REU 2019 Day 8

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Matrix groups over finite fields (and their representation theory)

The earlier problem #2 was secretly motivated by rep'n theory.

$s_\lambda(\underline{z})$ = Schur polynomials are characters of rep'ns "V_λ" of $GL_n(\mathbb{C})$

(= $n \times n$ invertible matrices with $\mathbb{C}$-entries)

In this project, want to study repns of groups like $GL_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ = finite field with q elements

$$q = p^k,\ p \text{ a prime}$$

Very brief motivation

① They give examples of finite simple groups

no nontrivial normal subgroups

classified (in thousands of pages), finished around 2004, as...

- cyclic groups of prime order
- alternating groups A_n, $n \geq 5$
- finite groups of Lie type
- 26 more examples ("sporadic")

  1860's Mathieu (5)

  ⋮

  ~1980 Griess - "Monster group"

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② Number theory

Diophantine equations

(= ~~integer~~ rational solutions to ~~integer~~ rational coefficient equations) e.g. $E: \ y^2 = x^3 + x + 1$

easier: $E(\mathbb{F}_p)$: (x,y) $y^2 \equiv x^3 + x + 1$ mod p

crazy idea: make a generating function

$$L(E,s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \qquad s \in \mathbb{C}, \ \mathrm{Re}(s) \gg 0$$

$a(p)$ made from $\#E(\mathbb{F}_p)$

$a(n)$'s made from $a(p)$'s

Amazing conjecture
(Birch – Swinnerton-Dyer)

Order of vanishing of $L(E,s)$ at $s=1$ gives (up to finitely many solutions) the # of solutions to $E(\mathbb{Q})$

Known: $E(\mathbb{Q}) = \mathbb{Z}^r \oplus \underbrace{\text{finite group}}_{|\cdot| \le 12}$

Study L-functions from Diophantine equations using L-functions from rep'n theory

made from rep'ns of groups like $GL_n(\mathbb{Q}_p)$

Completion of $\mathbb{Q}$ w.r.t. new abs. value counting divisibility by p

rep'n theory of $GL_n(\mathbb{Q}_p)$ is a lot like rep'n theory of $GL_n(\mathbb{F}_p)$

- What groups will we study?
- What do we know about rep'n theory?

Begin with $GL_2(\mathbb{F}_p) = 2\times 2$ matrices with $\det \neq 0$ in $\mathbb{F}_p$

If G is finite,

- $$\sum_{\rho \text{ irred. rep'n}} |\dim(\rho)|^2 = |G|$$
- # of irred rep'ns of G = # conjugacy classes of G

$GL_2(\mathbb{F}_p) \supset B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ "Borel subgroup"

FACT: (Bruhat decomposition)

$GL_2(\mathbb{F}_p) = B \sqcup BwB$

disjoint union

where $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and $BwB = \{ b_1 w b_2 \mid b_1, b_2 \in B \}$

"double coset"

(so $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ can move to the other B in the double coset)

More generally,

$$GL_n(\mathbb{F}_q) = \bigsqcup_{w \in W} BwB$$

where $B = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}$ = upper $\triangle$ subgroup

W = permutation matrices in $GL_n(\mathbb{F}_q)$

$\cong S_n$ symmetric group

REU EXERCISE 18

(a) Show $GL_2(\mathbb{F}_q) = B \sqcup B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B$

(b) Show that, for $GL_n(\mathbb{F}_q)$,

if $T := \left\{ \begin{pmatrix} * & & 0 \\ & * & \\ & & \ddots \\ 0 & & & * \end{pmatrix} \right\}$ (torus)

normalizer of T in G

$$\text{then } \underbrace{N_G(T)/T}_{= W \text{ Weyl group}} \cong S_n$$

(c) Compute $|GL_2(\mathbb{F}_q)|$

REU EXERCISE 19

(a) Determine the conjugacy classes of $GL_2(\mathbb{F}_p)$

What groups will we study?

A: Groups G with Borel subgroups $B \supseteq T$ a tori (for some defn of B)

so that $G = \bigsqcup_{w \in W} BwB$ with $W = N_G(T)/T$

Which groups W are possible?

__A__: A subset of the finite Coxeter groups.

Coxeter groups are groups generated by reflections.

They have presentations given by Coxeter diagrams:

graphs with vertices $\underset{i}{\bullet}$ $\leftrightarrow$ generating reflection s_i

edges labeled m $\underset{i}{\bullet}\overset{m}{—}\underset{j}{\bullet}$ $\leftrightarrow$ $(s_i s_j)^m = 1$

Hip kids don't write m if m=3

$$s_1 s_2 s_1 = s_2 s_1 s_2$$
$$(\Leftrightarrow (s_1 s_2)^3 = 1)$$

and don't write the edge at all if m=2 $(s_1 s_2 = s_2 s_1)$

There are 4 infinite families of graphs/groups whose graph is connected (irreducible) where the group ends up being finite...

A_n $\quad 1 - 2 - 3 - \cdots - n \quad \cong S_{n+1}$

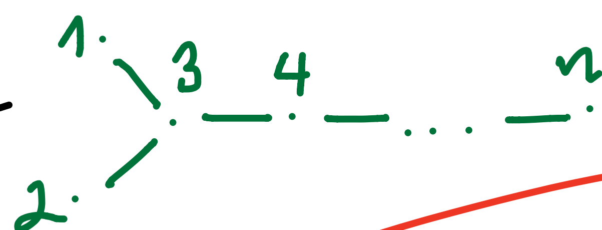
e.g. $A_2 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, \; s_1 s_2 s_1 = s_2 s_1 s_2 \rangle \cong S_3$

$s_1 \longmapsto (1,2)$

$s_2 \longmapsto (2,3)$

B_n / C_n $\quad 1 \overset{4}{-} 2 - 3 - \cdots - n$

D_n



not crystallographic so no Lie group

$\cancel{I_2(m)}$ $\quad 1 \overset{m}{-} 2 \quad$ dihedral groups

$F_4, E_6, E_7, E_8, G_2, \cancel{H_2}, \cancel{H_3}, \cancel{H_4}$

We study groups of this type, having Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$ and W from the above list.

There are nice presentations of these groups using "root systems" acted on by W (see Rehmann reference).

The groups G are really ~~fixed points of Frobenius acting on reductive algebraic groups defined over $\mathbb{F}_p$~~ have a nice presentation in terms of W and its associated root system.

Rep'n theory of such G

E.g. $GL_2(\mathbb{F}_p)$

Pick a (big) abelian subgroup.

Consider $T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$

Its irreducible rep'ns are 1-dim'l

$T \cong (\mathbb{F}_p^\times)^2 \qquad \chi_1, \chi_2 : \mathbb{F}_p^\times \to \mathbb{C}^\times$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = t \qquad \chi(t) = \chi_1(a)\chi_2(b)$

To get more representations, induce up to G, i.e., try $\mathrm{Ind}_T^G(\chi)$.

Not so good, because it's reducible.

Instead do something trickier:

think of χ as rep'n of

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \longrightarrow \mathbb{C}^\times$$

$\forall\, b \in B$, write $b = tu$, $t \in T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$

$u \in U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$

and define $\chi(b) := \chi(t)$.

Now do $\mathrm{Ind}_B^G(\chi)$.
This turns out (not obvious) to be an irreducible rep'n of dimension $[G:B] = p+1$, unless $\chi_1 = \chi_2$.
(then reducible, get 1-dim'l and a p-dim'l irreducible).

REM Exercise 19(b)
Assuming the above facts about $\mathrm{Ind}_B^G(\chi)$, how many irreducible rep'ns of $GL_2(\mathbb{F}_p)$ are left?

Upshot: Begin with characters of torus T
$\rightsquigarrow$ irred. G-rep'ns

Macdonald (1968)
- conjectured there is a correspondence between characters of tori and irreducible rep'ns of reductive split algebraic groups over $\mathbb{F}_q$

(there are more tori inside $GL_2(\mathbb{F}_p)$ and its relatives

e.g. $\left\{ \begin{pmatrix} x & y \\ Dy & x \end{pmatrix} \middle| x^2 - Dy^2 \neq 0 \right\}$ $\sqrt{D} \notin \mathbb{F}_p$ $\leftarrow$ $\cong$ a torus $E^\times$ with $E = \mathbb{F}_p(\sqrt{D})$)

REU Problem 8

Study the rep'n theory of central extensions of $G(\mathbb{F}_q)$ where G is a split connected reductive algebraic group

i.e. $1 \to S \to \tilde{G} \to G(\mathbb{F}_q) \to 1$

turns out to be a finite abelian group

with $S \subseteq Z(\tilde{G})$

"center of"

THM (Steinberg '81)

$\tilde{G} \longrightarrow G(\mathbb{F}_q)$ is trivial

except for 11 exceptions

(G = simply connected, simple algebraic group defined over $\mathbb{F}_q$)

e.g. $A_1(4) \cdot \longleftrightarrow SL_2(\mathbb{F}_4)$

$A_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

(Atlas of Finite Group Rep'ns says

$|F_4(\mathbb{F}_2)| \sim 3 \times 10^{15}$)