alpha \varepsilon_j = \varepsilon_i$, and $s_\alpha \varepsilon_k = \varepsilon_k$ for $k \neq i, j$. Since $\dim_{\mathbf{R}} X = n - 1$, the matrix $s_\alpha \in \mathbf{O}(X)$ with respect to the basis $\{\varepsilon_1, \ldots, \varepsilon_{n-1}\}$ is not a permutation matrix. Nevertheless, the permutation in S_n which corresponds to s_α is again (ij). The integers $n_{\beta,\alpha}$ are exactly as in the case $\mathbf{GL}_n$.

Example 4.16 Let $G = \mathbf{Sp}_n$ where $n = 2l$. Use the notation of Example 4.9 and the inner product in Example 4.13. We compute the action of the $s_\alpha \in \mathbf{O}(X)$ on the basis $\{\varepsilon_1, \ldots, \varepsilon_l\}$ as well as the corresponding permutations $\pi \in C \subseteq S_n$. If $\alpha = \varepsilon_i - \varepsilon_j$ then (39) gives $s_\alpha \varepsilon_i = \varepsilon_j$, $s_\alpha \varepsilon_j = \varepsilon_i$, and $s_\alpha \varepsilon_k = \varepsilon_k$ for $k \neq i, j$. The matrix for s_α is the permutation matrix corresponding to the transposition (ij). Remember here that $1 \leq i \neq j \leq l$ so that $s_\alpha \in P_l$. The corresponding permutation in C is $\pi = (ij)(\bar{i}\bar{j})$. If $\alpha = \varepsilon_i + \varepsilon_j$ then (39) gives $s_\alpha \varepsilon_i = -\varepsilon_j$, $s_\alpha \varepsilon_j = -\varepsilon_i$, and $s_\alpha \varepsilon_k = \varepsilon_k$ for $k \neq i, j$. The corresponding permutation in C is $\pi = (i\bar{j})(\bar{i}j)$. If $\alpha = 2\varepsilon_i$ then $s_\alpha \varepsilon_i = -\varepsilon_i$ and $s_\alpha \varepsilon_k = \varepsilon_k$ for $k \neq i$. The corresponding permutation in

C is $\pi = (i\bar{i})$. The permutation matrices of size l together with the diagonal matrices $\mathrm{diag}(1, \dots, -1, \dots, 1)$ generate W when viewed as subgroup of $\mathbf{GL}(X)$ relative to the basis $\varepsilon_1, \dots, \varepsilon_l$. This is the assertion (37) in our present example. We omit the list of integers $2(\beta, \alpha)/(\alpha, \alpha)$, but some of them are given later in (4.24).

Let X be a real vector space of finite dimension, with positive definite inner product. Let $W \subset \mathbf{O}(X)$ be a finite group generated by reflections. Say that W is *crystallographic* if it satisfies the condition (40), called the *crystallographic restriction* [16, 2.8-2.9]. There are finite groups generated by reflections which are not crystallographic. For example if $\mathcal{P}$ is a regular n-gon in the Euclidean plane $\mathbf{R}^2$ then the symmetry group $W(\mathcal{P})$ is dihedral and hence generated by reflections, but $W(\mathcal{P})$ is crystallographic only for $n \in \{3, 4, 6\}$. If $\mathcal{P}$ is one of the regular polytopes in $\mathbf{R}^3$ then $W(\mathcal{P})$ is generated by reflections, but $W(\mathcal{P})$ is not crystallographic if $\mathcal{P}$ is the icosahedron or dodecahedron. In fact, a finite group generated by reflections is crystallographic if and only if it is the Weyl group of some reductive group. [30]

Now return to the context of roots in a reductive group. Since W permutes Φ and $s_\alpha \alpha = -\alpha$, it follows that if $\alpha \in \Phi$ then $-\alpha \in \Phi$. We will decompose the set of roots as a disjoint union $\Phi = \Phi^+ \cup \Phi^-$ in such a way that (i) $\alpha \in \Phi^+ \Leftrightarrow -\alpha \in \Phi^-$ and (ii) if $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$. This may be done with an elementary geometric argument in the context of a real vector space X and finite subgroup of $\mathbf{GL}(X)$ generated by reflections [14, 4.1], [16, 1.3]. However, in the context of reductive groups, we must relate the decomposition $\Phi = \Phi^+ \cup \Phi^-$ to the structure of the group G. This depends on the notion of a Borel subgroup, which has been suppressed so far in this exposition, but is used in the proof of Theorem 4.1.

Let G be a connected reductive group. A *Borel subgroup* of G is a maximal element B in the set of closed connected solvable subgroups of G. Then B is maximal in the set of solvable subgroups of G [15, Corollary 23.1.A]. Borel subgroups exist and are uniquely determined up to conjugacy in G [3, Theorem 11.1], [15, Theorem 21.3]. [31] Let T be a maximal torus of G and let $\mathcal{B}^T$ denote the set of Borel subgroups of G which include T. Then $\mathcal{B}^T$ is finite, nonempty, and in fact [3, Proposition 11.19], [15, Proposition 24.1.A]

$$|\mathcal{B}^T| = |W(G,T)| \,. \tag{42}$$

If $B, B' \in \mathcal{B}^T$ then $B' = \omega B \omega^{-1}$ for some $\omega \in N$ and, for given B, the coset $w = \omega T$ is uniquely determined by B' [3, Proposition 11.19], [15, Proposition 24.1.A]. Since ωB and $B\omega$ depend only on w we may write $\omega B = wB$ and $B\omega = Bw$. Thus the Borel subgroups which include T have the form wBw^{-1} for some unique $w \in W$:

---

[30] It follows from (41) that the Weyl group of a reductive group is a crystallographic reflection group. For the converse, one must prove the existence of reductive groups. This can be done "case by case" using the classification of finite reflection groups. Such argument is of necessity long, since one must construct groups corresponding to the exceptional root systems E_6, E_7, E_8, F_4, G_2 [15, 33.6]. See [50, Chapter 12] for a case free argument.

[31] Borel subgroups may be defined in any connected algebraic group and are uniquely determined up to conjugacy [3, Theorem 11.1], [15, Theorem 21.3].

$$B^T = \{wBw^{-1} \mid w \in W\} . \tag{43}$$

Let $U = \mathcal{R}_u(B)$ be the unipotent radical of B. Since B is solvable, U is the set of unipotent elements of B, and B may be decomposed as a semidirect product [3, Theorem 10.6(4)], [15, Theorems 19.3(b) and 19.5]

$$B = TU = T\mathcal{R}_u(B) . \tag{44}$$

Example 4.17 Let $G = \mathbf{GL}_n$ and $T = \mathbf{T}_n$ be as in Example 4.1. Then $B = \mathbf{B}_n$ is a Borel subgroup which includes T. We have remarked in Example 3.4 that $\mathcal{R}_u(\mathbf{B}_n) = \mathbf{U}_n$ where $\mathbf{U}_n$ is as in (13). Clearly $\mathbf{B}_n = \mathbf{T}_n\mathbf{U}_n$.

Example 4.18 Let $G = \mathbf{SL}_n$. Choose $T = G \cap \mathbf{T}_n$ as in Example 4.2. Then $B = G \cap \mathbf{B}_n$ is a Borel subgroup of G which includes T, and $U = \mathcal{R}_u(B) = G \cap \mathbf{U}_n$.

Example 4.19 Let $G = \mathbf{Sp}_n$ where $n = 2l$. Choose $T = G \cap \mathbf{T}_n$ as in Example 4.3. Define subgroups B_1, B_2 of G by

$$B_1 = \{ \begin{bmatrix} I & a \\ 0 & I \end{bmatrix} \mid a \in \mathbf{M}_l \text{ is symmetric } \}, \qquad B_2 = \{ \begin{bmatrix} b & 0 \\ 0 & (b^{-1})^{\top} \end{bmatrix} \mid b \in \mathbf{B}_l \} .$$

Then $B = B_1B_2$ is a Borel subgroup of $\mathbf{Sp}_n$ which includes T, and $U = \mathcal{R}_u(B) = B \cap \mathbf{U}_n$.

Let G be a connected reductive group, let T be a maximal torus, and let $B \supset T$ be a Borel subgroup. There exists a unique Borel subgroup $B^- \supset T$ such that $B \cap B^- = T$ [3, Theorem 14.1], [15, Theorem 26.2.B]. The Borel subgroups B, B^- are said to be *opposite.* For example if $G = \mathbf{GL}_n$ and $B = \mathbf{B}_n$ is the group of upper triangular matrices then B^- is the group $\mathbf{B}_n^-$ of lower triangular matrices. Define

$$\Phi^+(B) = \{\alpha \in \Phi \mid U_\alpha \subset B\}, \qquad \Phi^-(B) = \{\alpha \in \Phi \mid U_\alpha \subset B^-\} . \tag{45}$$

The elements of $\Phi^+(B)$ are called *positive roots relative to* B; the elements of $\Phi^-(B)$ are called *negative roots relative to* B. In the rest of this paper we fix a Borel subgroup $B \supset T$ and write $\Phi^+ = \Phi^+(B)$ and $\Phi^- = \Phi^-(B)$. Elements of Φ^+ are called *positive roots*; elements of Φ^- are called *negative roots.* Then [3, Theorems 13.18(5)(a) and 14.1.III]

$$\Phi = \Phi^+ \cup \Phi^- \text{ and } \Phi^- = -\Phi^+ \tag{46}$$

where the union is disjoint. The group $X(T)$ may be given the structure of a totally ordered abelian group such that Φ^+ is the set of positive elements in Φ [3, Theorem 13.18(5)(d)], [15, 28.1]. Thus

$$\alpha, \beta \in \Phi^+ \text{ and } \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Phi^+ . \tag{47}$$

Furthermore

$$U = \mathcal{R}_u(B) = \prod_{\alpha \in \Phi^+} U_\alpha \,. \tag{48}$$

The groups U_α in the product may be taken in any order [3, Proposition 14.4], [15, Proposition 28.1].

Example 4.20 Let $G = \mathbf{GL}_n$. Let B be as in Example 4.17. If $\alpha = \varepsilon_i - \varepsilon_j$ then $U_\alpha \subset B$ precisely when $i < j$. Thus $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$ and $\Phi^- = \{\varepsilon_j - \varepsilon_i \mid 1 \le i < j \le n\}$. Assertion (46) is clear. As for (47), if $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$ and $\beta = \varepsilon_k - \varepsilon_l \in \Phi^+$ then $i < j$ and $k < l$ so $\alpha + \beta = \varepsilon_i - \varepsilon_j + \varepsilon_k - \varepsilon_l \in \Phi$ precisely when $j = k$, in which case $\alpha + \beta = \varepsilon_i - \varepsilon_l \in \Phi^+$ because $i < l$. Formula (48) asserts in this case that every upper unitriangular matrix may be written in the form $\prod_{1 \le i < j \le n}(I + c_{ij}E_{ij})$ for suitable $c_{ij} \in K$. The product may be taken in any order. Once the order is fixed, the coefficients c_{ij} are in fact uniquely determined. If $G = \mathbf{SL}_n$ and B is as in Example 4.18, the conclusions are exactly as in $\mathbf{GL}_n$.

Example 4.21 Let $G = \mathbf{Sp}_n$. Let B be as in Example 4.19. It follows from the shape of the matrices in (26)-(28) and the list Φ in Example 4.12 that

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le l\} \cup \{\varepsilon_i + \varepsilon_j \mid 1 \le i < j \le l\} \cup \{2\varepsilon_i \mid 1 \le i \le l\} \,.$$

Assertion (46) is clear. To verify (47), use $(\varepsilon_i - \varepsilon_j) + (\varepsilon_j - \varepsilon_k) = (\varepsilon_i - \varepsilon_k)$, $(\varepsilon_i - \varepsilon_j) + (\varepsilon_i + \varepsilon_j) = 2\varepsilon_i$ and $(\varepsilon_i - \varepsilon_j) + 2\varepsilon_j = \varepsilon_i + \varepsilon_j$. To verify (48) in this case, note that U has a semidirect product decomposition $U = U_1U_2$ analogous to the decomposition $B = B_1B_2$ in Example 4.19.

We say that a positive root is a *simple root* if it cannot be written as a sum of two positive roots [3, 14.7], [15, 27.3]. Let Δ denote the set of simple roots. Then Δ is a linearly independent subset of the real vector space X. Furthermore if $\alpha \in \Phi^+$ then α may be written uniquely as a $\mathbf{Z}$-linear combination of elements in Δ with non-negative coefficients [3, Corollary 14.8.1],

Example 4.22 Let $G = \mathbf{GL}_n$ with Φ^+ as in Example 4.20. Then

$$\Delta = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \ldots,\ \varepsilon_{n-1} - \varepsilon_n\} \,.$$

This set is clearly linearly independent. If $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$, so that $i < j$, then $\alpha = \sum_{k=i}^{j-1}(\varepsilon_k - \varepsilon_{k+1})$ is the desired linear combination. If $G = \mathbf{SL}_n$ the formulas are exactly as in $\mathbf{GL}_n$.

Example 4.23 Let $G = \mathbf{Sp}_n$ and let Φ^+ be as in Example 4.21. Then

$$\Delta = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \ldots,\ \varepsilon_{l-1} - \varepsilon_l,\ 2\varepsilon_l\} \,.$$

The desired linear combinations may be found using the formulas (i) $\varepsilon_i - \varepsilon_j = \sum_{k=1}^{j-1}(\varepsilon_k - \varepsilon_{k+1})$ for $1 \le i < j < l-1$, (ii) $\varepsilon_j + \varepsilon_l = (\varepsilon_j - \varepsilon_l) + 2\varepsilon_l$ for $1 \le j < l$, (iii) $2\varepsilon_j = (\varepsilon_j - \varepsilon_l) + (\varepsilon_j + \varepsilon_l)$ for $1 \le j < l$, (iv) $\varepsilon_i + \varepsilon_j = (\varepsilon_i - \varepsilon_j) + 2\varepsilon_j$ for $1 \le i < j < l$.

It is understood in what follows that $\Phi = \Phi(G,T)$ is the set of roots of a reductive group, that Δ is the set of simple roots with respect to a chosen Borel subgroup $B \supset T$ and that $W = N/T$ is the Weyl group. We introduce the Cartan matrix, Coxeter diagram and Dynkin diagram. Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ denote the set of simple roots. Thus $l = \dim \sum_{\alpha\in\Phi} \mathbf{R}\alpha$ is the semisimple rank of G. The *Cartan matrix* of Φ [5, 3.6], [15, p.230] is the $l \times l$ matrix with (i,j) entry

$$n_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = n_{\alpha_i,\alpha_j} \in \mathbf{Z} \ . \tag{49}$$

The n_{ij} are integers by (40). Clearly $n_{ii} = 2$. Clearly $n_{ij} = 0$ if and only if $n_{ji} = 0$. It happens that $n_{ij} \in \{0,-1,-2,-3\}$ and that $n_{ij}n_{ji} = \max\{|n_{ij}|, |n_{ji}|\}$. Thus if $n_{ij} \neq 0$ then at least one of n_{ij}, n_{ji} is equal to -1. In view of (38), the Cartan matrix determines the action of W on T and hence on $X(T)$.

Example 4.24 If $G = \mathbf{SL}_n$ with $l = n-1$ and Δ is as in (4.22), or $G = \mathbf{Sp}_n$ with $n = 2l$ and Δ is as in (4.23), then the Cartan matrices are given by

Cartan matrix

$\mathbf{SL}_n$

$$\begin{bmatrix} 2 & -1 & 0 \cdots & 0 & 0 \\ -1 & 2 & -1 \cdots & 0 & 0 \\ 0 & -1 & 2 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 2 & -1 \\ 0 & 0 & 0 \cdots & -1 & 2 \end{bmatrix}$$

$\mathbf{Sp}_n$

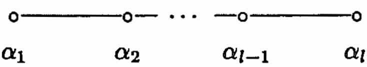
$$\begin{bmatrix} 2 & -1 & 0 \cdots & 0 & 0 \\ -1 & 2 & -1 \cdots & 0 & 0 \\ 0 & -1 & 2 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 2 & -1 \\ 0 & 0 & 0 \cdots & -2 & 2 \end{bmatrix}$$

The Dynkin diagram is a graph with vertex set $\Delta = \{\alpha_1, \dots, \alpha_l\}$, no loops, and directed possibly multiple edges, defined as follows. Suppose $i \neq j$. If $n_{ij} = 0$ then $n_{ji} = 0$. The multiplicity of the edge connecting α_i to α_j is $n_{ij}n_{ji}$; if this number is 0 it is understood that there is no edge. If $(\alpha_i,\alpha_i) > (\alpha_j,\alpha_j)$ then the edge is directed from α_i to α_j. If $(\alpha_i,\alpha_i) = (\alpha_j,\alpha_j)$ then the edge is undirected. These diagrams, with undirected edges and slightly different notation for the multiplicities (see below) were first introduced by Coxeter [8] in his enumeration of finite subgroups of $\mathbf{GL}_l(\mathbf{R})$ generated by reflections; then Dynkin introduced them in the classification of simple Lie algebras over $\mathbf{C}$ [11, p.1]. Their ancestry may be traced to Schläfli who used a similar notation in his enumeration of the regular polytopes in $\mathbf{R}^l$. [32]

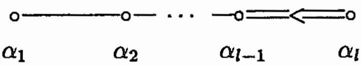
[32] The labeling of multiple edges in the Dynkin diagram exhibits the integers $n_{ij}n_{ji}$. The directed edges exhibit the n_{ij} as well. Tits [52, p.24, p.243] was the first to use the directed diagrams; "... à un détail près, celui utilisé par Dynkin ... ". Tits called them "Figures de Schläfli" following a terminology introduced by Borel and de Siebenthal. Plus ça change, plus c'est la même chose.

Example 4.25 The diagrams for $\mathbf{SL}_n$ and $\mathbf{Sp}_n$ are

$\mathbf{SL}_n$



$\mathbf{Sp}_n$



Now let's describe Coxeter's presentation of W as a *Coxeter group*, given in (53) below. Define integers m_{ij} for $1 \leq i, j \leq l$ as follows. Let $m_{ii} = 1$. For $i \neq j$ let m_{ij} be the order of $s_i s_j \in W$. Thus $m_{ij} = m_{ji}$. The $l \times l$ matrix with (i, j) entry equal to m_{ij} is called the *Coxeter matrix* [4, n° 1.9].

Example 4.26 The Coxeter matrices for the Weyl groups of $\mathbf{SL}_n$ and $\mathbf{Sp}_n$ are

Coxeter matrix

$\mathbf{SL}_n$

$$\begin{bmatrix} 1 & 3 & 2 \cdots & 2 & 2 \\ 3 & 1 & 3 \cdots & 2 & 2 \\ 2 & 3 & 1 \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 \cdots & 1 & 3 \\ 2 & 2 & 2 \cdots & 3 & 1 \end{bmatrix}$$

$\mathbf{Sp}_n$

$$\begin{bmatrix} 1 & 3 & 2 \cdots & 2 & 2 \\ 3 & 1 & 3 \cdots & 2 & 2 \\ 2 & 3 & 1 \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 \cdots & 1 & 4 \\ 2 & 2 & 2 \cdots & 4 & 1 \end{bmatrix}$$

The connection between the Coxeter matrix and the Cartan matrix is given by the formula

$$4\cos^2(\pi/m_{ij}) = n_{ij}n_{ji}\,. \tag{50}$$

The explicit values of the m_{ij} in terms of the $n_{ij}n_{ji}$ are given by the table

(51)

m_{ij}	2	3	4	6
$n_{ij}n_{ji}$	0	1	2	3

To get the Coxeter diagram from the Dynkin diagram proceed as follows: if $i \neq j$ and $n_{ij}n_{ji} > 1$, replace the multiple edge in the Dynkin diagram, and its arrow, by a single edge adorned with the integer m_{ij}. Thus the Coxeter diagram and Dynkin diagram for $\mathbf{SL}_n$ are identical and the Coxeter diagram for $\mathbf{Sp}_n$ is obtained from the Dynkin diagram by replacing its unique multiple edge by $\circ\!\overset{4}{\text{—}}\!\circ$.

Let $s_i = s_{\alpha_i} \in \mathbf{O}(X)$ be the reflection corresponding to $\alpha_i \in \Delta$. Let

$$S = \{s_1, \dots, s_l\} \,. \tag{52}$$

Then S generates W, as we have remarked in (37). The set S is called a set of *Coxeter generators* of W. It depends on Φ^+ and hence on choice of the Borel subgroup B. The s_i satisfy the relations

$$(s_i s_j)^{m_{ij}} = 1 \,, \; 1 \leq i, j \leq l \tag{53}$$

where $[m_{ij}]$ is the Coxeter matrix. It is a theorem of Coxeter that these relations (53) are a set of defining relations for W [8], [9, Theorem 64.26], [15, Theorem 29.4].

Example 4.27 Let W be the Weyl group of $\mathbf{SL}_n$. Then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$. It follows from the formula for s_α given in Example 4.15 that the permutation corresponding to the reflection s_i under the isomorphism $W \simeq S_n$ is the transposition $\tau_i = (i, i+1)$. These transpositions generate S_n. It was shown by E.H. Moore in 1897 that S_n has a presentation in terms of these transpositions with defining relations

$$\begin{array}{lll} \tau_i^2 & = 1 & 1 \leq i \leq n-1 \\ (\tau_i \tau_{i+1})^3 & = 1 & 1 \leq i \leq n-2 \\ (\tau_i \tau_j)^2 & = 1 & 1 \leq i < j-1 \leq n-1 \,. \end{array}$$

This is the presentation in (53) with $S = \{s_1, \dots, s_n\}$ and s_i in place of τ_i.

Let $w \in W$. Since S generates W we may write $w = s_{i_1} \cdots s_{i_p}$ where $s_{i_1}, \dots, s_{i_p} \in S$. Define the *length* of w to be the least such p:

$$l(w) = \min\{p \mid w = s_{i_1} \cdots s_{i_p}\}. \tag{54}$$

For example $l(1) = 0$ and $l(s_i) = 1$. This length function appears in various parts of the theory of reductive groups, from combinatorics to cohomology. In general, if a group is given by generators and relations, it is hard to determine the length of a group element as a word in the given set of generators. For the Weyl group we have the following combinatorial description of length due to Iwahori [17], [5, Theorem 2.2.2]. If $w \in W$ let

$$\Phi_w^- = \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\} \tag{55}$$

be the set of positive roots mapped to negative roots by w. Then [17, Lemma 2.2], [9, Theorem 64.16(iii)]

$$l(w) = |\Phi_w^-| \,. \tag{56}$$

Example 4.28 Let $G = \mathbf{GL}_n$. If $\alpha \in \Phi^+$ then $\alpha = \varepsilon_i - \varepsilon_j$ where $i < j$. If $w \in W$ then $w\alpha = \varepsilon_{\pi i} - \varepsilon_{\pi j}$ where $\pi \in S_n$ is the corresponding permutation. If $w\alpha \in \Phi^-$ then $\pi i > \pi j$. Thus $n(w)$ is the number of pairs (i, j) with $1 \leq i < j \leq n$ and $\pi i > \pi j$. This is the number of *inversions* in the permutation π. Thus $l(w)$ is the number of inversions in the permutation $\pi \in S_n$ which corresponds to w. In particular, the permutation $\pi_0 : 123 \cdots n \mapsto n \cdots 321$ has order 2 and is the unique permutation in S_n with the maximum number of inversions. The corresponding Weyl group element w_0 maps Φ^+ to Φ^- and is the unique element with this property. If $B = \mathbf{B}_n$ is our chosen Borel subgroup, then $w_0 B w_0$ is the group $\mathbf{B}_n^-$ of *lower* triangular matrices, the opposite Borel subgroup.

The remarks in Example 4.28 are a special case of a general theorem about finite reflection groups: If X is a real vector space and $W \subset \mathbf{GL}(X)$ is a finite group generated by reflections, then W contains a unique element w_0, called the *opposition element*, with the property $w_0\Phi^+ = \Phi^-$ [5, Proposition 2.2.6], [9, Theorem 64.16(vi)], [16, 1.8]. It follows from (56) that w_0 is the unique element in W of maximal length, and that if $w \in W$ maps no positive root into a negative root then $w = 1$. Since $\Phi^- = -\Phi^+$ we have $w_0^2\Phi^+ = \Phi^+$, so w_0^2 maps no positive root into a negative root, and thus $w_0^2 = 1$. Since $w_0\Phi^+ = \Phi^-$, it follows from (36), (45) and (48) that

$$w_0 B w_0 = B^- \tag{57}$$

is the Borel subgroup opposite to our chosen Borel subgroup B.

Let G be a connected reductive group. A subgroup of G is a *parabolic subgroup* [3, Corollary 11.2], [15, 29.3] if it includes some Borel subgroup. A parabolic subgroup is closed [3, 14.16], [15, 21.3] and connected [3, Theorem 11.16], [15, Corollary 23.1.B]. A parabolic subgroup is a *standard parabolic subgroup* [3, 14.17], [15, 30.1] if it includes our chosen Borel subgroup B.

Example 4.29 Let $G = \mathbf{GL}_n$. A *composition* or *ordered partition* γ of n, is a k-tuple $(\gamma_1, \dots, \gamma_k)$ of positive integers with $\gamma_1 + \cdots + \gamma_k = n$ where $1 \leq k \leq n$. The γ_i are called the *parts* of γ. There are 2^{n-1} compositions of n. Let P_γ be the group of all block diagonal matrices of the shape

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{bmatrix} \tag{58}$$

where A_{ij} is a matrix of size $\gamma_i \times \gamma_j$ and A_{ii} is invertible. Then $P_\gamma \supseteq B$ so P_γ is a standard parabolic subgroup of G. It is a consequence of Theorem 4.2 that each standard parabolic subgroup has the form P_γ for some composition γ of n. Thus the mapping $\gamma \to P_\gamma$ is bijective from the set of compositions of n to the set of standard parabolic subgroups of G. In particular, G has 2^{n-1} standard parabolic subgroups.

Let's examine, in the example $G = \mathbf{GL}_n$, the connection between the standard parabolic subgroups of G and certain subgroups of the Weyl group W. To avoid notational clutter, we agree to identify W with the symmetric group and with the group of permutation matrices. Thus $W = S_n \subset \mathbf{GL}_n = G$. With this identification let $W_\gamma = W \cap P_\gamma$. The elements of W_γ have the form $\mathrm{diag}(\pi_1, \ldots, \pi_k)$ where π_i is a permutation matrix of size γ_i so that $W_\gamma \simeq S_{\gamma_1} \times \cdots \times S_{\gamma_k}$. Thus the groups W_γ, when viewed as groups of permutations, are the *partition subgroups* or *Young subgroups* of S_n. It is an elementary but important fact that

$$P_\gamma = BW_\gamma B \tag{59}$$

where $B = \mathbf{B}_n$ is the group of upper triangular matrices. Let's see why this is so. Let $B^- = \mathbf{B}_n^-$ be the group of lower triangular matrices. Recall from linear algebra that if $a \in \mathbf{M}_n$ and the process of Gaussian elimination has no zero pivots, then a is the product of a lower triangular matrix and an upper triangular matrix. Thus if $g \in G$ has no zero pivots then $g \in B^-B$. A generalization of this fact, which takes zero pivots into account, is the formula $G = B^-WB$. This is sometimes called the *Birkhoff decomposition* of G. As in Example 4.28 let $\pi_0 \in W$ be the permutation with the maximum number of inversions. Since $\pi_0 G = G$, $\pi_0 W = W$ and $\pi_0 B \pi_0 = B^-$, it follows from the Birkhoff decomposition that $G = BWB$. This is called the *Bruhat decomposition* of G, and is the desired formula (59) when $\gamma = (n)$ has a single part. The general formula may be deduced from this special case by a matrix calculation. For example suppose $k = 2$ and $\gamma = (\gamma_1, \gamma_2)$. Given A_{ij} of size $\gamma_i \times \gamma_j$ with A_{ii} invertible, and $1 \le i \le j \le 2$ we may use the Bruhat decomposition for matrices of size γ_1, γ_2 to write $A_{ii} = B_{ii}\pi_i B_{ii}'$ where B_{ii}, B_{ii}' are invertible upper triangular matrices of size γ_i and π_i is a permutation matrix of size γ_i. Then the equation

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix} \begin{bmatrix} B_{11}' & 0 \\ 0 & B_{22}' \end{bmatrix} \tag{60}$$

may be solved for a matrix B_{12} of size $\gamma_1 \times \gamma_2$.

To get the correct general formulation of these facts about $G = \mathbf{GL}_n$, let S be the set of Coxeter generators for W as in Example 4.27. If $I \subseteq S$ let W_I be the subgroup of W generated by I. If γ is a composition of n there exists a unique subset $I \subseteq S$ such that $W_I = W_\gamma$. Thus $I \mapsto BW_IB$ is a bijection between subsets of S and standard parabolic subgroups of $G = \mathbf{GL}_n$.

In 1962, J.Tits [53] introduced the notion of a group G with BN-pair. His intent, inspired by arguments of Chevalley [7, Théorème 2], was to study certain properties of the parabolic subgroups of a reductive group from an axiomatic point of view. The axioms are few and the consequences are many. The axioms concern a group G, two subgroups B, N and a set S of generators for $N/B \cap N$. The given data G, B, N, S together with the prescribed axioms are now called a *Tits system* [3, 14.15], [9, 65.1], [15, 29.1]. The axioms are satisfied in case G is a reductive group, B is a Borel subgroup, N is the normalizer of a maximal torus $T \subset B$, and S is the set of Coxeter generators for the Weyl group $W = N/T$ [3, 14.15], [15, 29.1]. They have had profound impact on the theory of groups, in particular on the classification of finite simple groups, and on geometry. They have also influenced the theory of reductive monoids see for example [32], [43], [48], and Section 5 of this paper.

Here are the axioms for a Tits system $\mathcal{T} = (G, B, N, S)$. Let G be a group and let B, N be two subgroups which generate G. Let $T = B \cap N$. Assume that $T \lhd N$ and that $W := N/T$ is generated by a set $S = \{s_1, \dots, s_l\}$ of involutory elements, assumed distinct. If $\omega \in N$ represents $w \in W$ we write $\omega B = wB$ and $B\omega = Bw$ as in (57). The group W is called the *Weyl group* of $\mathcal{T}$. Assume that

$$\text{(61)} \qquad \begin{aligned} &\text{Axiom 1: If } s \in S \text{ and } w \in W \text{ then } sBw \subseteq BwB \cup BswB\,. \\ &\text{Axiom 2: If } s \in S \text{ then } sBs \neq B\,. \end{aligned}$$

Axiom 2 excludes, for example, the possibility that $G = S_n$, $B = \{1\}$, and $N = S_n$. It takes some work to check Axiom 1 even in the prototype $G = \mathbf{GL}_n$, $B = \mathbf{B}_n$, $N = \mathbf{N}_n$ [9, Theorem 65.10]. It is remarkable how much follows from these Axioms. For example:

Theorem 4.2 **([5, 8.3], [9, §65], [15, Theorem 29.3])** *Let $\mathcal{T} = (G, B, N, S)$ be a Tits system. If $I \subseteq S$ let W_I be the subgroup of W generated by I. Then*
(1) $P_I = BW_IB$ is a group.
(2) P_I is equal to its normalizer in G.
(3) If $I \neq J$ then P_I and P_J are not conjugate in G; in particular $P_I \neq P_J$.
(4) $P_I \cap P_J = P_{I\cap J}$.
(5) The map $I \to P_I$ is bijective from the set of subsets of S to the set of subgroups of G which include B.

Since the subgroups P_I are the standard parabolic subgroups of G, the subgroups W_I are often called *standard parabolic subgroups* of W. Thus we have a one-to-one correspondence between standard parabolic subgroups P_I of G and standard parabolic subgroups W_I of W. The index set I is called the *type* of the parabolic subgroup P_I. It follows from Theorem 4.2(1) with $I = S$ that

$$\text{(62)} \qquad G = BWB = \bigsqcup\nolimits_{w\in W} BwB \quad \text{and} \quad BwB = Bw'B \Rightarrow w = w'.$$

This double coset decomposition is called the *Bruhat decomposition* of G. As in (54) we define the length $l(w)$ for $w \in W$ to be the least integer p such that w may be written as a word of length p in the generating set S. It follows from the axioms that if $s \in S$ and $w \in W$ then $l(sw) = l(w) \pm 1$ and [15, Lemma 29.3.A]

$$\text{(63)} \qquad BsB \cdot BwB = \begin{cases} BswB & \text{if } l(sw) = l(w) + 1 \\ BswB \sqcup BwB & \text{if } l(sw) = l(w) - 1\,. \end{cases}$$

We emphasize that Theorem 4.2 and formulas (62) and (63) apply to any reductive group G where G, B, N, S have their usual meaning. In this case Theorem 4.2 shows that every standard parabolic subgroup of G, as defined prior to Example 4.29 has the form P_I for some subset I of the set S of Coxeter generators. In the context of reductive groups we often use the same index I for the corresponding subset $\{\alpha \in \Delta \mid s_\alpha \in I\}$ of the set of simple roots. If $G = \mathbf{GL}_n$ the correspondence $I \mapsto P_I$ is given explicitly in Example 4.29. See (122) for a monoid analogue of (63).

We have already remarked that a parabolic subgroup P of G is closed and connected. It is not in general reductive. It splits as a semidirect product

$$P = L\mathcal{R}_u(P) \tag{64}$$

where L is a closed connected reductive subgroup called a *Levi subgroup* or *Levi factor* of P [3, 11.22, Corollary 14.19], [15, Theorem 30.2]. The Levi factor is a connected reductive group [3, 11.22]. It is not in general unique, but any two Levi factors are conjugate by an element of $\mathcal{R}_u(P)$ [3, Proposition 14.21], [15, Theorem 30.2].

Example 4.30 Let $G = \mathbf{GL}_n$ and let $P = P_\gamma$ be as in Example 4.29. Then

$$\mathcal{R}_u(P) = \{ \begin{bmatrix} I & A_{12} & \dots & A_{1k} \\ 0 & I & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} \}, \qquad L = \{ \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{bmatrix} \}.$$

where A_{ij} has size $\gamma_i \times \gamma_j$. Although L is not uniquely determined, the L pictured here is the most natural choice. Note that $L \simeq \mathbf{GL}_{\gamma_1} \times \cdots \times \mathbf{GL}_{\gamma_k}$.

Two parabolic subgroups of a connected reductive group G are said to be *opposite parabolic subgroups* if their intersection is a Levi subgroup of both [3, 14.20]. This generalizes the notion of opposite Borel subgroups B, B^- in which case $B = T\mathcal{R}_u(B)$ as in (44), $B^- = T\mathcal{R}_u(B^-)$ and $B \cap B^- = T$ is the common Levi subgroup. If P is a parabolic subgroup with Levi factor L there is a unique opposite parabolic subgroup P^- with Levi factor L [3, Proposition 14.21]. If $P = P_\gamma$ is as in Example 4.29 then the group P_γ^- consisting of transposes of elements in P_γ is an opposite parabolic subgroup with common Levi subgroup pictured in (4.30).

Now return to Example 3.6 and re-examine it in current terms. We have a group $G = \mathbf{GL}_n$ and an idempotent $e = e_r$ in the Zariski closure $M = \overline{G} = \mathbf{M}_n$. Let γ be the composition $\gamma = (r, n-r)$ of n. Then the right centralizer $P(e) = P_\gamma$ of e in $\mathbf{GL}_n$ is a maximal parabolic subgroup of $\mathbf{GL}_n$, and the left centralizer $P^-(e) = P_\gamma^-$ of e in $\mathbf{GL}_n$ is also a maximal parabolic subgroup of $\mathbf{GL}_n$. These two parabolic subgroups are opposite. The common Levi subgroup is $L(e) = P(e) \cap P^-(e)$. It is this elementary observation which connects the idempotents in $\mathbf{M}_n$ to the group structure of $\mathbf{GL}_n$ and allows one to build a theory of reductive monoids on the theory of reductive groups.

5. Reductive Monoids

Let M be a reductive monoid with unit group G. Thus G is by definition a connected reductive group. In the theory of connected reductive groups, a main guiding principle is to reduce problems, as far as possible, to a study of the Weyl group W. In the theory of reductive monoids, the analogous guiding principle is to reduce to a study of W and its action on $E(\overline{T})$, where T is a maximal torus. Since T is conjugate to a subgroup of $\mathbf{T}_n$ [3, Theorem 8.5] we may assume that $T \subseteq \mathbf{T}_n$. Thus $\overline{T} \subseteq \mathbf{D}_n$, the monoid of diagonal matrices.

The group G acts on M by conjugation: $a \mapsto gag^{-1}$ for $a \in M$ and $g \in G$. The set $E(M)$ is stable under conjugation. The first main problem is to determine the G-orbits on $E(M)$. This is simplified by Theorem 5.2 which shows that every G-orbit on $E(M)$ meets $E(\overline{T})$. Theorems 5.2 and 5.3 allow us to reduce the study of $E(M)$ to that of W on $E(\overline{T})$. Theorem 5.1 shows that it suffices to prove Theorem 5.2 in case G is solvable. Note that Theorem 5.1 and Theorem 5.2 concern an irreducible algebraic monoid M which is not assumed to be reductive.

Theorem 5.1 **([23, Theorem 1.4], [31, Proposition 6.3])** *Let M be an irreducible algebraic monoid with unit group G. Let $B \subseteq G$ be a Borel subgroup of G. Then*

$$M = \bigcup_{g \in G} g\overline{B}g^{-1}.$$

Theorem 5.2 **([23, Corollary 1.6], [31, Corollary 6.10])** *Let M be an irreducible algebraic monoid with unit group G. Let $T \subseteq G$ be a maximal torus. Then*

$$E(M) = \bigcup_{g \in G} gE(\overline{T})g^{-1} .$$

Example 5.1 Suppose $M = \mathbf{M}_n$ so that $G = \mathbf{GL}_n$. Let $B = \mathbf{B}_n$ be the group of invertible upper triangular matrices, let $T = \mathbf{T}_n$ be the group of invertible diagonal matrices, and let $\mathbf{D}_n$ be the monoid of diagonal matrices. Theorem 5.1 asserts that every matrix is conjugate under $\mathbf{GL}_n$ to an element of $\mathbf{B}_n$. Theorem 5.2 asserts that every idempotent matrix is conjugate under $\mathbf{GL}_n$ to an element of $\mathbf{D}_n$.

Assume now that M is a reductive monoid with group of units G. As in Section 4, choose a maximal torus T and Borel subgroup B of G with $T \subset B$. Let $N = N_G(T)$ be the normalizer of T and let W be the Weyl group. If $\omega \in N$ and $e \in E(\overline{T})$, then $\omega e \omega^{-1} \in \overline{T}$. If $w = \omega T \in W$ then $\omega e \omega^{-1}$ depends only on w so we may define $wew^{-1} = \omega e \omega^{-1}$. Thus W acts, by conjugation, on $E(\overline{T})$.

Theorem 5.3 **([31, Theorem 6.25])** *Let M be a reductive monoid. If $e, f \in E(\overline{T})$ are conjugate under G they are conjugate under W.* [33]

Example 5.2 Suppose $M = \mathbf{M}_n$, $G = \mathbf{GL}_n$ and $T = \mathbf{T}_n$. Suppose $e, f \in E(\overline{T})$. Write $e = e_I$ and $f = e_J$ as in Example 2.1 where I, J are subsets of $\{1, \ldots, n\}$. If e, f are conjugate under G then $|I| = |J|$. Choose a permutation $\pi \in S_n$ such

[33] This theorem is true for any irreducible algebraic monoid. The Weyl group $W(G,T)$ is by definition $N_G(T)/Z_G(T)$; see the footnote which follows (32).

that $\pi I = J$. If $w \in W$ corresponds to π under the isomorphism $W \simeq S_n$ then $f = wew^{-1}$.

In view of Theorems 5.2, 5.3 and 3.1 we have bijections

$$G\backslash M/G = \mathcal{U}(M) \leftrightarrow E(M)/G \leftrightarrow E(\overline{T})/W \tag{65}$$

given by

$$GeG \leftrightarrow \{geg^{-1} \mid g \in G\} \leftrightarrow \{wew^{-1} \mid w \in W\} \tag{66}$$

for $e \in E(\overline{T})$ where $E(M)/G$ denotes the set of G-conjugacy classes in $E(M)$ and $E(\overline{T})/W$ denotes the set of W-conjugacy classes in $E(\overline{T})$. This gives us an explicit description of the $\mathcal{J}$-classes, alias $G \times G$ orbits, provided we can compute $E(\overline{T})$ and the Weyl group action.

In Example 2.3 we computed some idempotents in $E(\overline{T})$ for a monoid $M = M(\rho)$ defined by a representation ρ of $\mathbf{SL}_m$. At that time we did not have the terms "reductive group" and "maximal torus" at our disposal. Now we can put that computation in its proper context. It is a remarkable fact that if M is any reductive monoid with unit group G and T is a maximal torus of G then $E(\overline{T})$ is anti-isomorphic, by a W-equivariant map, to the lattice of faces of a rational convex cone. If $G = K^*\rho(G_0)$ where G_0 is a semisimple group and $\rho : G_0 \to \mathbf{GL}(V) = \mathbf{GL}_n$ is a rational representation then the cone may be replaced by a rational convex polytope.

To see how this comes about we need some general remarks about one-parameter subgroups of tori. Let G be a connected reductive group of rank r and let T be a maximal torus of G. A *one-parameter subgroup* of T is a homomorphism $\lambda : K^* \to T$ of algebraic groups [15, 16.1], [3, 8.6]. For example if $a_1, \dots, a_n \in \mathbf{Z}$ then the map $K^* \to \mathbf{T}_n$ defined in (5) by $t \mapsto \operatorname{diag}(t^{a_1}, \dots, t^{a_n})$ is a one-parameter subgroup of $\mathbf{T}_n$. Let $X_*(T)$ denote the set of one-parameter subgroups of T. Since T is abelian, $X_*(T)$ has the structure of abelian group with product defined by $(\lambda\lambda')(t) = \lambda(t)\lambda'(t)$ for $t \in K^*$. The group $X_*(T)$ is free abelian of rank $r = \dim T$. If T' is a subtorus of T there is a natural injection $X_*(T') \hookrightarrow X_*(T)$ which we view as inclusion. As with $X(T)$, we often use additive notation and write $\lambda + \lambda'$ rather than $\lambda\lambda'$. Let $X_* = X_*(T) \otimes \mathbf{R}$. We identify $X_*(T)$ with a $\mathbf{Z}$-submodule of the real vector space X_* via $\lambda = \lambda \otimes 1$. Then $\dim_{\mathbf{R}} X_* = \dim T$. If $\chi \in X(T)$ and $\lambda \in X_*(T)$ then $\chi \circ \lambda : K^* \to K^*$ is a rational homomorphism and thus has the form $t \mapsto t^a$ for some $a \in \mathbf{Z}$ and all $t \in K^*$. Write $\langle \chi, \lambda \rangle = a$. Thus we have a $\mathbf{Z}$-bilinear dual pairing [3, Proposition 8.6]

$$\langle\ ,\ \rangle : X(T) \times X_*(T) \to \mathbf{Z} \tag{67}$$

defined by

$$(\chi \circ \lambda)(t) = t^{\langle \chi, \lambda \rangle} \tag{68}$$

for $\chi \in X(T)$, $\lambda \in X_*(T)$ and $t \in K^*$ This pairing extends to a nondegenerate $\mathbf{R}$-bilinear pairing $X \times X_* \to \mathbf{R}$ of vector spaces where $X = X(T) \otimes \mathbf{R}$. Define a $\mathbf{Z}$-linear action of W on $X_*(T)$ as follows. If $w = \omega T$ with $\omega \in N(T)$ and $\lambda \in X_*(T)$ set

(69) $$(w\lambda)(t) = \omega\lambda(t)\omega^{-1}$$

for $t \in K^*$. Then $w\lambda \in X_*(T)$ and

(70) $$\langle w\chi, w\lambda\rangle = \langle \chi, \lambda\rangle$$

for $w \in W$, $\chi \in X(T)$ and $\lambda \in X_*(T)$. The action of W on $X_*(T)$ extends to an $\mathbf{R}$-linear action on X_* such that (70) holds for $\chi \in X$ and $\lambda \in X_*$. Let $\mathbf{Z}_*^n$ denote the set of *row* vectors over $\mathbf{Z}$.

Example 5.3 Suppose $G = \mathbf{GL}_n$ and $T = \mathbf{T}_n$. If $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}_*^n$ define $\lambda^{\mathbf{a}} \in X_*(\mathbf{T}_n)$ by

(71) $$\lambda^{\mathbf{a}}(t) = \operatorname{diag}(t^{a_1}, \ldots, t^{a_n})$$

for $t \in K^*$. The map $\mathbf{a} \mapsto \lambda^{\mathbf{a}}$ is an isomorphism $\mathbf{Z}_*^n \to X_*(\mathbf{T}_n)$ of abelian groups. For $1 \le i \le n$ define $\eta_i \in X_*(\mathbf{T}_n)$ by $\eta_i(t) = \operatorname{diag}(1, \ldots, t, \ldots, 1)$ with t in the i-th position. Then $\lambda^{\mathbf{a}} = \eta_1^{a_1} \cdots \eta_n^{a_n}$ and $\{\eta_1, \ldots, \eta_n\}$ is a basis for $X_* = X_*(\mathbf{T}_n) \otimes \mathbf{R}$. Let $\varepsilon_j \in X(\mathbf{T}_n)$ be as in (9). Since $\varepsilon_j(\eta_i(t)) = t^{\delta_{ij}}$ we have $\langle \varepsilon_j, \eta_i\rangle = \delta_{ij}$. Thus $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\eta_1, \ldots, \eta_n\}$ are dual bases. The Weyl group actions on $X(T)$ and $X_*(T)$ are given by $w\varepsilon_j = \varepsilon_{\pi j}$ and $w\eta_i = \eta_{\pi i}$ where $\pi \in S_n$ corresponds to $w \in W$.

In general we have a commutative diagram

(72) $$\begin{array}{ccc} T & \to & \mathbf{T}_n \\ \downarrow & & \downarrow \\ \overline{T} & \to & \mathbf{D}_n \end{array}$$

where all the maps are inclusions. A *rational character* of $\overline{T}$ is a morphism $\overline{T} \to K^\times$ of algebraic monoids where $K^\times$ denotes the field K viewed as monoid under multiplication. The set $X(\overline{T})$ of rational characters of $\overline{T}$ is a commutative monoid under multiplication. Restriction defines a monoid homomorphism $X(\overline{T}) \to X(T)$ which is injective [31, p.80]. If $\chi \in X(T)$ extends to a character of $\overline{T}$, let $\overline{\chi}$ denote the unique extension. We have an injection

(73) $$X(\mathbf{D}_n) = \mathbf{N}\overline{\varepsilon}_1 \oplus \cdots \oplus \mathbf{N}\overline{\varepsilon}_n \to \mathbf{Z}\varepsilon_1 \oplus \cdots \oplus \mathbf{Z}\varepsilon_n = X(\mathbf{T}_n)$$

where $\mathbf{N}$ is the set of non-negative integers and ε_j is the j-th coordinate function on $\mathbf{T}_n$. It is tempting to write ε_j rather than $\overline{\varepsilon}_j$ on the left side of (73) and view the map (73) as inclusion, but it is in fact restriction. Let χ_j denote the restriction of ε_j to T and let $\overline{\chi}_j$ denote the restriction of $\overline{\varepsilon}_j$ to $\overline{T}$. We have a commutative diagram

$$\begin{array}{lccccccc} (74) & X(\mathbf{D}_n) & \to & X(\overline{T}) & \qquad & \overline{\varepsilon}_j & \mapsto & \overline{\chi}_j \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & X(\mathbf{T}_n) & \to & X(T) & & \varepsilon_j & \mapsto & \chi_j \end{array}$$

where all the maps are restrictions. We have already remarked that the vertical maps are injective. The horizontal map in the bottom row is surjective by [3, Proposition 8.2(c)]. The horizontal map in the top row is also surjective; this is shown in the proof of Lemma 2.2 in [23]. The inclusion $T \subseteq \mathbf{T}_n$ thus provides us with a set $\{\overline{\chi}_1, \dots, \overline{\chi}_n\}$ of generators for the monoid $X(\overline{T})$ of characters of $\overline{T}$ and we have an injection

$$(75) \qquad X(\overline{T}) = \mathbf{N}\overline{\chi}_1 + \cdots + \mathbf{N}\overline{\chi}_n \to \mathbf{Z}\chi_1 + \cdots + \mathbf{Z}\chi_n = X(T)\ .$$

Since the coordinate ring of $\mathbf{D}_n$ is $K[\mathbf{D}_n] = K[\overline{\varepsilon}_1, \dots, \overline{\varepsilon}_n]$ we see that the coordinate ring of $\overline{T}$ is $K[\overline{T}] = K[\overline{\chi}_1, \dots, \overline{\chi}_n]$. Note that $\mathbf{N}\chi_1 + \cdots + \mathbf{N}\chi_n \subseteq X(T)$ is the set of characters of T which may be extended to characters of $\overline{T}$. Define a cone σ in X_* by

$$(76) \qquad \sigma = \{\lambda \in X_* \mid \langle \chi_j, \lambda \rangle \geq 0 \text{ for } 1 \leq j \leq n\}\ .$$

The cone σ is the key to the structure of $E(\overline{T})$. To see why this might be so, suppose that $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{Z}_*^n$ and let $\lambda^{\mathbf{a}}$ be as in (71). If $\lambda^{\mathbf{a}} \in \sigma$ then $\langle \chi_j, \lambda^{\mathbf{a}} \rangle \geq 0$ for all $j = 1, \dots, n$ so $\mathbf{a} \in \mathbf{N}_*^n$. Thus we may set $t = 0$ in $\mathrm{diag}(t^{a_1}, \dots, t^{a_n})$. This gives an element of $\overline{T} \subseteq \mathbf{D}_n$, which is suggestively written as $\lambda(0)$ or $\lim_{t\to 0} \lambda(t)$. Since the diagonal entries of $\lambda(0)$ are 0 or 1 we have

$$(77) \qquad \lambda(0) = \lim_{t \to 0} \lambda(t) \in E(\overline{T})\ .$$

This is the construction we used in Example 2.3.

To proceed further, we need some elementary facts about convex polyhedral cones. Fulton [12, Section 1.2] gives a summary of these facts, with proofs, which is made to order for the present setup. Let M, N be free (additively written) abelian groups of rank r. Assume there is given a $\mathbf{Z}$-bilinear dual pairing $M \times N \to \mathbf{Z}$. Imbed $M \subseteq M_{\mathbf{R}} = M \otimes \mathbf{R}$, imbed $N \subseteq N_{\mathbf{R}} = N \otimes \mathbf{R}$, and extend the pairing to a nondegenerate $\mathbf{R}$-bilinear pairing $M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$. Let $\mathbf{R}^+$ be the set of non-negative real numbers. A *convex polyhedral cone* in $N_{\mathbf{R}}$ is a set of the form

$$(78) \qquad \sigma = \mathbf{R}^+\nu_1 + \cdots + \mathbf{R}^+\nu_s$$

where $\nu_1, \dots, \nu_s \in N_{\mathbf{R}}$ are called *generators* of σ. We usually omit "convex polyhedral" and simply say that σ is a *cone*. Then

(79) $$\sigma^\vee = \{\chi \in M_{\mathbf{R}} \mid \langle \chi, \lambda \rangle \geq 0 \text{ for } \lambda \in \sigma\}$$

is a cone in $M_{\mathbf{R}}$ called the *dual* of σ [12, 1.2.9]. We have $(\sigma^\vee)^\vee = \sigma$ [12, 1.2.1]. Since we may interchange $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$ in (78) and (79), it follows that if σ is a cone in $N_{\mathbf{R}}$ then there exist $\chi_1, \ldots, \chi_n \in M_{\mathbf{R}}$ such that

(80) $$\sigma = \{\lambda \in N_{\mathbf{R}} \mid \langle \chi_j, \lambda \rangle \geq 0 \text{ for } 1 \leq j \leq n\} .$$

Let $\mathcal{F}(\sigma)$ denote the set of faces of σ partially ordered by inclusion. The poset $\mathcal{F}(\sigma)$ is finite [12, 1.2.2]. If $\tau \in \mathcal{F}(\sigma)$ then τ is itself a cone [12, 1.2.2]. If $\tau, \tau' \in \mathcal{F}(\sigma)$ then $\tau \cap \tau' \in \mathcal{F}(\sigma)$ [12, 1.2.3]. Thus $\mathcal{F}(\sigma)$ is a finite lattice. The minimal element of $\mathcal{F}(\sigma)$ is $\sigma \cap (-\sigma)$. A cone σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$ or, equivalently, if $\sigma^\vee$ spans $M_{\mathbf{R}}$ [12, 1.2.13]. If σ is strongly convex and different from $\{0\}$ it has a minimal set $\{\nu_1, \ldots, \nu_s\}$ of generators which are unique up to multiplication by positive real numbers; the rays $\mathbf{R}^+\nu_i$ are the one-dimensional faces of σ [12, 1.2.13].

A convex polyhedral cone σ in $N_{\mathbf{R}}$ is by definition *rational* if it has the form (80) where $\chi_1, \ldots, \chi_n \in M$ [12, p.12]. If σ is rational then $\sigma^\vee$ is rational [12, p.12]. If σ is rational then it may be written in the form (78) where $\nu_i \in N$ for $i = 1, \ldots, s$. Thus if σ is rational then $\sigma = \mathbf{R}^+(\sigma \cap N)$. A face of a rational cone is rational [12, p.13]. Let $\operatorname{relint}(\sigma)$ denote the relative interior of σ. If σ is rational then $\operatorname{relint}(\sigma) \cap N$ is non-empty. Thus, since a face of a rational cone is rational, $\operatorname{relint}(\tau) \cap N$ is non-empty for every $\tau \in \mathcal{F}(\sigma)$.

Henceforth we assume that $M = X(T)$ and $N = X_*(T)$ where T is a maximal torus in a connected reductive group G. Thus $M_{\mathbf{R}} = X$ and $N_{\mathbf{R}} = X_*$ in our earlier notation. Let σ be as in (76). Since $\sigma = (\sigma^\vee)^\vee$ it follows from (76) that

(81) $$\sigma^\vee = \mathbf{R}^+\chi_1 + \cdots + \mathbf{R}^+\chi_n .$$

In particular, we see that $\sigma^\vee$ spans X so σ is strongly convex and thus the minimal element of $\mathcal{F}(\sigma)$ is $\{0\}$. Suppose $\tau \in \mathcal{F}(\sigma)$. Since τ is a rational cone the set $\operatorname{relint}(\tau) \cap X_*(T)$ is non-empty. Suppose that $\lambda, \lambda' \in \operatorname{relint}(\tau) \cap X_*(T)$. If $1 \leq j \leq n$ then $\langle \chi_j, \lambda \rangle > 0$ if and only if $\langle \chi_j, \lambda' \rangle > 0$. Thus $\lambda(0) = \lambda'(0)$ depends only on τ. Let $e_\tau \in E(\overline{T})$ be the common value of the $\lambda(0)$ as λ ranges over $\operatorname{relint}(\tau) \cap X_*(T)$. Note that $T\overline{T} \subseteq \overline{T}$ so that T acts by multiplication on $\overline{T}$. The following theorem describes the orbits for this action, and the orbit closures, in terms of $\mathcal{F}(\sigma)$. [34]

[34] Theorem 5.4 is part of the theory of toric varieties [18, Theorem 2] and [12, Section 3.1]. If the toric variety is $\overline{T}$, the word "idempotent" is implicit, but not explicit in the general theory. In the early stages of his work on algebraic monoids Putcha [23], [31, Chapter 8] observed the connection between idempotents in $\overline{T}$ and rational convex polytopes; he proved what he needed from first principles. Renner [39, Chapter 4], [41, Section 3] remarked that the structure of $E(\overline{T})$ follows from known facts about toric varieties. A reader who checks these references should note that our $\sigma \subseteq X_*(T) \otimes \mathbf{R}$ and $\sigma^\vee \subseteq X(T) \otimes \mathbf{R}$ agree with [18] and [12] but do not agree with [41, p.201] where $\sigma \subseteq X(T) \otimes \mathbf{R}$ means the dual cone.

Theorem 5.4 ([18, Theorem 2(c,d)]) *Let $T \subseteq \mathbf{T}_n$ be a torus and let $\chi_1, \dots, \chi_n \in X(T)$ be the restrictions of the coordinate functions on $\mathbf{T}_n$ to T. Let*

$$\sigma = \{\lambda \in X_* \mid \langle \chi_j, \lambda \rangle \geq 0 \text{ for } 1 \leq j \leq n\} .$$

If $\tau \in \mathcal{F}(\sigma)$ let $e_\tau \in E(\overline{T})$ be the corresponding idempotent. Then (1) the map $\tau \mapsto Te_\tau$ is bijective from $\mathcal{F}(\sigma)$ to the set of T-orbits on $\overline{T}$, and (2) if $\tau, \tau' \in \mathcal{F}(\sigma)$ then $\tau \subseteq \tau'$ if and only if $\overline{Te_\tau} \supseteq Te_{\tau'}$.

It follows from (1) that each T-orbit on $\overline{T}$ contains an idempotent e. This idempotent is unique. For suppose $te \in E(\overline{T})$ for some $t \in T$. Since e is idempotent we have $\varepsilon_j(e) \in \{0,1\}$ for $1 \leq j \leq n$. In either case $\varepsilon_j(te) = \varepsilon_j(e)$ so $te = e$. Thus e is unique. Suppose that $\tau, \tau' \in \mathcal{F}(\sigma)$ and $\tau \subseteq \tau'$. The set $\{a \in \overline{T} \mid ae_\tau = a\}$ is closed and includes Te_τ because e_τ is idempotent, so it includes $\overline{Te_\tau}$ and hence contains $e_{\tau'}$ by (2). Thus $e_{\tau'}e_\tau = e_{\tau'}$ so that $e_\tau \geq e_{\tau'}$ in the partial order (3) on $E(\overline{T})$. Thus

Corollary 5.5 *The map $\tau \to e_\tau$ is a lattice anti-isomorphism from $\mathcal{F}(\sigma)$ to $E(\overline{T})$.*

We can give a more explicit formula for e_τ. Choose $\lambda_\tau \in X_*(T) \cap \operatorname{relint}(\tau)$. Since $\lambda_\tau(t) = \operatorname{diag}(t^{\langle \chi_1, \lambda_\tau \rangle}, \dots, t^{\langle \chi_n, \lambda_\tau \rangle})$,

$$e_\tau = \sum_{\langle \chi_j, \lambda_\tau \rangle = 0} E_{jj} . \tag{82}$$

The vertex $\{0\}$ of σ corresponds to the idempotent $1 \in E(\overline{T})$. The one-dimensional faces of σ correspond to the maximal idempotents of $E(\overline{T})$. Since every idempotent is a product of maximal idempotents, it suffices, in computing $E(\overline{T})$, to find the idempotents corresponding to the one-dimensional faces of σ.

The group W permutes $\mathcal{F}(\sigma)$ and $\mathcal{F}(\sigma^\vee)$. To see this, note that the inclusion map $T \to \mathbf{T}_n$ is a representation of T with weights $\chi_1, \dots, \chi_n$ where χ_j is the restriction to T of the j-th coordinate function ε_j on $\mathbf{T}_n$. It follows from (34) that W permutes $\{\chi_1, \dots, \chi_n\}$. Thus, from (81) we have $W\sigma^\vee = \sigma^\vee$. If $\lambda \in \sigma$ and $w \in W$ then, by (76) and (70) we have $\langle \chi_j, w\lambda \rangle = \langle w^{-1}\chi_j, \lambda \rangle \geq 0$ so $w\lambda \in \sigma$. Thus $W\sigma = \sigma$. It follows that

$$W\mathcal{F}(\sigma) = \mathcal{F}(\sigma) \text{ and } W\mathcal{F}(\sigma^\vee) = \mathcal{F}(\sigma^\vee) . \tag{83}$$

Suppose $\tau \in \mathcal{F}(\sigma)$ and $w \in W$. Write $w = \omega T$ for some $\omega \in N$ and write $e_\tau = \lambda(0)$ for some $\lambda \in \operatorname{relint}(\tau) \cap X_*(T)$. Then $\lambda' = \omega\lambda\omega^{-1} \in \operatorname{relint}(w\tau) \cap X_*(T)$. Since $\lambda'(0) = e_{w\tau}$

$$we_\tau w^{-1} = \omega e_\tau \omega^{-1} = e_{w\tau} . \tag{84}$$

Example 5.4 Suppose $G = \mathbf{GL}_n$ and $T = \mathbf{T}_n$ as in Example (5.3). Then $\chi_j = \varepsilon_j$. The faces of $\sigma = \mathbf{R}^+\eta_1 + \dots + \mathbf{R}^+\eta_n$ have the form $\tau_I = \sum_{i \in I} \mathbf{N}\eta_i$ where I runs over the subsets of $\{1, \dots, n\}$. Thus $\mathcal{F}(\sigma)$ is isomorphic to the lattice of subsets of

$\{1, \ldots, n\}$. Since elements of $X_*(\mathbf{T}_n)$ have the form $\eta_1^{a_1} \cdots \eta_n^{a_n}$ with $a_i \in \mathbf{N}$ we have $\sigma \cap X_*(T) = \mathbf{N}\eta_1 + \cdots + \mathbf{N}\eta_n$ in additive notation. If $\lambda \in \text{relint}(\tau_I)$ then, since $\text{relint}(\tau_I) \cap X_*(T) = \{\prod_{i \in I} \eta_i^{a_i} \mid a_i \in \mathbf{N} \text{ and } a_i > 0\}$, we have $\lambda(0) = \sum_{k \notin I} E_{kk}$. Maximal idempotents in $\overline{T}$ correspond to one-dimensional faces $\mathbf{R}^+\eta_1, \ldots, \mathbf{R}^+\eta_n$ of σ and are thus given by $\eta_i(0) = \sum_{k \neq i} E_{kk}$. The $\mathbf{T}_n$-orbit on $\mathbf{D}_n$ corresponding to the face τ_I is $\{\sum_{k \notin I} t_k E_{kk} \mid t_k \in K^*\}$. This orbit has closure $\{\sum_{k \notin I} t_k E_{kk} \mid t_k \in K\}$. The Weyl group action is given by permutation of coordinates.

Assume until further notice that $\dim Z(G) = 1$ and that $0 \in M$. These monoids are said to be *semisimple* [38, p.313]. Renner [41] has classified the semisimple monoids in the spirit of the classification of algebraic groups by root systems Φ [50, Chapters 11,12], under the additional hypothesis that M is a normal algebraic variety. The replacement for Φ is a combinatorial datum called a *polyhedral root system* [41, Definition 3.6] which is an abstraction for the triple $(X(T), \Phi, X(\overline{T}))$. Our assumptions insure that $0 \in E(\overline{T})$ and that the semisimple rank l of G is given by $l = r - 1$. [35] Let $Z = Z(G)^\circ$. Write $G' = (G, G)$. Since G is reductive, G' is semisimple and $G = G'Z$ where $G' \cap Z$ is finite. Let $T' = G' \cap T$. Since $Z \subseteq T$ we have $T = T'Z$ where $T' \cap Z$ is finite. Thus $\dim T' = \dim T - \dim Z = r - 1 = l$. Thus T' is a maximal torus of G' and, since $T' \cap Z$ is finite, $X_*(Z) \cap X_*(T') = \{0\}$, in additive notation. Let $N' = N_{G'}(T')$. The inclusion $N' \subset N$ induces an isomorphism $W(G', T') \simeq W(G, T)$.

Let $X'_* = X_*(T') \otimes \mathbf{R}$. Then $\dim_{\mathbf{R}} X_* = 1 + \dim_{\mathbf{R}} X'_*$. Since $X_*(T') \subset X_*(T)$ we have $X'_* \subseteq X_*$. Since $\dim Z(G) = 1$ we may choose a non-trivial element $\lambda_0 \in X_*(Z) \subseteq X_*(T)$. Then $\lambda_0 \notin X'_*$ so

$$X_* = \mathbf{R}\lambda_0 \oplus X'_* \,. \tag{85}$$

Since G' is semisimple we have $G' = (G', G')$ so that $G' \subseteq \mathbf{SL}_n$. Thus, if $\lambda' \in X_*(T')$ and $t \in K^*$ then $\prod_{j=1}^n \chi_j(\lambda'(t)) = \det \lambda'(t) = 1$. Thus $\sum_{j=1}^n \langle \chi_j, \lambda' \rangle = 0$ for $\lambda' \in X_*(T')$. By $\mathbf{R}$-linearity the same holds for $\lambda' \in X'_*$. Suppose $\lambda \in X_*$. Write $\lambda = r\lambda_0 + \lambda'$ with $r \in \mathbf{R}$ and $\lambda' \in X'_*$. The map $f : \lambda \mapsto r$ is an $\mathbf{R}$-linear form on X_* and we have $\sum_{j=1}^n \langle \chi_j, \lambda \rangle = f(\lambda) \sum_{j=1}^n \langle \chi_j, \lambda_0 \rangle$. Since $0 \in E(\overline{T})$ it follows from Theorem 5.4 that we may choose our $\lambda_0 \in X_*(Z)$ so that $\langle \chi_j, \lambda_0 \rangle > 0$ for $1 \leq j \leq n$ and hence $\sum_{j=1}^n \langle \chi_j, \lambda_0 \rangle > 0$. It follows from the definition of σ that if $\lambda \in \sigma$ then $f(\lambda) \geq 0$ with equality if and only if $\lambda = 0$. Thus, if we set $\sigma(1) = \{\lambda \in \sigma \mid f(\lambda) = 1\}$ then $\sigma = \mathbf{R}^+\sigma(1)$ is a cone over $\sigma(1)$. In Example 5.4 $\sigma(1)$ is a simplex with vertices at $\eta_1, \ldots, \eta_n$. In general, since $\sigma(1)$ is the convex hull of its intersections with the one-dimensional faces of σ it is a *rational polytope*: its vertices are in $X_*(T) \otimes \mathbf{Q}$. [36] If $\lambda' \in X'_*$ and $|\langle \chi_j, \lambda' \rangle| < 1$ for all j then $\langle \chi_j, \lambda_0 + \lambda' \rangle = 1 + \langle \chi_j, \lambda' \rangle > 0$ so that $\lambda_0 + \lambda' \in \sigma(1)$. Thus $\dim_{\mathbf{R}} \sigma(1) = l$ so that $\dim_{\mathbf{R}} \sigma = l + 1 = r$. Let $\mathcal{F}(\sigma(1))$ be the lattice of faces of $\sigma(1)$ including the empty face. It follows from Corollary 5.5 that there is a lattice anti-isomorphism $\mathcal{F}(\sigma(1)) \simeq E(\overline{T})$.

Now consider the action of the Weyl group. Suppose $w = \omega T \in W$ where $\omega \in N = N_G(T)$. Then ω normalizes T' because $G' \lhd G$. Suppose $\lambda \in X^*(T')$.

[35] In what follows we use the remarks about semisimple groups which precede Example 4.10.

[36] See [12, Section 1.5] for a discussion of rational polytopes in the present context.

Then $(w\lambda)(t) = \omega\lambda(t)\omega^{-1} \in T'$ for $t \in K^*$ so that $w\lambda \in X_*(T')$. Thus $WX_*(T') = X_*(T')$, so $WX'_* = X'_*$ and the second summand in (85) is stable under W. Since $\lambda_0(t) \in Z(G)$ we have $\omega\lambda_0(t)\omega^{-1} = \lambda_0(t)$ so that $w\lambda_0 = \lambda_0$ and the first summand in (85) is fixed by W. Thus $W\sigma(1) = \sigma(1)$. The lattice anti-isomorphism between $\mathcal{F}(\sigma(1))$ and $E(\overline{T})$ is thus W-equivariant.

The hypothesis that $0 \in M$ and dim $Z(G) = 1$ is satisfied in the following important special case. Let G_0 be a connected semisimple group of rank l and let $\rho : G_0 \to \mathbf{GL}_n$ be a rational representation with *finite kernel.* Let $G = K^*\rho(G_0)$ and let $M = M(\rho) = \overline{G}$. Since G_0 and K^* are connected, so is G. The group $\rho(G_0)$ is semisimple by [3, Proposition 14.10.1(c)] so G is reductive. The rank of $\rho(G_0)$ is l because ρ has finite kernel. Since $\rho(G_0)$ is semisimple its center $Z(\rho(G_0))$ is finite. Since $Z(G) = K^*Z(\rho(G_0))$ it follows that dim $Z(G) = 1$. In fact $Z = \{tI \mid t \in K^*\}$ so that we may choose $\lambda_0(t) = \text{diag}(t, \dots, t)$ for $t \in K^*$. Let T_0 be a maximal torus of G_0. If T', G' are as in the preceding paragraphs then $T' = \rho(T_0)$ and $G' = \rho(G_0)$. Let $N_0 = N_{G_0}(T_0)$. Then $\rho(N_0) \subset N$. The homomorphism $W(G_0, T_0) \to W(G, T)$ defined by $\omega_0 T_0 \mapsto \rho(\omega_0)T$ for $\omega_0 \in N_0$ is an isomorphism since ρ has finite kernel. In what follows we identify $W(G_0, T_0) = W(G, T)$ and let W denote either of these groups. With this identification we have $w(\rho \circ \theta) = \rho \circ (w\theta)$ for $w \in W$ and $\theta \in X_*(T_0)$. This is the general construction $(G_0, \rho) \rightsquigarrow M(\rho)$ promised in Section 2, where we chose $G_0 = \mathbf{SL}_n$ and considered $M(\rho)$ for various representations ρ.

To compute σ and $\sigma(1)$ in concrete cases choose $\lambda_0, \lambda_1, \dots, \lambda_l \in X_*(T)$ such that

$$T = \{\lambda_0(t_0)\lambda_1(t_1)\cdots\lambda_l(t_l) \mid t_0, t_1, \dots, t_l \in K^*\} \subseteq \mathbf{T}_n \tag{86}$$

where, as before, $\lambda_0(t) = \text{diag}(t, \dots, t)$. Such λ_i exist. Since dim $T = r = l + 1$ it follows that $\lambda_0, \lambda_1, \cdots, \lambda_l$ are linearly independent over $\mathbf{Z}$ and hence over $\mathbf{R}$ when viewed in X_*. If $G = K^*\rho(G_0)$ we choose $\lambda_1, \dots, \lambda_l$ as follows. Let $\theta_1, \dots, \theta_l$ be a $\mathbf{Z}$-basis for $X_*(T_0)$. [37] Then $T_0 = \{\theta_1(t_1)\cdots\theta_l(t_l) \mid t_1, \dots, t_l \in K^*\}$. Define $\lambda_i = \rho \circ \theta_i$. Then $\rho(T_0) = \{\lambda_1(t_1)\cdots\lambda_l(t_l) \mid t_1, \dots, t_l \in K^*\}$ so the conditions in (86) are satisfied. We do not assume uniqueness of expression in (86) since this may not hold if $G = K^*\rho(G_0)$. [38] In the rest of this section we distinguish row and column vectors over $\mathbf{R}$. Let $\mathbf{R}^r = \mathbf{R}^{1+l}$ denote the space of column vectors with standard basis written $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_l$. Let $\mathbf{R}^r_* = \mathbf{R}^{1+l}_*$ denote the space of row vectors with standard basis written $\mathbf{e}_{0*}, \mathbf{e}_{1*}, \dots, \mathbf{e}_{l*}$ where $\mathbf{e}_{i*} = \mathbf{e}_i^\top$. We will see that choice of λ_i as in (86) leads to vector space isomorphisms $\chi \mapsto \chi^*$ from $X \to \mathbf{R}^r$ and $\beta \mapsto \beta_*$ from $\mathbf{R}^r_* \to X_*$. The notation is chosen so that $\langle \chi, \beta_* \rangle = \langle \chi^*, \beta \rangle$ where the pairing on the left is given by (67) and the pairing on the right is the natural pairing of column and row vectors $\mathbf{R}^r \times \mathbf{R}^r_* \to \mathbf{R}$; see (100).

Since $T \subseteq \mathbf{T}_n$ there exist $\mathbf{a}_i = (a_{i1}, \dots, a_{in}) \in \mathbf{Z}^n_*$ for $0 \leq i \leq l$ such that

$$\lambda_i(t) = \lambda^{\mathbf{a}_i}(t) = \text{diag}(t^{a_{i1}}, \dots, t^{a_{in}}) . \tag{87}$$

[37] There is a natural choice for $\theta_1, \dots, \theta_l$ namely the *co-roots* $\alpha_1^\vee, \dots, \alpha_l^\vee$ as defined, for example, in [50, Lemma 9.1.5]. If $G_0 = \mathbf{SL}_m$ then $l = m - 1$ and $\alpha_i^\vee(t) = \text{diag}(1, \dots, t, t^{-1}, \dots, 1)$ where t, t^{-1} are in positions $i, i+1$. We do not define the co-roots here.

[38] This happens in the simplest example $G_0 = \mathbf{SL}_2$ with $\rho(g) = g$.

Note that $\mathbf{a}_0 = (1, \dots, 1)$. Let

(88)
$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{l1} & \dots & a_{ln} \end{bmatrix} \in \mathbf{M}_{l,n}(\mathbf{Z})$$

be the $l \times n$ integer matrix with rows $\mathbf{a}_1, \dots, \mathbf{a}_l$. Since $\lambda_0, \lambda_1, \dots, \lambda_l$ are linearly independent, so are $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_l$. Thus the $r \times n$ matrix A_0 with rows $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_l$ has rank r. In particular, rank $A = l$. The general element of T has the form

(89)
$$\lambda_0(t_0)\lambda_1(t_1)\cdots\lambda_l(t_l) = \operatorname{diag}(\ \dots,\ \textstyle\prod_{i=0}^{l} t_i^{a_{ij}}\ ,\ \dots) .$$

Thus $\chi_j(\lambda_i(t)) = t^{a_{ij}}$ for $t \in K^*$, so that $\langle \chi_j, \lambda_i \rangle = a_{ij}$. If $\beta = (b_0, b_1, \dots, b_l) \in \mathbf{Z}_*^r = \mathbf{Z}_*^{1+l}$ define $\beta_* \in X_*(T)$ by $\beta_* = \prod_{i=0}^{l} \lambda_i^{b_i}$. The map $\beta \mapsto \beta_*$ is a homomorphism $\mathbf{Z}_*^r \to X_*(T)$ of abelian groups. Extend this map to an $\mathbf{R}$-linear map $\beta \mapsto \beta_*$ from $\mathbf{R}_*^r \to X_*$. In additive notation

(90)
$$\beta_* = \sum_{i=0}^{l} b_i \lambda_i$$

for $\beta = (b_0, b_1, \dots, b_l) \in \mathbf{R}_*^r$. Define $\mathbf{R}$-linear forms $\xi_1, \dots, \xi_n$ on $\mathbf{R}_*^r$ by

(91)
$$\xi_j(\beta) = \langle \chi_j, \beta_* \rangle = \sum_{i=0}^{l} a_{ij} b_i .$$

Define a rational convex polyhedral cone $\gamma \subset \mathbf{R}_*^r$ by

(92)
$$\gamma = \{\beta \in \mathbf{R}_*^r \mid \xi_j(\beta) \geq 0 \text{ for } 1 \leq j \leq n\} .$$

Since $\beta \mapsto \beta_*$ is an isomorphism of vector spaces, it follows from (76) that it defines an isomorphism $\gamma \simeq \sigma$ of rational convex polyhedral cones. Let $\gamma(1) = \{(b_0, b_1, \dots, b_l) \in \gamma \mid b_0 = 1\}$ be the preimage of $\sigma(1)$ under the isomorphism. Then $\gamma(1) \subset \mathbf{R}_*^r$ is a rational convex polytope isomorphic to $\sigma(1)$.

Example 5.5 Let $G_0 = \mathbf{Sp}_n$. Then $T_0 = G_0 \cap \mathbf{T}_n$ is a maximal torus of G_0.[39] Let $G = K^* G_0 \subset \mathbf{GL}_n$. Then G is a connected reductive group and $T = K^* T_0$ is a maximal torus of G. The group G bears the same relation to $\mathbf{Sp}_n$ that $\mathbf{GL}_n$ does to $\mathbf{SL}_n$. Recall from (24) that elements of T_0 have the shape $\operatorname{diag}(t_1, \dots, t_l, t_1^{-1}, \dots, t_l^{-1})$ where $n = 2l$ and $t_1, \dots, t_l$ are arbitrary in K^*. The group G has rank $r = l + 1$ and

[39] The change in notation from G, T in Example 4.3 to G_0, T_0 here is forced by the following definitions of G, T.

semisimple rank l. For $1 \le i \le l$ define $\lambda_i \in X_*(T_0) \subset X_*(T)$ by $\lambda_i = \eta_i \eta_{l+i}^{-1}$ where η_i is as in Example 5.3. Then T is given by (86). In the notation of (71) and (87) we have $\mathbf{a}_i = \mathbf{e}_{i*} - \mathbf{e}_{l+i,*}$ for $1 \le i \le l$. The $l \times n$ matrix A is given by

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 \end{bmatrix}. \tag{93}$$

The linear forms $\xi_j : \mathbf{Z}_*^r \to \mathbf{Z}$ are given for $1 \le j \le n$ by

$$\begin{aligned} \xi_i(\beta) &= b_0 + b_i \quad \text{for } 1 \le i \le l \\ \xi_{l+i}(\beta) &= b_0 - b_i \quad \text{for } 1 \le i \le l\,. \end{aligned} \tag{94}$$

To describe the cone γ, note that $\gamma(1)$ is defined by the inequalities $|b_i| \le 1$ for $1 \le i \le l$ and is thus a cube with 2^l vertices at the points $\beta = (1, \pm 1, \ldots, \pm 1) \in \mathbf{Z}_*^r$. Since the maximal idempotents in $\overline{T}$ are in one-to-one correspondence with the 1-dimensional faces of σ it follows that $E(\overline{T})$ has 2^l maximal idempotents. If $\beta = (1, b_1, \ldots, b_l)$ is a vertex of $\gamma(1)$ then $\beta_*(t) = \operatorname{diag}(t^{1+b_1}, \cdots, t^{1+b_l}, t^{1-b_1}, \ldots, t^{1-b_l})$. The corresponding maximal idempotent is

$$\beta_*(0) = \sum_{j \in I(\beta)} E_{jj} \tag{95}$$

where $I(\beta) = \{1 \le j \le l \mid b_j = -1\} \cup \{l+1 \le j \le n \mid b_{j-l} = +1\}$. For example if $l = 3$ and $(b_1, b_2, b_3) = (-1, +1, +1)$ then $\beta_*(0) = \operatorname{diag}(1, t^2, t^2, t^2, 1, 1)$ so $I(\beta) = \{1, 5, 6\}$ and $\beta_*(0) = E_{11} + E_{55} + E_{66}$. The set $E(\overline{T})$ may be described as follows. Say that a subset $I \subseteq \{1, \ldots, n\}$ is *admissible* if $j \in I$ implies $\bar{j} \notin I$ where $j \mapsto \bar{j}$ is the involutory permutation of $\{1, \ldots, n\}$ introduced in Example 4.6. The sets $I(\beta)$ for β a vertex of $\gamma(1)$, are the maximal admissible subsets of $\{1, \ldots, n\}$. The map

$$I \mapsto e_I := \sum_{j \in I} E_{jj} \tag{96}$$

is bijective from admissible subsets of $\{1, \ldots, n\}$ to $E(\overline{T}) - \{1\}$. It remains to describe the Weyl group action on $E(\overline{T})$. Use the notation in Examples 4.6 and 4.9. If $w \in W_1$ and $\pi \in S_n$ corresponds to w, there exists a subset I of $\{1, \ldots, l\}$ such that $\pi = \prod_{i \in I} (i\bar{i})$. Then for $1 \le i \le l$ we have $w\lambda_i = \lambda_i^{-1}$ if $i \in I$ and $w\lambda_i = \lambda_i$ if $i \notin I$. If $w \in W_2$ and $\pi \in S_n$ corresponds to w then $\pi\{1, \ldots, l\} \subseteq \{1, \ldots, l\}$ and the action of w on $X_*(T)$ is given by $w\lambda_i = \lambda_{\pi i}$ for $1 \le i \le l$. Thus the action of W on $X_*(T)$ is like that on $X(T)$. Since W centralizes the involution $i \mapsto \bar{i}$ it maps admissible sets to admissible sets. The action of W on $E(\overline{T})$ is given by $w e_I w^{-1} = e_{wI}$. The group W acts transitively on faces of σ of any given dimension. Thus the W-orbits on $E(\overline{T})$, excluding the orbits of 0 and 1, are in one-to-one correspondence with the set $\{1,\dots,l\}$. The number of $\mathcal{J}$-classes in the reductive monoid $M(\rho)$, excluding $\{0\}$ and $\{1\}$, is thus l. The poset $\mathcal{J}$ is linearly ordered.

It is sometimes convenient to work with the dual cone $\sigma^\vee \subset X$. If $\tau \in \mathcal{F}(\sigma)$ define a subset τ^* of X by

$$\tau^* = \{\chi \in X \mid \langle \chi, \lambda\rangle \geq 0 \text{ for } \lambda \in \sigma \text{ and } \langle \chi, \lambda\rangle = 0 \text{ for } \lambda \in \tau\}\ . \tag{97}$$

The map $\tau \mapsto \tau^*$ is a lattice anti-isomorphism $\mathcal{F}(\sigma) \to \mathcal{F}(\sigma^\vee)$ [12, 1.2.10] which is W-equivariant by (70). It follows from Corollary 5.5 that the map $\tau^* \mapsto e_\tau$ is a lattice isomorphism $\mathcal{F}(\sigma^\vee) \to E(\overline{T})$. Define $\chi_0 \in X$ by $\chi_0 = \frac{1}{n}(\chi_1 + \cdots + \chi_n)$ where χ_j is the restriction of ε_j to T. Thus, in multiplicative notation, $\chi_0^n \in X(T)$ is the determinant. Define a subspace X' of X by $X' = \{\chi \in X \mid \langle \chi, \lambda_0\rangle = 0\}$. Since $\dim_{\mathbf{R}} X' = l$ and $\langle \chi_0, \lambda_0\rangle = 1$ we have

$$X = \mathbf{R}\chi_0 \oplus X'. \tag{98}$$

As in (85) the Weyl group fixes the first summand and stabilizes the second. If $1 \leq j \leq n$ then $\langle \chi_j - \chi_0, \lambda_0\rangle = 1 - 1 = 0$ so the decomposition of χ_j according to (98) is $\chi_j = \chi_0 + (\chi_j - \chi_0)$. Since $G' \subseteq \mathbf{SL}_n$ we have $\langle \chi_0, \lambda\rangle = 0$ for $\lambda \in X_*'$. Thus $X_*' = \{\lambda \in X_* \mid \langle \chi_0, \lambda\rangle = 0\}$. The restriction of the pairing $\langle\ ,\ \rangle$ to $X' \times X_*'$ is nondegenerate. If $\chi \in X$ write $\chi = r\chi_0 + \chi'$ with $r \in \mathbf{R}$ and $\chi' \in X'$. The map $f : \chi \mapsto r$ is an $\mathbf{R}$-linear form on X. The argument which follows (85) shows that if we set $\sigma^\vee(1) = \{\chi \in \sigma^\vee \mid f(\chi) = 1\}$, then $\sigma^\vee(1)$ is stable under W and $\sigma^\vee = \mathbf{R}^+\sigma^\vee(1)$ is a cone over $\sigma^\vee(1)$. Let $\chi \in X$. By (81) we may write $\chi = \sum_{j=1}^n r_j\chi_j$ where $r_j \in \mathbf{R}^+$. Then $\chi = (\sum_{j=1}^n r_j)\chi_0 + \sum_{j=1}^n r_j(\chi_j - \chi_0)$ so $f(\chi) = \sum_{j=1}^n r_j$. Thus $\sigma^\vee(1) = \{\sum_{j=1}^n r_j\chi_j \mid r_j \in \mathbf{R}^+ \text{ and } \sum_{j=1}^n r_j = 1\}$ is the convex hull of the χ_j. [40]

Define an $\mathbf{R}$-linear map $\chi \mapsto \chi^*$ from $X \to \mathbf{R}^r$ by

$$\chi^* = (\langle \chi, \lambda_0\rangle, \langle \chi, \lambda_1\rangle, \dots, \langle \chi, \lambda_l\rangle)^\top\ . \tag{99}$$

Then $\chi \mapsto \chi^*$ is an isomorphism of vector spaces. It follows from (90) and (99) that

$$\langle \chi, \beta_*\rangle = \sum_{i=0}^{l} b_i\langle \chi, \lambda_i\rangle = \langle \chi^*, \beta\rangle \tag{100}$$

for $\chi \in X$ and $\beta \in \mathbf{R}_*$, where the pairing on the right is the natural pairing of column and row vectors $\mathbf{R}^r \times \mathbf{R}_*^r \to \mathbf{R}$. We may use the isomorphisms (90) and (99) to transport the action of W on X, X_* to $\mathbf{R}^r, \mathbf{R}_*^r$. Thus we have

[40] The χ_j are the weights of the representation $T \hookrightarrow \mathbf{T}_n$. Since $T_0 \to \rho(T_0) \to T$ where the first map is ρ and the second map is inclusion, there is a homomorphism $\pi : X(T) \to X(T_0)$ of abelian groups. The functions $\pi \circ \chi_j$ are the weights of the representation ρ. Since the restriction of π to $X(T) \cap X'$ is injective we may say, with slight abuse of terminology that $\sigma^\vee(1)$ is the convex hull of the set $\Phi(\rho)$ of weights of ρ; compare [41, Section 3].

$$w(\chi^*) = (w\chi)^*, \qquad (w\beta)_* = w(\beta_*) \tag{101}$$

for $\chi \in X$ and $\beta \in \mathbf{R}^r_*$. It follows from these formulas and (70) that

$$\langle w(\chi^*), w\beta \rangle = \langle \chi^*, \beta \rangle \tag{102}$$

so that W respects the pairing on $\mathbf{R}^r \times \mathbf{R}^r_*$. To compute matrix representations using these actions note that if $\beta = \mathrm{e}_{i*}$ then $\beta_* = \lambda_i$. Thus the matrix, say $\theta(w)$, for the action of w on $\mathbf{R}^r_*$ is precisely the matrix for its action on X_* with respect to the basis $\lambda_0, \ldots, \lambda_l$. It follows from (102) that the matrix for the action of w on $\mathbf{R}^r$ is the contragredient $\theta(w^{-1})^\top$. The columns of A_0 are $\chi_1^*, \ldots, \chi_n^*$. Thus left multiplication of A_0 by $\theta(w^{-1})^\top$ permutes the columns of A_0 as w permutes the set $\Phi(\rho)$ of weights.

We have already remarked that the polytope $\sigma^\vee(1)$ is the convex hull of the set $\{\chi_1, \ldots, \chi_n\}$. Thus its image, say $\mathcal{P}_0 \subset \mathbf{R}^r$, under the map $\chi \mapsto \chi^*$ is the convex hull of the columns of A_0 and is thus isomorphic, by a translation, to the convex hull $\mathcal{P}$ in $\mathbf{R}^l$ of the columns of A. The Weyl group action on $\mathcal{P}_0$ and on $\mathcal{P}$ is given by $w \mapsto \theta(w^{-1})^\top$. The lattice anti-isomorphism $\mathcal{F}(\sigma) \to \mathcal{F}(\sigma^\vee)$ induces a lattice anti-isomorphism $\mathcal{F}(\sigma(1)) \to \mathcal{F}(\sigma^\vee(1))$ [12, p.24] and hence a lattice anti-isomorphism $\mathcal{F}(\sigma) \to \mathcal{F}(\mathcal{P})$. We conclude from Corollary 5.5 and formula (82) that the map

$$F \mapsto e_F := \sum_{\chi_j^* \in F} E_{jj}, \tag{103}$$

is a lattice isomorphism $\mathcal{F}(\mathcal{P}_0) \to E(\overline{T})$. Note that $\chi_j^* = (a_{1j}, \ldots, a_{lj})^\top$ is the j-th column of the matrix A. This is the form in which Putcha [23], [31, Chapter 8] proved Corollary 5.5.

The polytopes $\sigma(1)$ and $\sigma^\vee(1)$ are *dual polytopes* [12, p.24]. For example, if $G = K^*\mathbf{Sp}_n$ where $n = 2l$ and $\mathrm{e}_1, \ldots, \mathrm{e}_l$ is the standard basis for $\mathbf{R}^l$ then (93) shows that the set of columns of A is $\{\mathrm{e}_1, \ldots, \mathrm{e}_l, -\mathrm{e}_1, \ldots, -\mathrm{e}_l\}$. The polytope $\mathcal{P} \simeq \sigma^\vee(1)$ is thus the cross polytope, which is dual to the cube we found in Example 5.5.

Example 5.6 Let's reconsider Example 2.3 in case $m = 4$ and complete the calculation begun there. We will compute in detail, so that the reader may enjoy reproducing these calculations in other examples. Let $G_0 = \mathbf{SL}_4$ and let $T_0 = G_0 \cap \mathbf{T}_4$. Let $V = K^4 \otimes K^4$. Define $\rho : G_0 \to \mathbf{GL}(V)$ by $\rho(g)(v \otimes v') = gv \otimes (g^\top)^{-1}v'$. Define $\theta_1, \theta_2, \theta_3 \in X_*(T_0)$ by $\theta_1(t) = \mathrm{diag}(t, t^{-1}, 1, 1)$, $\theta_2(t) = \mathrm{diag}(1, t, t^{-1}, 1)$, $\theta_3(t) = \mathrm{diag}(1, 1, t, t^{-1})$. The general element of T_0 has the shape $t_0 = \theta_1(t_1)\theta_2(t_2)\theta_3(t_3)$. Let $\lambda_i = \rho \circ \theta_i \in X_*(\rho(T_0)) \subseteq X_*(\rho(T))$. The vectors $v_j \otimes v_k$ where $1 \le j, k \le 4$, are a basis of weight vectors. We view $\rho(T_0) \subseteq \mathbf{T}_4 \times \mathbf{T}_4$. Let $\varepsilon_{jk} \in X(\mathbf{T}_4 \times \mathbf{T}_4)$ denote the coordinate functions on $\mathbf{T}_4 \times \mathbf{T}_4$ and let χ_{jk} denote the restriction of ε_{jk} to $T = K^*\rho(T_0)$ or to $\rho(T_0)$. Since $t_0(v_j \otimes v_k) = (\chi_{jk} \circ \rho)(t_0)(v_j \otimes v_k)$ the functions $\chi_{jk} \circ \rho \in X(T_0)$ are the weights of ρ. For example since $\rho(t_0)(v_1 \otimes v_2) =$

$t_0 v_1 \otimes t_0^{-1} v_2 = t_0 v_1 \otimes (t_1 t_2^{-1})^{-1} v_2 = t_1^2 t_2^{-1}(v_1 \otimes v_2)$ we have $(\chi_{12} \circ \rho)(t_0) = t_1^2 t_2^{-1}$. Write $(\chi_{jk} \circ \rho)(t_0) = t_1^{a_1} t_2^{a_2} t_3^{a_3}$ for suitable integers $a_i = a_{ijk}$. The list of weights is:

j k	a_1	a_2	a_3	j k	a_1	a_2	a_3
1 1	0	0	0	3 1	−1	−1	1
1 2	2	−1	0	3 2	1	−2	1
1 3	1	1	−1	3 3	0	0	0
1 4	1	0	1	3 4	0	−1	2
2 1	−2	1	0	4 1	−1	0	−1
2 2	0	0	0	4 2	1	−1	−1
2 3	−1	2	−1	4 3	0	1	−2
2 4	−1	1	1	4 4	0	0	0

The matrix A of size 3×16 has the vectors $(a_1, a_2, a_3)^\top$ as its columns. The polytope $\mathcal{P}$ is the convex hull of these columns. The 12 nonzero columns are not the midpoints of the edges of a cube with vertices $\pm \mathrm{e}_1 \pm \mathrm{e}_2 \pm \mathrm{e}_3$ that we found in Example 2.3. We haven't lost it though. For $1 \le j \ne k \le 4$ let $\mathrm{e}_{jk} = \mathrm{e}_j - \mathrm{e}_k \in \mathbf{R}^4$. The e_{jk} lie in the hyperplane $H = \{\sum_{i=1}^4 x_i \mathrm{e}_i \mid \sum_{i=1}^4 x_i = 0\}$. Map $H \to \mathbf{R}^3$ by $(x_1, x_2, x_3, x_4)^\top \mapsto (x_1 - x_2, x_2 - x_3, x_3 - x_4)^\top$; this is not the map ϕ we used in Example 2.3. The e_{jk} map to the vectors $(a_1, a_2, a_3)^\top$ so $\mathcal{P}$ is the cuboctahedron after all.

We compute the W-conjugacy classes in $E(\overline{T})$. Since the correspondence $\mathcal{F}(\mathcal{P}) \leftrightarrow E(\overline{T})$ respects the W-action, we may compute the orbits for the W-action on vertices, edges, and faces of $\mathcal{P}$. The reader should look at the picture of the cuboctahedron given in Example 2.3. The symmetry group of the cube, and of the inscribed cuboctahedron, has order 48. The Weyl group W of $\mathbf{SL}_4$ has order 24, and is isomorphic as abstract group to the symmetric group on four letters. The group W has two inequivalent irreducible representations in $\mathbf{R}^3$, one as the group of rotations of the cube, the other as the group generated by reflections in the six symmetry planes of the cube which are different from the coordinate planes. Our representation of W in Example 2.3 is the one generated by reflections. Thus W is transitive on vertices of the cuboctahedron, has 2 orbits on edges, and has 3 orbits on faces of dimension 2, one orbit consisting of squares and two orbits consisting of triangles. We conclude that the number of W-congugacy classes of idempotents in $E(\overline{T})$ is, excluding the classes $\{0\}, \{1\}$, equal to $1+2+3$. In view of (65) this is the number of $G \times G$ orbits on M or the number of G orbits on $E(M)$ excluding $\{0\}, \{1\}$.

Example 5.7 Let $G_0 = \mathbf{SL}_4$ and let $T_0 = G_0 \cap \mathbf{T}_4$. Let $V_p = \bigwedge^p K^4$ be the p-th exterior power of K^4. Thus V_1 has basis $\{v_i \mid 1 \le i \le 4\}$, V_2 has basis $\{v_j \wedge v_k \mid 1 \le j < k \le 4\}$, and V_3 has basis $\{v_p \wedge v_q \wedge v_r \mid 1 \le p < q < r \le 4\}$. Let $V = V_1 \otimes V_2 \otimes V_3$. Then $\dim V = 96$. Define $\theta_1, \theta_2, \theta_3$ as in Example 5.6 and let $\lambda_i = \rho \circ \theta_i$. The matrix A has size 3×96. It is hard to get a picture of $\mathcal{P}$ from this matrix. However, there is a trick we can use which is implicit in the computation of Example 5.6. Write the elements of $\mathbf{SL}_4$ as $\operatorname{diag}(t_1, t_2, t_3, t_4)$ where $t_1t_2t_3t_4 = 1$. Ignore, for a moment, the restriction $t_1t_2t_3t_4 = 1$, compute a congruent polytope in $\mathbf{R}^4$ and return to the desired polytope in $\mathbf{R}^3$ via the mapping $(x_1, x_2, x_3, x_4) \mapsto (x_1 - x_2, x_2 - x_3, x_3 - x_4)$. The vector v_i has weight t_i, the vector $v_j \wedge v_k$ has weight t_jt_k and the vector $v_p \wedge v_q \wedge v_r$ has weight $t_pt_qt_r$. Thus $v_i \otimes (v_j \wedge v_k) \otimes (v_p \wedge v_q \wedge v_r)$ has weight $t_it_jt_kt_pt_qt_r$. This gives us a matrix of size 4×96. If we write $t_it_jt_kt_pt_qt_r = t_1^at_2^bt_3^ct_4^d$ we find that the only column vectors $(a, b, c, d)^\top$ which occur in this matrix are gotten by permuting the components of a vector in the list: $(3, 2, 1, 0)^\top$, $(3, 1, 1, 1)^\top$, $(2, 2, 2, 0)^\top$, $(2, 2, 1, 1)^\top$. These all live in the hyperplane $x_1 + x_2 + x_3 + x_4 = 6$. When we pass to the convex hull, only the vector $(3, 2, 1, 0)^\top$ and its permutations survive. The polytope $\mathcal{P}$ is the convex hull of the 24 points which are permutations of $(3, 2, 1, 0)^\top$. This polytope is often called the *permutohedron* because its vertices correspond to permutations on 4 letters. Here is a picture (Figure 5.1), adapted from [2, p. 136].

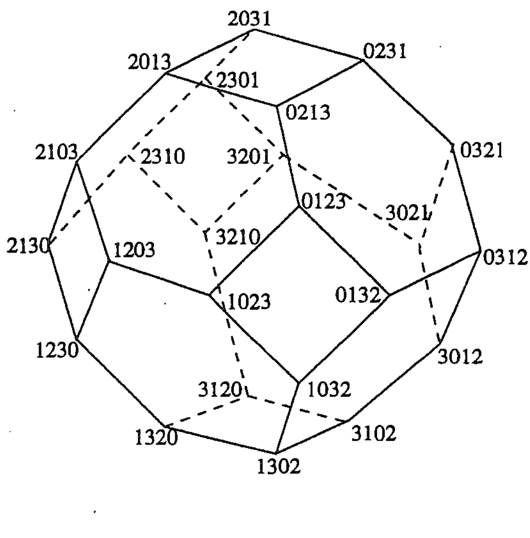


Figure 5.1. Permutohedron

In this example W is transitive on vertices, has 3 orbits on edges, and 3 orbits on faces of dimension 2. Thus the number of $\mathcal{J}$-classes in the corresponding monoid $M(\rho)$, excluding the classes $\{0\}$, $\{1\}$, is $1 + 3 + 3 = 2^3 - 1$. If ρ is *any* representation of $\mathbf{SL}_4$ such that W is transitive on the vertices of the corresponding polytope $\mathcal{P}$ then the number of $\mathcal{J}$-classes, excluding the classes $\{0\}$, $\{1\}$, is at most $2^3 - 1$. The proper context for this fact lies in the notion of a *canonical monoid* for $\mathbf{SL}_4$; see [38], [35] and the remarks which follow Theorem 5.9.

Now return to the general theory of a reductive monoid M with unit group G. We drop the assumption that dim $Z(G) = 1$ but, since some of the theorems we state require that $0 \in M$, we ***assume henceforth*** that $0 \in M$. Note that $0 \in M(\rho)$ for any representation ρ. Theorems 5.6 and 5.7 show how to recapture some of the standard constructs in the theory of a reductive group, for example Borel subgroup, maximal torus, parabolic subgroup, Levi factor, in terms of the idempotent set $E(M)$. A ***chain of idempotents*** is a linearly ordered subset $\Gamma = \{e_1 < \cdots < e_k\}$ of the poset $E(M)$. If Γ is a chain of idempotents then there exists a maximal torus T of G such that $\Gamma \subset \overline{T}$ [31, Corollary 6.10]. [41] The length of a maximal chain in $E(M)$ or $E(\overline{T})$ is $\dim T$ [31, Theorem 6.20] which is also the length of a maximal chain in $G\backslash M/G$ [31, Theorem 6.20]. If $\Gamma \subseteq E(M)$, define the right and left centralizers $P(\Gamma)$, $P^-(\Gamma)$ by

$$P(\Gamma) = \textstyle\bigcap_{e\in\Gamma} P(e), \qquad P^-(\Gamma) = \textstyle\bigcap_{e\in\Gamma} P^-(e)\ , \tag{104}$$

where $P(e), P^-(e)$ are as in (14). Define the centralizer $C_G(\Gamma)$ by

$$C_G(\Gamma) = P(\Gamma) \cap P^-(\Gamma) = \textstyle\bigcap_{e\in\Gamma} C_G(e)\ . \tag{105}$$

Theorem 5.6 ([31, Theorem 7.1]) *Let M be a reductive monoid. Let Γ be a maximal chain in $E(M)$. Then $C_G(\Gamma)$ is a maximal torus of G and $P(\Gamma), P^-(\Gamma)$ are a pair of opposite Borel subgroups relative to $C_G(\Gamma)$.*

Every maximal torus and Borel subgroup of G can be obtained in this way [26, Theorem 4.5]. Thus the Borel subgroups and maximal tori in G may be recovered from $E(M)$. How about the parabolic subgroups?

Theorem 5.7 ([26, Theorem 4.6], [27, Theorem 4]) *Let M be a reductive monoid. Let Γ be a chain in $E(M)$. Then $P(\Gamma)$ and $P^-(\Gamma)$ are a pair of opposite parabolic subgroups with common Levi factor $C_G(\Gamma)$.*

Every parabolic subgroup P of G has the form $P = P(\Gamma)$ for some chain $\Gamma \subset E(M)$ [30, Theorem 7.2]. [42] The simple statement of Theorem 5.7 contains implicit information. For example it says, by the definition (64) of Levi subgroup, that $C_G(\Gamma)$ is a connected reductive group. In particular, if $\Gamma = \{e\}$ then $C_G(e)$ is a connected reductive group and $P(e), P^-(e)$ are a pair of opposite parabolic subgroups. If Γ is a maximal chain then Theorem 5.7 reduces to Theorem 5.6.

Example 5.8 Let $M = \mathbf{M}_n$ so that $G = \mathbf{GL}_n$. Let $e_r \in \mathbf{D}_n$ be as in Example 3.6. Let $\gamma = (\gamma_1, \ldots, \gamma_k)$ be a composition of n as in (4.29). Then $\Gamma = \{e_{\gamma_1}, e_{\gamma_1+\gamma_2}, \ldots, e_{\gamma_1+\gamma_2+\cdots+\gamma_k}\}$ is a chain of idempotents and $P(\Gamma)$ is the parabolic subgroup pictured in (58). The opposite parabolic subgroup $P^-(\Gamma)$ is the transpose of $P(\Gamma)$. The intersection $C_G(\Gamma) = P(\Gamma) \cap P^-(\Gamma)$ is the Levi subgroup pictured in Example 4.30. If $\Gamma = \{e_r\}$ then we are back in Example 3.6.

[41] If $\Gamma = \{e\}$ this is Theorem 5.2, which serves to start an induction.

[42] The proof uses the notion of cross section lattice, introduced below, as well as an algebro-geometric result of Renner [39].

Let M be a reductive monoid with unit group G and maximal torus T. It follows from (66) that if $\Lambda \subseteq E(\overline{T})$ represents the set of W-orbits on T then the map $\Lambda \to G\backslash M/G$ defined by $e \mapsto GeG$ is bijective. Since both $E(\overline{T})$ and $G\backslash M/G$ are partially ordered sets it is natural to ask if one can choose Λ so that the bijection respects the partial orders. This question leads to the following important definition [26, §6], [31, Definition 9.1]:

Definition 3 A set $\Lambda \subseteq E(\overline{T})$ is a *cross section lattice* if Λ is a set of representatives for the W-orbits on $E(\overline{T})$ *and* the bijection $\Lambda \to G\backslash M/G$ is order preserving.

Since $G\backslash M/G$ is a lattice so is Λ. It is not clear that cross section lattices exist. In fact they do exist by a remarkable theorem of Putcha, which shows in addition that there is a bijection between the set of cross section lattices and the set $\mathcal{B}^T$ of Borel subgroups of G which include T. Recall that if $e \in E(\overline{T})$ then the right centralizer $P(e)$ is a parabolic subgroup of G. Since $\overline{T}$ is commutative we have $P(e) \supset T$.

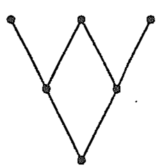
Theorem 5.8 **([30, Theorem 1.1], [31, Theorem 9.10])** *If $B \in \mathcal{B}^T$, define a subset $\Lambda(B)$ of $E(\overline{T})$ by $\Lambda(B) = \{e \in E(\overline{T}) \mid P(e) \supseteq B\}$. The map $B \mapsto \Lambda(B)$ is bijective from $\mathcal{B}^T$ to the set of cross section lattices in $E(\overline{T})$.*

We may reconstruct the corresponding Borel subgroup from a given cross section lattice Λ according to the recipe $B = \bigcap_{e\in\Lambda} P(e)$. Since $P(wew^{-1}) = wP(e)w^{-1}$ we have $\Lambda(wBw^{-1}) = w\Lambda(B)w^{-1}$. It follows, in view of (43), that if $B \in \mathcal{B}^T$ is a fixed Borel subgroup then the map $w \to w\Lambda(B)w^{-1}$ is bijective from W to the set of all cross section lattices in $E(\overline{T})$.

Example 5.9 Suppose $M = \mathbf{M}_n$, $G = \mathbf{GL}_n$ and $T = \mathbf{T}_n$. Let $e \in E(\overline{T})$. Then $P(e) \supseteq \mathbf{B}_n$ if and only if $e \in \{e_0, e_1, \dots, e_n\}$ with e_r as in Example 3.6. Thus the cross section lattice corresponding to the standard Borel subgroup $\mathbf{B}_n$ is $\Lambda(\mathbf{B}_n) = \{e_0, e_1, \dots, e_n\}$ where $e_0 = 0$ and $e_n = 1$. If $I \subseteq \{1, \dots, n\}$ write $e_I = \sum_{i\in I} E_{ii}$ as in Example 2.1. Thus $e_r = e_{\{1,\dots,r\}}$. Since $we_Iw^{-1} = e_{\pi I}$ where $\pi \in S_n$ is the permutation corresponding to w, we see that the cross section lattices in $E(\overline{T})$ have the shape $\{e_{I_0}, e_{I_1}, e_{I_2}, \dots, e_{I_n}\}$ where $\emptyset = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_n = \{1, \dots, n\}$ and $|I_k| = k$.

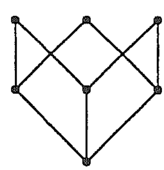
Theorem 5.8 allows us to describe the partial order in $G\backslash M/G$ in terms of idempotents which may be computed using the geometric method in (103). The partial order structure in $G\backslash M/G$ in Examples 5.5, 5.6, 5.7 is, omitting the orbits $\{0\}, \{1\}$, given by:

cuboctahedron



cross polytope

permutohedron



Now fix a Borel subgroup $B \supset T$ and let $\Lambda = \Lambda(B)$. If $e \in \Lambda$ then $P(e) \supseteq B$ is a standard parabolic subgroup and hence, by Theorem 4.2, has the form P_I for some $I \subseteq S$, called the type of $P(e)$. This gives a map

$$\lambda : \Lambda \to 2^S \tag{106}$$

called the *type map*, defined by $\lambda(e) = \text{type } P(e)$. In fact [30, Lemma 2.4], [35, p.645]

$$\text{type } P(e) = \{s \in S \mid ses^{-1} = e\} , \tag{107}$$

is the set of Coxeter generators which fix e. See [36, Section 2] and the references given there for more about the type map. In Example 5.9 the image of the type map consists, aside from S and $\emptyset$, of the sets $\{s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_{n-1}\}$ for $1 \leq r \leq n-1$. The corresponding parabolic subgroups are, in view of Theorem 4.2(b), the maximal parabolic subgroups given in Example 3.6. On the other hand, in Example 5.7 the type map $\lambda : \Lambda \to 2^S$ is surjective, and is in fact bijective when restricted to $\Lambda - \{0\}$. Note in both examples that the type map is injective, and that W acts transitively on the set of vertices of $\mathcal{P}$ and hence acts transitively on the set of minimal nonzero idempotents of $E(\overline{T})$.

Theorem 5.9 ([41, Corollary 8.3.3], [31, Corollary 15.3]) *Let M be a reductive monoid. Then W acts transitively on the set of minimal nonzero idempotents of $E(\overline{T})$ if and only if M has an irreducible idempotent separating representation ρ.*[43]

Monoids with the property of Theorem 5.9 are said to be *$\mathcal{J}$-irreducible* because transitivity of W on the set of minimal nonzero idempotents of $E(\overline{T})$ is equivalent, in view of Theorem 65, to the assertion that M has a unique minimal non-zero $\mathcal{J}$-class. If M is $\mathcal{J}$-irreducible then dim $Z(G) = 1$ so that M is a semisimple monoid [31, Corollary 15.3], [41, Lemma 8.3.2]. Thus the remarks which follow 5.4 may be applied to M. If $\rho : G_0 \to \mathbf{GL}_n$ is an irreducible representation of a semisimple group G_0, then the inclusion map $M(\rho) \to \mathbf{M}_n$ is a faithful irreducible representation of $M(\rho)$. Thus, if ρ is an irreducible representation of G_0 then $M(\rho)$ is a $\mathcal{J}$-irreducible monoid. All the examples we have given to illustrate the geometry of the cone σ and the polytope $\mathcal{P}$ are $\mathcal{J}$-irreducible. If M is $\mathcal{J}$-irreducible and $e, f \in E(M)$ then $P(e) = P(f)$ if and only if $eG = fG$ [37, Proposition 4.3]. Thus if M is $\mathcal{J}$-irreducible then the type map is injective. If M is $\mathcal{J}$-irreducible, there is an effective method for computing the type map [37, Corollary 4.11,Theorem 4.16] which by-passes the geometric arguments given earlier in this Section. This argument has been used to compute the posets $\Lambda \simeq G\backslash M/G$ if G_0 is a simple algebraic group, in case ρ is the adjoint representation, in case the highest weight of ρ is a fundamental weight, and

[43] A representation is *idempotent separating* if it is faithful on $E(M)$. The condition that ρ be idempotent separating may be replaced by the condition that ρ is a finite morphism in the sense of algebraic geometry [41, Corollary 8.3.3]; the hypothesis that M is a normal variety is not used in the proof of [41, Lemma 8.3.2].

in some other cases [37, Section 6]. [44] If the highest weight is fixed by no nonidentity element of W, then the type map $\lambda : \Lambda - \{0\} \to 2^S$ is surjective and hence bijective as in Example 5.7. Monoids with this property play a special role in the representation theory of "finite monoids of Lie type"; see the remarks which follow Example 5.11.

The Bruhat decomposition and Tits system in a reductive group G are central in the structure theory. They may be used to reduce many questions about G to questions about the Weyl group. Renner [43] found an analogous decomposition for reductive monoids with striking consequences. [45] Let M be a reductive monoid and let $G, B \supset T, N, W$ be as before. Let $\overline{N}$ be the Zariski closure of N in M. Then $\overline{N}$ is a monoid which normalizes T so $R = \overline{N}/T$ is a monoid. Thus

$$R = \overline{N}/T \supset N/T = W. \tag{108}$$

Example 5.10 Let $M = \mathrm{M}_n$. Then, as in Example 4.4, N consists of all monomial matrices, while $\overline{N}$ consists of matrices with *at most* one nonzero entry in each row and column. Thus R may be identified with the monoid of all zero-one matrices which have at most one entry equal to 1 in each row and column. This is a finite monoid isomorphic to the *symmetric inverse semigroup* I_n which consists, by definition, of all injective partial maps from $\{1, \dots, n\} \to \{1, \dots, n\}$. It is sometimes useful, from a combinatorial point of view, to think of these matrices as placements of non-attacking rooks on an $n \times n$ board. In [48] R is called the rook monoid. The order of R is $|R| = \sum_{r=0}^{n} \binom{n}{r}^2 r!$. The group of units of I_n is the group P_n of permutation matrices.

Theorem 5.10 *Let M be a reductive monoid. Let $\Lambda = \Lambda(B)$ be a cross section lattice for $E(\overline{T})$ and let $e \in \Lambda$. Then*
(1) R is a finite inverse monoid.
(2) The group of units of R is W, and $R = WE(R)$.
(3) $E(R) \simeq E(\overline{T})$.
(4) $M = \bigsqcup_{\rho \in R} B\rho B$, and $B\rho B = B\rho' B \Rightarrow \rho = \rho'$.
(5) If $s \in S$ is a Coxeter generator then $BsB \cdot B\rho B \subseteq Bs\rho B \cup B\rho B$.
(6) $GeG = \bigsqcup_{\rho \in WeW} B\rho B$.
(7) If $w_0 \in W$ is the opposition element then Bw_0eB is open and dense in GeG.

When Renner proved these facts, they were new even in case $M = \mathrm{M}_n(K)$; no-one had noticed the significance of the symmetric inverse semigroup for the structure theory of $\mathrm{M}_n(K)$, although it is easy to prove (4) using elementary row and column operations in this case: just note that for any field K a square matrix over K is equivalent to a rook matrix by elementary row and column operations in which *rows* are moved *up* and *columns* are moved to the *right*. As an analogue of the Weyl

[44] We have not defined "highest weight" or "fundamental weight" in this paper; see for example [15, Theorem 31.3]. If G_0 is a semsimple group and $\rho : G_0 \to \mathrm{GL}_n$ is an irreducible representation with highest weight μ then the polytope $\mathcal{P}$ is the convex hull of the set $W\mu$. This is shown, with slightly different formulation, in [41, Proposition 3.5].

[45] Monoids corresponding to classical groups in their natural representations were first studied by Grigor'ev [13], who looked for an analogue of the Bruhat decomposition.

group, the *Renner monoid* R plays a central role in the theory of reductive monoids. Note that (4) may be viewed as an analogue of the Bruhat decomposition of G, that (5) gives M a structure like a Tits system, that (6) in case $e = 1$ is the Bruhat decomposition of G and that (7) in case $e = 1$ asserts the existence of Chevalley's "big cell" Bw_0B where w_0 is the longest element of W. The "big cell" should be familiar in case $G = \mathbf{GL}_n$ under a different name, since $Bw_0B = w_0B^-B$ and the set B^-B consists of the matrices for which the process of Gaussian elimination involves no zero pivots; see the remarks which follow (62).

Let K be the algebraic closure of $\mathbf{F}_q$. The rest of this paper concerns those finite submonoids of $\mathbf{M}_n(K)$ which are fixed points under the Frobenius map $\sigma : \mathbf{M}_n(K) \to \mathbf{M}_n(K)$ defined by $\sigma : [c_{ij}] \mapsto [c_{ij}^q]$. These monoids and more general fixed point monoids were introduced by Renner in [44]; see also [38, Section 4], [46]. To agree with Renner's paper in these Proceedings we *change notation*; this looks unnecessarily complicated to start but is ultimately simpler. Henceforth we write $\mathbf{M}_n(K)$ rather than $\mathbf{M}_n$ and let $\underline{M} \subseteq \mathbf{M}_n(K)$ denote a reductive algebraic monoid with 0 such that $\sigma\underline{M} = \underline{M}$. We let $M = \{a \in \underline{M} \mid \sigma a = a\}$ denote the finite submonoid of fixed points. Thus, for example, if $\underline{M} = \mathbf{M}_n(K)$ then $M = \mathbf{M}_n(\mathbf{F}_q)$.[46] We call M a *finite reductive monoid.* The remarks in the following paragraph say, roughly, that many of the constructs in this paper descend from the reductive monoid $\underline{M}$ to the finite reductive monoid M.

Let $\underline{G}$ be the group of units of $\underline{M}$. According to Steinberg [51, Corollary 10.10] we may choose a Borel subgroup $\underline{B}$ and maximal torus $\underline{T} \subset \underline{B}$, both stable under σ. Let $\underline{N}$ denote the normalizer of $\underline{T}$ in $\underline{G}$. Let $\underline{W}$ denote the Weyl group, let $\underline{R}$ denote the Renner monoid, and let $\underline{\Lambda}$ denote the cross section lattice defined in Theorem 5.8. Let $G = \underline{G} \cap M$ denote the finite group of units of M, let $B = \underline{B} \cap M$, let $T = \underline{T} \cap M$, and let $N = \underline{N} \cap M$. The Frobenius automorphism acts on $\underline{W}, \underline{R}$ and $\underline{\Lambda}$. Let W, R, Λ denote the corresponding fixed point sets. Then $R = \underline{R}, \Lambda = \underline{\Lambda}$ and $W = \underline{W} \simeq N/T$ [44], [46]. [47] We identify $N/T = W$. Then (G, B, N, S) is a Tits system with Weyl group W. There is a decomposition

$$(109) \qquad M = BRB = \bigsqcup_{\rho \in R} B\rho B$$

analogous to the Bruhat decomposition $G = \sqcup_{w\in W} BwB$ of the finite reductive group G. The union in (109) is disjoint and $B\rho B = B\rho' B$ implies $\rho = \rho'$.

Example 5.11 If $\underline{M} = \mathbf{M}_n(K)$ then $M = \mathbf{M}_n(\mathbf{F}_q)$, $G = \mathbf{GL}_n(\mathbf{F}_q)$, B is the group of invertible upper triangular matrices over $\mathbf{F}_q$, and T is the group of invertible diagonal matrices over $\mathbf{F}_q$. Here R is isomorphic to the symmetric inverse semigroup (rook monoid), W is isomorphic to the symmetric group S_n, and Λ is as in Example 5.9.

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[46] Renner allows for a more general setup which includes, for example, monoid analogues of the finite unitary groups. His endomorphisms σ are analogous to Steinberg's endomorphisms [51] in the theory of reductive groups. This setup is more subtle than the one we consider here.

[47] The simplicity of this statement depends on our assumption that σ is the q-th power map; see [46, 3.3] for an example in which G is a finite unitary group.

There is a close connection between the modular representation theories of G and M. If G is any finite reductive group over $\mathbf{F}_q$, there exists [38, Section 4] a reductive monoid $\underline{M}$, over the algebraically closure K of $\mathbf{F}_q$ such that (1) $G(M)$ is a finite central extension of G and (2) the type map $\lambda : \Lambda - \{0\} \to 2^S$ is an order preserving bijection. The finite monoid M has a central extension $\mathcal{M}$ whose representation theory over K is intimately related to the modular representation theory of G. In particular, [38, Theorem 2.2], every irreducible modular representation of $\mathcal{M}$ restricts to an irreducible representation of G and the number of extensions of a modular representation of G to $\mathcal{M}$ is determined explicitly in terms of the Curtis-Richen representation theory of G. See Renner's paper [47] in these Proceedings for an explicit statement and further references.

Since the Bruhat decomposition is at the heart of the theory of finite reductive groups, one might expect the same of Renner's decomposition (109) for finite reductive monoids. With this in mind, we sketch the analogue for a finite reductive monoid M of Iwahori's double coset ring construction for G [9, §67], [17]. Let $\mathbf{C}M = \bigoplus_{a\in M} \mathbf{C}a$ be the monoid algebra, a finite dimensional associative algebra with identity which contains the group algebra $\mathbf{C}G$. If X is any subset of M write $[X] = \sum_{a\in X} a \in \mathbf{Z}M$. Thus $\epsilon = |B|^{-1}[B] \in \mathbf{C}G$ is the idempotent corresponding to the subgroup B. The Iwahori algébra [48] is by definition

$$\mathcal{H}_{\mathbf{C}}(G,B) = \epsilon \mathbf{C}G\epsilon = \bigoplus_{w\in W} \mathbf{C}[BwB] \ . \tag{110}$$

It is semisimple because $\mathbf{C}G$ is semisimple. If we define

$$T_w = \frac{1}{|B|}[BwB] \tag{111}$$

then, on general group theoretic grounds [9, 11.30],

$$\mathcal{H}(G,B) = \bigoplus_{w\in W} \mathbf{Z}T_w \tag{112}$$

is a ring. Iwahori studied this ring [17] because $\mathcal{H}_{\mathbf{C}}(G,B)$ controls the decomposition of the permutation representation of G on G/B. In particular, he found the multiplication table for the basis elements T_w [17, Corollary 4.2], [9, Theorem 67.2]:

Theorem 5.11 *Let G be a finite reductive group. The ring $\mathcal{H}(G,B)$ is generated by the T_s for $s \in S$. The multiplication of the T_w is determined by the formulas*

[48] This is usually called the Iwahori Hecke algebra or simply Hecke algebra; see [9, 11.22] and the Introduction to [48] for historical remarks on the terminology. See [9, §11] for the theory of Hecke algebras or double coset rings in the context of general representation theory of finite groups and [9, §§67,68] for these algebras in the context of finite groups with a Tits system. Iwahori [17] was the first to notice that the double coset ring of a finite reductive group G with respect to a Borel subgroup B has a remarkable structure. The examples he studied are the ones which have surfaced as "Hecke algebras" in other parts of mathematics.

(113) $$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w)+1 \\ qT_{sw} + (q-1)T_w & \text{if } l(sw) = l(w)-1 \end{cases}$$

where $s \in S$ and $w \in W$.

Here $w \mapsto l(w)$ is the length function given by (54). If $w = s_{i_1} \cdots s_{i_p}$ where $s_{i_1}, \ldots, s_{i_p} \in S$ and $l(w) = p$, it follows from (113) that $T_w = T_{s_{i_1}} \cdots T_{s_{i_p}}$. Thus, in view of (113), if $w', w'' \in W$ we have $T_wT_{w'} = \sum_{w'' \in W} c_{w,w',w''} T_{w''}$ where the $c_{w,w',w''}$ are polynomials in q with integer coefficients.

Is there an analogous theorem for a reductive monoid M? If M is *any* finite monoid with group of units G, and B is a subgroup of G, then we may define a $\mathbf{C}$-algebra $\mathcal{H}_{\mathbf{C}}(M,B)$ by

(114) $$\mathcal{H}_{\mathbf{C}}(M,B) = \epsilon\mathbf{C}[M]\epsilon = \bigoplus_D \mathbf{C}[D] \supseteq \mathcal{H}_{\mathbf{C}}(G,B)$$

where D ranges over the $B \times B$ orbits on M. These orbits are the natural replacement for the double cosets. The algebra $\mathcal{H}_{\mathbf{C}}(M,B)$ won't, in general, be semisimple. So let's suppose that M is a finite reductive monoid, and let G, B be as before. Then $\mathbf{C}[M]$ is semisimple and hence so is $\mathcal{H}_{\mathbf{C}}(M,B)$. This is a theorem of Okniński and Putcha [20]. The proof is not easy; it uses the Harish-Chandra induction theory for characters of G. To gauge the depth of this result one should note that there are no general theorems about semisimplicity of monoid algebras which will prove the semisimplicity of $\mathbf{C}[M]$ in case $M = \mathrm{M}_n(\mathbf{F}_q)$; Kovaćs [19] has argued this special case from first principles. Since $\mathbf{C}[M]$ is semisimple, so is $\mathcal{H}_{\mathbf{C}}(M,B)$. What can we say about the structure of $\mathcal{H}_{\mathbf{C}}(M,B)$ in the direction of Iwahori's work for G? Renner's decomposition (109) gives

(115) $$\mathcal{H}_{\mathbf{C}}(M,B) = \bigoplus_{\rho \in R} \mathbf{C}[B\rho B] \,.$$

Putcha [33], [36, Section 4] studied $\mathcal{H}_{\mathbf{C}}(M,B)$ with remarkable results. Fix an orbit, alias $\mathcal{J}$-class, $\mathcal{O} \in G\backslash M/G$ and consider the *local monoid* $M(\mathcal{O}) = G \cup \mathcal{O} \cup \{0\}$ [35, p.638] where the product in $M(\mathcal{O})$ of two elements of $\mathcal{O}$ is defined to be 0 if their product in M is not in $\mathcal{O}$. Since $M(\mathcal{O}) \supset G$ we may consider the monoid algebra $\mathcal{H}_{\mathbf{C}}(M(\mathcal{O}),B) \supset \mathcal{H}_{\mathbf{C}}(G,B)$. Putcha determined the structure of these algebras $\mathcal{H}_{\mathbf{C}}(M(\mathcal{O}),B)$, and, by piecing them together over the various orbits $\mathcal{O}$, proved an isomorphism [33, Theorem 4.1]

(116) $$\mathcal{H}_{\mathbf{C}}(M,B) \simeq \mathbf{C}[R]$$

analogous to Tits' *deformation theorem* $\mathcal{H}_{\mathbf{C}}(G,B) \simeq \mathbf{C}[W]$ in the theory of finite groups with a Tits system [9, Theorem 68.21]. The proof involves an interesting application of the Kazhdan-Lusztig "R-polynomials."

There is no known general theorem for finite monoids, like [9, Theorem 11.30] for finite groups, which guarantees the existence of constants $a_D \in \mathbb{Q}$ corresponding to $B \times B$ orbits D on M, such that $\oplus_D \mathbb{Z}a_D[D]$ is a ring. If M is a finite reductive monoid one may hope for special dispensation. Thus the question is: are there elements $T_\rho = a_\rho[B\rho B]$ with $\rho \in R$ and $a_\rho \in \mathbb{Q}$, and formulas analogous to those in (113), which show that the structure constants for the multiplication table in $\mathcal{H}_{\mathbb{C}}(M, B)$ relative to the basis T_ρ are integers given by polynomials in q? The answer is "yes". To see how this comes about, we recall Iwahori's setup in $\mathcal{H}_{\mathbb{C}}(G, B)$. The trivial representation $g \mapsto 1$ for $g \in G$ of $\mathbb{C}G$ restricts to a representation ind : $\mathcal{H}_{\mathbb{C}}(G, B) \to \mathbb{C}$. The double coset BwB may be written in Chevalley normal form [9, 69.7]

$$BwB = BwU_w^- \tag{117}$$

where

$$U_w^- = \prod_{\alpha \in \Phi_w^-} U_\alpha \tag{118}$$

with U_α as in (21) and $\Phi_w^- = \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}$ as in (55). The groups U_w^- were defined by Chevalley in [7, p.42]. The expression of an element of BwB as bwu with $b \in B$, $w \in W$, and $u \in U_w^-$ is unique. Thus, in view of (56) we have

$$|BwB| = |B|q^{l(w)} \tag{119}$$

where $l(w)$ is the length function on W. Thus

$$\mathrm{ind}(T_w) = q^{l(w)} . \tag{120}$$

We would like monoid analogues of (117)-(120). If we find them, we have a good shot at the structure of $\mathcal{H}(M, B)$. To begin, we need a definition of length. In the case of reductive groups there are various ways to view the length function on W, all of them closely related:

(L1) Words: $l(w) = \min\{p \mid w = s_1 \dots s_p \text{ with } s_i \in S\}$ as in (54) .

(L2) Roots: $l(w) = |\Phi_w^-|$ as in (56).

(L3) Geometry of G/B: $l(w) = \dim BwB/B = \dim BwB - \dim B$ is the dimension of a Bruhat cell; in this formulation we are in the context of algebraic groups over an algebraically closed ground field K.

The monoid analogue of Iwahori's theorem (113) and of the deformation theorem (116) was proved in [48] in case $M = \mathrm{M}_n(\mathbb{F}_q)$. The argument there, defines a length function $\rho \mapsto l(\rho)$ in the spirit of (L1) and (L2). Each $W \times W$ orbit on R has its own length function. If the orbit is $W \cdot 1 \cdot W$ then the length function agrees with the length function $w \mapsto l(w)$ on W. The construction depends on the set

$\mathcal{N} = \{1, \nu, \nu^2, \ldots, \nu^{n-1}, \nu^n = 0\}$ where $\nu = E_{12} + E_{23} + \ldots + E_{n-1,n}$ is a Jordan block. The set $\mathcal{N}$ is a cross section for $G\backslash M/G$ and for $W\backslash R/W$; the elements of $G\nu^i G$ have rank $n-i$. If $\rho \in W\nu^i W$ write $\rho = w\nu^i w'$ with $w, w' \in W$ and define $l(\rho) = \min(l(w) + l(w'))$ over all such expressions. Thus the element ν^i is the unique element of length zero in its $W \times W$-orbit. The main combinatorial idea in [48] is to find a substitute for Chevalley's two part partitions of the set of positive roots: those positive roots which change sign under a given element $w \in W$ and those which don't. The corresponding partitions for $\rho \in R$ have five (some possibly empty) parts. In view of later developments [49] these partitions are a piece of the general theory. Renner [45] discovered the following marvelous fact, which proves the existence of a set analogous to $\mathcal{N} \subseteq \mathrm{M}_n(\mathbf{F}_q)$ in any reductive monoid.

Theorem 5.12 *Let M be a reductive monoid.* [49] *Let $\mathcal{N} = \dot{\mathcal{N}}(B) = \{\nu \in R \mid B\nu = \nu B\}$. The set $\mathcal{N}$ is a cross section for $G\backslash M/G$ as well as $W\backslash R/W$.*

If M is the monoid of all matrices over *any* field, and B is the group of upper triangular matrices, and $\nu = E_{12} + E_{23} + \ldots + E_{n-1,n}$ is a Jordan block then $\mathcal{N} = \{1, \nu, \ldots, \nu^{n-1}\}$.

Renner used his set $\mathcal{N}$ to define length in the spirit of (L3) as follows. Suppose $\rho \in R$. Choose $\nu \in \mathcal{N}$ with $\rho \in W\nu W$ and define

$$l(\rho) = \dim B\rho B - \dim B\nu B\ . \tag{121}$$

Thus the elements of $\mathcal{N}$ are, by definition, the elements of length zero. With this definition Renner [44] proved the monoid analogue of Tits' formulas (63) for the multiplication of double cosets, namely,

$$BsB \cdot B\rho B = \begin{cases} B\rho B & \text{if } l(s\rho) = l(\rho) \\ Bs\rho B & \text{if } l(s\rho) = l(\rho) + 1 \\ Bs\rho B \sqcup B\rho B & \text{if } l(s\rho) = l(\rho) - 1 \end{cases} \tag{122}$$

where $s \in S$ and $\rho \in R$. These formulas agree with those in [48, Proposition 3.14] if $M = \mathrm{M}_n(\mathbf{F}_q)$ which were proved using (L1) and (L2). The case $l(s\rho) = l(\rho)$ where $BsB \cdot B\rho B = B\rho B$ is impossible for groups. It turns out that one can also define length for the Renner monoid R of any reductive monoid M in the spirit of (L1) and (L2). One also has (L3) since there is a Chevalley normal form for the orbits $B\rho B$ with $\rho \in R$ which is analogous to (117) [49]. This leads, in particular, to the formula

$$|B\rho B| = |eBe| q^{l(\rho)} \tag{123}$$

[49] In this theorem, and in (122) one may choose M to be a reductive monoid over an algebraically closed field K *or* the finite monoid of fixed points under the Frobenius map in case K is the algebraic closure of $\mathbf{F}_q$. In (121) it is understood that the field is algebraically closed.

where $e \in \Lambda$ is the idempotent in the $\mathcal{J}$-class, alias $G \times G$ orbit, of ρ. As in the case of groups the restriction of the trivial representation $a \mapsto 1$ for $a \in M$ of $\mathbf{C}M$ restricts to a representation ind : $\mathcal{H}_{\mathbf{C}}(M, B) \to \mathbf{C}$. Thus if we define

$$T_\rho = \frac{1}{|eBe|}[B\rho B] \tag{124}$$

we have a rational multiple of the double coset sum $[B\rho B]$ which satisfies the analogue of (120), namely

$$\operatorname{ind}(T_\rho) = q^{l(\rho)} . \tag{125}$$

This leads to a theorem [49] on the Iwahori ring of a finite reductive monoid:

Theorem 5.13 *Let M be a finite reductive monoid. Define a free $\mathbf{Z}$-module of rank $|R|$ by*

$$\mathcal{H}(M, B) = \bigoplus_{\rho \in R} \mathbf{Z}T_\rho .$$

Then $\mathcal{H}(M, B)$ is a ring generated by the T_s for $s \in S$, and T_ν for $\nu \in \mathcal{N}$. The multiplication table is determined by the formulas

$$T_sT_\rho = \begin{cases} qT_\rho & \text{if } l(s\rho) = l(\rho) \\ T_{s\rho} & \text{if } l(s\rho) = l(\rho) + 1 \\ qT_{s\rho} + (q-1)T_\rho & \text{if } l(s\rho) = l(\rho) - 1 \end{cases} \tag{126}$$

and

$$T_\nu T_\rho = q^{l(\rho) - l(\nu\rho)} T_{\nu\rho} \tag{127}$$

for $s \in S$, $\rho \in R$, and $\nu \in \mathcal{N}$. There are analogous formulas for right multiplication by T_s and the T_ν.

The structure constants in (127) are integers because $l(\rho) \geq l(\nu\rho)$ for all $\rho \in R$ and $\nu \in \mathcal{N}$. If $M = \mathbf{M}_n(\mathbf{F}_q)$ this inequality was proved in [48, Theorem 4.12]. The author first proved Theorem 5.13 with $\mathcal{H}(M, B)$ replaced by $\mathcal{H}_A(M, B) = \oplus_{\rho \in R} AT_\rho$ where A is any (commutative) coefficient ring in which q is invertible. The obstacle to replacement of A by $\mathbf{Z}$ lay in the conjectured inequality $l(\rho) \geq l(\nu\rho)$ for $\rho \in R$ and $\nu \in \mathcal{N}$, which one needs to control the exponent in (127). The inequality was proved by Putcha in a letter to the author. [50] Putcha [34] described the irreducible representations of $\mathcal{H}_{\mathbf{C}}(M, B)$, and found an explicit isomorphism $\mathcal{H}_{\mathbf{C}}(M, B) \simeq \mathbf{C}[R]$.

[50] The author would like to express his thanks.

The theory developed by Putcha and Renner seems to me to be the beginning of a long story. There is much to be done, on "Hecke rings", on finite reductive monoids, on reductive monoids over more general coefficient rings, and over special ones like C. If you have seen this subject here for the first time, want to work on it, and come to it from outside the theory of reductive groups, my advice would be: compute one difficult example in great detail and bring to bear the part of mathematics (for example semigroup theory) that you know best. If you are already comfortable with reductive groups, keep in mind the diagram:

$$\begin{array}{ccc} M\ (=\mathbf{M}_n) & \rightsquigarrow & R\ (= I_n) \\ \cup & & \cup \\ G\ (=\mathbf{GL}_n) & \rightsquigarrow & W\ (= S_n) \end{array}$$

Many aspects of the bottom row are well understood. Use what you know and work on the top.

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