A Virtually Complete Classification of Virtual Complete Intersections in $\mathbb{P}^1 \times \mathbb{P}^1$

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   - Free and Virtual Resolutions
   - Virtual Complete Intersections (VCIs)
2. Determination of VCIs
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   - Conditions on VCIs
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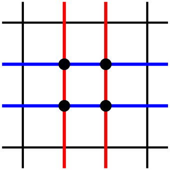

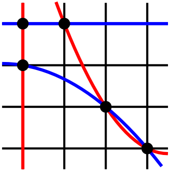

The Projective Space $\mathbb{P}^n$

Definition

A projective space $\mathbb{P}^n$ over the field $\mathbb{C}$ is the set of one-dimensional subspaces of the vector space $\mathbb{C}^{n+1}$.

- The coordinate ring of $\mathbb{P}^n$ is $S = \mathbb{C}[x_0, x_1, \ldots, x_n]$.
- Grading: Constants have degree 0. Each x_i has degree 1.

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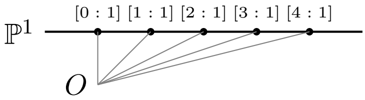

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Definition

A projective variety $X \subset \mathbb{P}^n$ is the zero locus of a collection of homogeneous polynomials $f_\alpha \in \mathbb{C}[x_0, x_1, \ldots, x_n]$.

The Biprojective Space $\mathbb{P}^1 \times \mathbb{P}^1$

Definition

The biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is the set of equivalence classes:

$$\mathbb{P}^1 \times \mathbb{P}^1 := \{((a_0, a_1), (b_0, b_1)) \in \mathbb{C}^2 \times \mathbb{C}^2 |_{\substack{(a_0,a_1)\neq(0,0) \\ \text{and } (b_0,b_1)\neq(0,0)}}\}/\sim$$

$x \sim y \iff x = \lambda y$, where $x, y \in \mathbb{P}^1, \lambda \in \mathbb{C}^*$

- Varieties $\leftrightarrow$ zero locus of bihomogenous $f \in \mathbb{C}[x_0, x_1, y_0, y_1]$
- Multigrading: $\deg(x_i) = (1, 0), \deg(y_i) = (0, 1)$
  ex. $x_0^2 y_0 + x_1 x_2 y_1$ has degree $(2, 1)$.

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  ex. $x_0^2 y_0 + x_1 x_2 y_1$ has degree $(2, 1)$.
- Irrelevant ideal: $B = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle \leftrightarrow V(B) = \emptyset$
- Saturation: $I : B^\infty = \{s \in S | sB^n \subset I \text{ for some } n\}$

The Nullstellensatz

The Nullstellensatz establishes a correspondence between ideals and varieties:

Theorem

Let X be a non-empty variety with the coordinate ring S and irrelevant ideal B. If $I \subseteq S$ is a homogeneous ideal, then there is an ***inclusion-reversing*** *bijective correspondence:*

$$\{V(I) \neq \emptyset\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{radical homogeneous } B\text{-saturated ideals} \subset S\}$$

- $V(I) :=$ zero locus of all $f \in I$
- $I(V(I)) = \sqrt{I}$

Varieties in $\mathbb{P}^1 \times \mathbb{P}^1$

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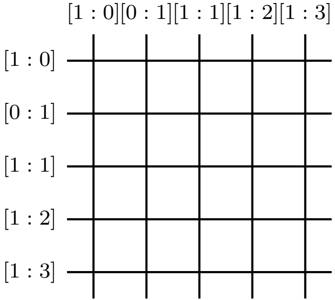

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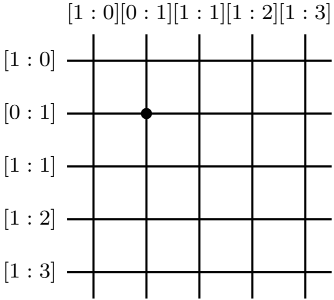

$X = ([0 : 1], [0 : 1])$

$I = \langle x_0, y_0 \rangle$

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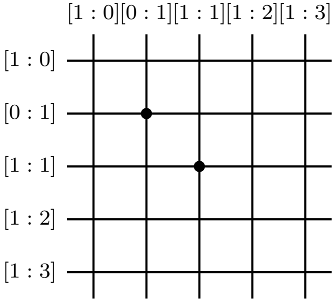


$X = ([0 : 1], [0 : 1]) \cup ([1 : 1], [1 : 1])$

$I = \langle x_0, y_0 \rangle \cap \langle x_0 - x_1, y_0 - y_1 \rangle$

Free Resolution

Definition

A free resolution of a module M is an exact sequence of homomorphisms:

$$0 \longleftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \longleftarrow \cdots,$$

- $\operatorname{im} \varphi_{i+1} = \ker \varphi_i$ at each step
- every $F_i \cong R^{r_i}$ is a free module

Minimal Free Resolution

Definition

A free resolution is minimal if for every $\ell \geq 1$, the nonzero entries of the graded matrix of φ_ℓ have positive degree.

- For each $\ell > 0$, φ_ℓ takes the standard basis of F_ℓ to a minimal generating set of $\text{im}(\varphi_\ell)$.
- Unique up to isomorphism.
- Depends on geometry of points (configuration/cross ratios)

Virtual Resolution

Definition

A virtual resolution for an ideal I in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is a free complex:

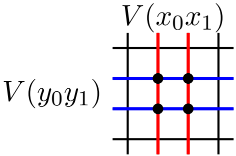
$$0 \longleftarrow I \xleftarrow{\varphi_0} S \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots$$

such that

- F_i are free modules for $i \geq 0$
- $\text{ann}\left(\frac{\ker(\varphi_i)}{\text{im}(\varphi_{i+1})}\right) \supseteq B^l$
- $\text{im}(\varphi_1) : B^\infty = I : B^\infty$.

Complete and Virtual Complete Intersection

- X is a **complete intersection** if $I(X)$ has 2 generators.



$$X = \begin{pmatrix} ([0:1],[1:0]), \\ ([1:0],[1:0]), \\ ([0:1],[0:1]), \\ ([1:0],[0:1]) \end{pmatrix}$$
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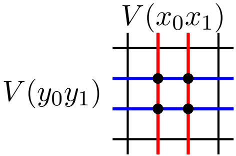

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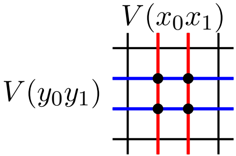

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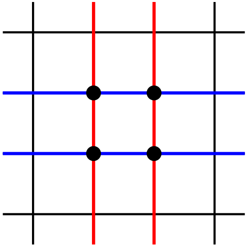
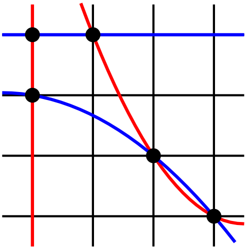
Definition

An ideal I of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is a virtual complete intersection (VCI) if I has a short virtual resolution that is Koszul.
In particular, $V(I) = V(f) \cap V(g)$.

VCI Examples



$$S^1 \leftarrow S^2 \leftarrow S^1 \leftarrow 0$$
$\implies$ Complete intersection



$$S^1 \leftarrow S^6 \leftarrow S^8 \leftarrow S^3 \leftarrow 0$$
$\implies$ Not complete intersection

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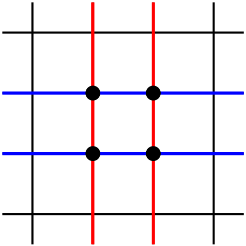

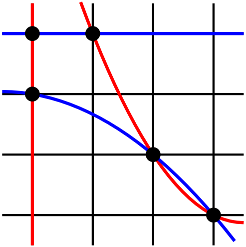

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$\implies$ Both are VCIs.

Generalized Bézout's Theorem

Theorem

Let $f, g \in k[x_0, x_1, y_0, y_1]$ be bihomogeneous forms. If f and g have multidegree (a, b) and (c, d), then $|V(f) \cap V(g)| = ad + bc$ counting multiplicities.

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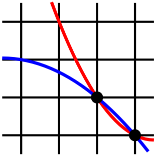
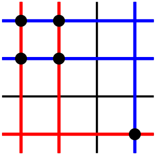


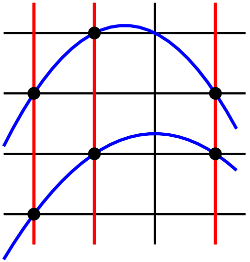
Red: $x_0y_1 + x_1y_0$: $(1, 1)$
Blue: $x_0y_1 - x_1y_0$: $(1, 1)$
$1 \cdot 1 + 1 \cdot 1 = 2$ points.



Red: $x_0x_1(y_0 - y_1)$: $(2, 1)$
Blue:$(x_0 - x_1)y_0y_1$: $(1, 2)$
$1 \cdot 1 + 2 \cdot 2 = 5$ points.

Our Main Results

Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.



This is a VCI: each vertical ruling has 2 points.

- Existence Case: Same number of points on each ruling.
- Non existence case: Bound on $|X|$ and maximal rulings form cross.
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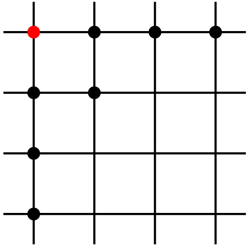

A $(4, 2, 1, 1)$-Ferrers Diagram

$|X| = 8$. We expect 16 points to have a VCI.

- Existence Case: Same number of points on each ruling.
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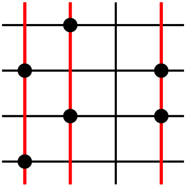
- Further conditions on VCIs.

Same Cardinality of Rulings

Theorem

If X has the same number (n) of points in each vertical (or each horizontal) ruling, it is a VCI.

- k vertical rulings each having n points $\implies \deg(f) = (n, \leq n), \deg(g) = (0, k)$.
- Idea: Use Lagrangian interpolation



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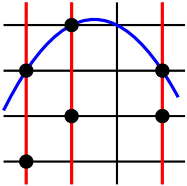

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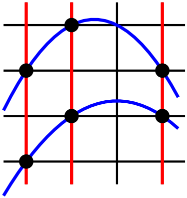


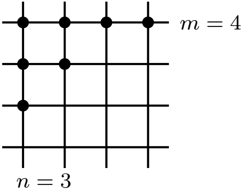
Degree Bound Lemma

Setup: f: (a,b)-form, g: (c,d)-form. Assume $X = V(f) \cap V(g)$.
$\leq m$ points collinear horizontally, $\leq n$ vertically

Lemma

$\max(a,c) \geq m$ *and* $\max(b,d) \geq n$.

When $|X| < mn$, *we must have* $a \geq m, b \geq n$ *(or* $c \geq m, d \geq n$*).*



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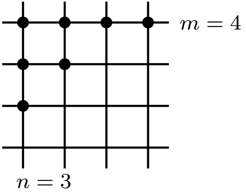


Two cases:

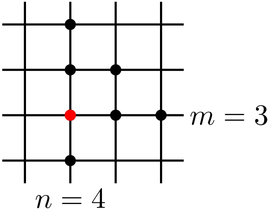
$\deg(f) = (\geq m, \geq n)$ $\deg(f) = (\geq m, ?\ )$

$\deg(g) = (\ ?\ ,\ ?\ )$ $\deg(g) = (\ ?\ , \geq n)$

Cross Point Condition

Theorem

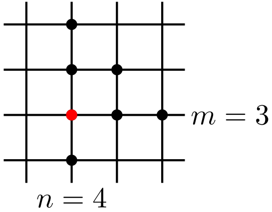
If $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points, then X is not a VCI.



Cross Point Condition: Proof Sketch

Theorem

$|X| < mn$ and cross point exists $\implies$ not VCI.



- Assume $V(f) \cap V(g) = X$. By Bézout, $|X| = ad + bc = 7$.
- $a \geq m, b \geq n$.

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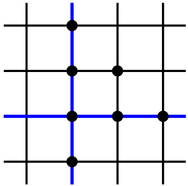

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- $a \geq m, b \geq n$.
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- Suppose $\deg(g_0) = (p, q)$.
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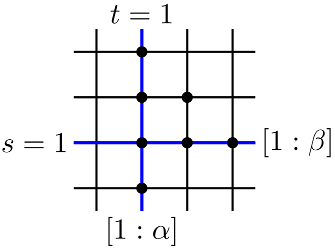

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- $a(s + q) + b(t + p) = |X|$ $\leq ms + nt - 1 + aq + bp$

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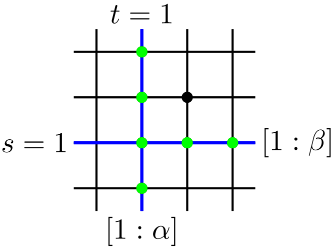

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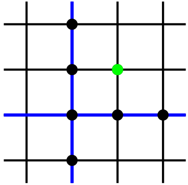

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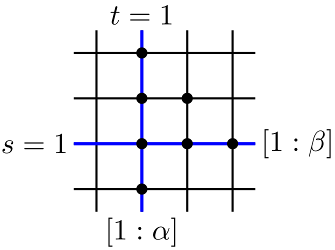

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- $as + bt \leq ms + nt - 1$

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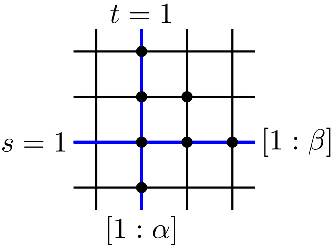

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- $as + bt \leq ms + nt - 1$ $\implies$ contradiction

Conditions on VCIs

Setup: f: (a,b)-form, g: (c,d)-form.
$\leq m$ points collinear horizontally, $\leq n$ vertically

Theorem

Let X be a VCI with $|X| < mn$.

- *f has degree (m,n) and g has vertical and horizontal components exactly on rulings with m and n points*
- $\gcd(m,n)$ *divides* $|X|$
- *If* $\gcd(m,n) = 1$*: g has degree:*

$$(n^{-1}|X| \mod m, \quad m^{-1}|X| \mod n)$$

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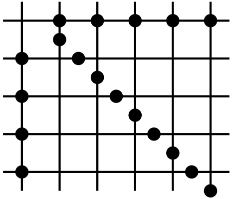
If $|X| < mn$: f has degree (m, n) and g has vertical and horizontal components exactly on rulings with m and n points



$m = 5$, $n = 4$, $|X| = 18$

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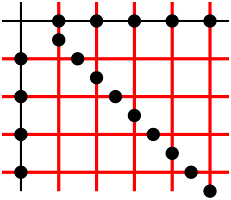

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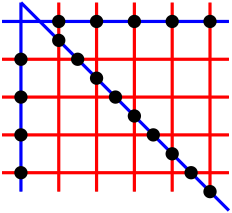

$m = 5$, $n = 4$, $|X| = 18$
f has degree $(5, 4)$
g has one $(1, 0)$ and one $(0, 1)$ part

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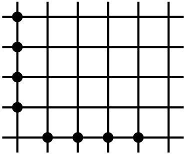
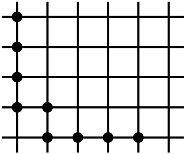
Theorem

If $|X| < mn$*:* $\gcd(m, n)$ *divides* $|X|$

- By Bézout and previous, $|X| = md + cn$



$m = 4, n = 4, |X| = 8$
Can be VCI



$m = 4, n = 4, |X| = 9$
Can not be VCI

Conditions on VCIs

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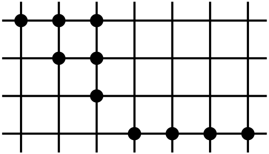
$\leq m$ points collinear horizontally, $\leq n$ vertically

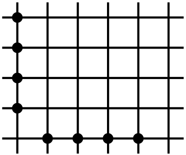
Theorem

If $|X| < mn$ *and* $\gcd(m, n) = 1$ g *has degree:*
$(n^{-1}|X| \mod m, \quad m^{-1}|X| \mod n)$

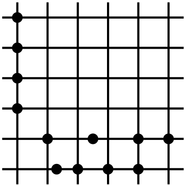


$m = 4$, $n = 3$, $|X| = 10$
g would have degree (2,1)
Impossible, so not VCI

Results in Action



8-point VCI



12-point VCI

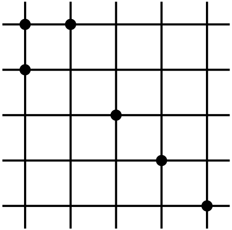
If $|X| < mn, m = 4, n = 4$, the only VCI configurations are as shown:

- By Cross Point Condition, m and n points share no coordinates
- By GCD condition, $|X|$ is 8 or 12
- f has degree $(4, 4)$ and g contains vertical and horizontal form
- If $|X| = 12 = 4c + 4d$, rest of g must be $(1, 0)$ or $(0, 1)$ form
- Each such form must have 4 points of X

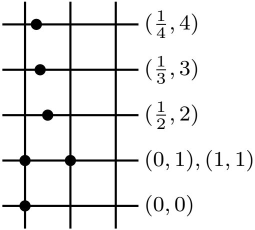
When values of coordinates matter...

Remark

Configuration does not always determine whether a set of points is a VCI. For instance,



In general, not a VCI.



Red:$(2, 1)$; Blue:$(2, 2)$.

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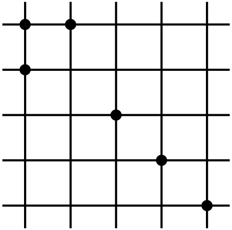

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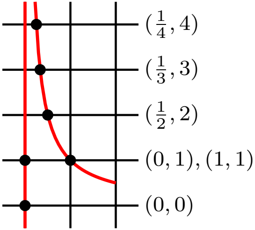

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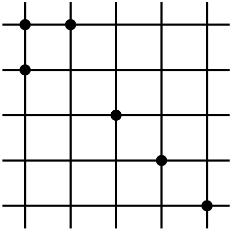

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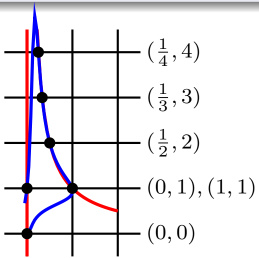

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- Our results:
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- Future work:
  1. Continue Classification
  2. Methods for finding f and g

Acknowledgements

We would like to thank Christine and Mike for their continual guidance, support, and encouragement.
Thank you to the other mentors and TAs for their help in the REU and to the NSF for funding us.