Chapter 1

Squarefree monomial ideals

We begin by studying ideals in a polynomial ring $\Bbbk[x_1, \ldots, x_n]$ that are generated by squarefree monomials. Such ideals are also known as *Stanley–Reisner ideals*, and quotients by them are called *Stanley–Reisner rings*. The combinatorial nature of these algebraic objects stems from their intimate connections to simplicial topology. This chapter explores various enumerative and homological manifestations of these topological connections, including simplicial descriptions of *Hilbert series* and *Betti numbers*.

After describing the relation between simplicial complexes and squarefree monomial ideals, this chapter goes on to introduce the objects and notation surrounding both the algebra of general monomial ideals as well as the combinatorial topology of simplicial complexes. Section 1.2 defines what it means for a module over the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ to be *graded by* $\mathbb{N}^n$ and what Hilbert series can look like in these gradings. In preparation for our discussion of Betti numbers in Section 1.5, we review simplicial homology and cohomology in Section 1.3 and free resolutions in Section 1.4. The latter section introduces *monomial matrices*, which allow us to write down $\mathbb{N}^n$-graded free resolutions explicitly.

1.1 Equivalent descriptions

Let $\Bbbk$ be a field and $S = \Bbbk[\mathbf{x}]$ the polynomial ring over $\Bbbk$ in n indeterminates $\mathbf{x} = x_1, \ldots, x_n$.

Definition 1.1 A **monomial** in $\Bbbk[\mathbf{x}]$ is a product $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ of nonnegative integers. An ideal $I \subseteq \Bbbk[\mathbf{x}]$ is called a **monomial ideal** if it is generated by monomials.

As a vector space over $\Bbbk$, the polynomial ring S is a direct sum

$$S \quad = \quad \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S_{\mathbf{a}},$$

where $S_{\mathbf{a}} = \Bbbk\{\mathbf{x}^{\mathbf{a}}\}$ is the vector subspace of S spanned by the monomial $\mathbf{x}^{\mathbf{a}}$. Since the product $S_{\mathbf{a}} \cdot S_{\mathbf{b}}$ of graded pieces equals the graded piece $S_{\mathbf{a}+\mathbf{b}}$ in degree $\mathbf{a}+\mathbf{b}$, we say that S is an $\mathbb{N}^n$-*graded* $\Bbbk$-algebra.

Monomial ideals are the $\mathbb{N}^n$-*graded ideals* of S, which means by definition that I can also be expressed as a direct sum, namely $I = \bigoplus_{\mathbf{x}^{\mathbf{a}}\in I} \Bbbk\{\mathbf{x}^{\mathbf{a}}\}$.

Lemma 1.2 *Every monomial ideal has a unique minimal set of monomial generators, and this set is finite.*

Proof. The Hilbert Basis Theorem says that every ideal in S is finitely generated. It implies that if I is a monomial ideal, then $I = \langle \mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r} \rangle$. The direct sum condition means that a polynomial f lies inside I if and only if each term of f is divisible by one of the given generators $\mathbf{x}^{\mathbf{a}_i}$. $\square$

Definition 1.3 A monomial $\mathbf{x}^{\mathbf{a}}$ is **squarefree** if every coordinate of $\mathbf{a}$ is 0 or 1. An ideal is **squarefree** if it is generated by squarefree monomials.

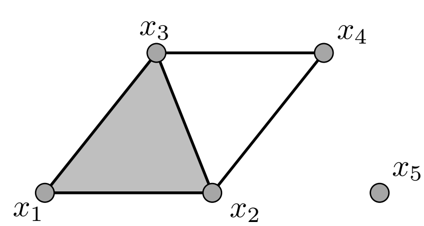
The information carried by squarefree monomial ideals can be characterized in many ways. The most combinatorial uses simplicial complexes.

Definition 1.4 An (abstract) **simplicial complex** Δ on the **vertex set** $\{1, \ldots, n\}$ is a collection of subsets called **faces** or **simplices**, closed under taking subsets; that is, if $\sigma \in \Delta$ is a face and $\tau \subseteq \sigma$, then $\tau \in \Delta$. A simplex $\sigma \in \Delta$ of cardinality $|\sigma| = i + 1$ has **dimension** i and is called an i-**face** of Δ. The **dimension** $\dim(\Delta)$ of Δ is the maximum of the dimensions of its faces, or it is $-\infty$ if $\Delta = \{\}$ is the **void complex**, which has no faces.

The empty set $\varnothing$ is the unique dimension -1 face in any simplicial complex Δ that is not the void complex $\{\}$. Thus the *irrelevant complex* $\{\varnothing\}$, whose unique face is the empty set, is to be distinguished from the void complex. The reason for this distinction will become clear when we introduce (co)homology as well as in numerous applications to monomial ideals.

We frequently identify $\{1, \ldots, n\}$ with the variables $\{x_1, \ldots, x_n\}$, as in our next example, or with $\{a, b, c, \ldots\}$, as in Example 1.8.

Example 1.5 The simplicial complex Δ on $\{1, 2, 3, 4, 5\}$ consisting of all subsets of the sets $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, and $\{5\}$ is pictured below:



The simplicial complex Δ

Note that Δ is completely specified by its *facets*, or maximal faces, by definition of simplicial complex. $\diamond$

Simplicial complexes determine squarefree monomial ideals. For notation, we identify each subset $\sigma \subseteq \{1, \ldots, n\}$ with its *squarefree vector* in $\{0,1\}^n$, which has entry 1 in the i^{th} spot when $i \in \sigma$, and 0 in all other entries. This convention allows us to write $\mathbf{x}^\sigma = \prod_{i\in\sigma} x_i$.

Definition 1.6 The **Stanley–Reisner ideal** of the simplicial complex Δ is the squarefree monomial ideal

$$I_\Delta \quad = \quad \langle \mathbf{x}^\tau \mid \tau \notin \Delta \rangle$$

generated by monomials corresponding to **nonfaces** τ of Δ. The **Stanley–Reisner ring** of Δ is the quotient ring S/I_Δ.

There are two ways to present a squarefree monomial ideal: either by its generators or as an intersection of monomial prime ideals. These are generated by subsets of $\{x_1, \ldots, x_n\}$. For notation, we write

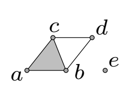
$$\mathfrak{m}^\tau \quad = \quad \langle x_i \mid i \in \tau \rangle$$

for the monomial prime ideal corresponding to τ. Frequently, τ will be the complement $\overline{\sigma} = \{1, \ldots, n\} \smallsetminus \sigma$ of some simplex σ.

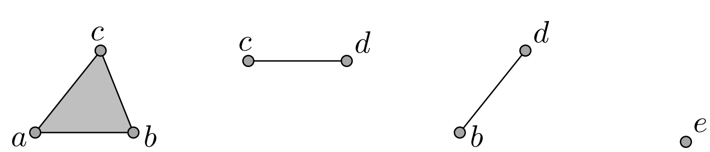
Theorem 1.7 *The correspondence $\Delta \rightsquigarrow I_\Delta$ constitutes a bijection from simplicial complexes on vertices $\{1, \ldots, n\}$ to squarefree monomial ideals inside $S = \Bbbk[x_1, \ldots, x_n]$. Furthermore,*

$$I_\Delta \quad = \quad \bigcap_{\sigma\in\Delta} \mathfrak{m}^{\overline{\sigma}}.$$

Proof. By definition, the set of squarefree monomials that have nonzero images in the Stanley–Reisner ring S/I_Δ is precisely $\{\mathbf{x}^\sigma \mid \sigma \in \Delta\}$. This shows that the map $\Delta \rightsquigarrow I_\Delta$ is bijective. In order for $\mathbf{x}^\tau$ to lie in the intersection $\bigcap_{\sigma\in\Delta} \mathfrak{m}^{\overline{\sigma}}$, it is necessary and sufficient that τ share at least one element with $\overline{\sigma}$ for each face $\sigma \in \Delta$. Equivalently, τ must be contained in no face of Δ; that is, τ must be a nonface of Δ. $\square$

Example 1.8 The simplicial complex $\Delta =$ 

from Example 1.5, after replacing the variables $\{x_1, x_2, x_3, x_4, x_5\}$ by $\{a, b, c, d, e\}$, has Stanley–Reisner ideal



$$\begin{aligned} I_\Delta \quad &= \quad \langle d, e \rangle \quad \cap \quad \langle a, b, e \rangle \quad \cap \quad \langle a, c, e \rangle \quad \cap \quad \langle a, b, c, d \rangle \\ &= \quad \langle ad, ae, bcd, be, ce, de \rangle. \end{aligned}$$

This expresses I_Δ via its prime decomposition and its minimal generators. Above each prime component is drawn the corresponding facet of Δ. $\diamond$

Remark 1.9 Because of the expression of Stanley–Reisner ideals I_Δ as intersections in Theorem 1.7, they are also in bijection with unions of coordinate subspaces in the vector space $\Bbbk^n$, or equivalently, unions of coordinate subspaces in the projective space $\mathbb{P}^{n-1}_\Bbbk$. A little bit of caution is warranted here: if $\Bbbk$ is finite, it is not true that I_Δ equals the ideal of polynomials vanishing on the corresponding collection of coordinate subspaces; in fact, this vanishing ideal will not be a monomial ideal! On the other hand, when $\Bbbk$ is infinite, the Zariski correspondence between radical ideals and algebraic sets does induce the bijection between squarefree monomial ideals and their zero sets, which are unions of coordinate subspaces. (The *zero set* inside $\Bbbk^n$ of an ideal I in $\Bbbk[\mathbf{x}]$ is the set of points $(\alpha_1, \ldots, \alpha_n) \in \Bbbk^n$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$ for every polynomial $f \in I$.)

1.2 Hilbert series

Even if the goal is to study monomial ideals, it is necessary to consider graded modules more general than ideals.

Definition 1.10 An S-module M is $\mathbb{N}^n$**-graded** if $M = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} M_\mathbf{b}$ and $\mathbf{x}^\mathbf{a} M_\mathbf{b} \subseteq M_{\mathbf{a}+\mathbf{b}}$. If the vector space dimension $\dim_\Bbbk(M_\mathbf{a})$ is finite for all $\mathbf{a} \in \mathbb{N}^n$, then the formal power series

$$H(M;\mathbf{x}) \quad = \quad \sum_{\mathbf{a}\in\mathbb{N}^n} \dim_\Bbbk(M_\mathbf{a}) \cdot \mathbf{x}^\mathbf{a}$$

is the **finely graded** or $\mathbb{N}^n$**-graded Hilbert series** of M. Setting $x_i = t$ for all i yields the ($\mathbb{Z}$**-graded** or **coarse**) **Hilbert series** $H(M; t, \ldots, t)$.

The ring of formal power series in which finely graded Hilbert series live is $\mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, \ldots, x_n]]$. In this ring, each element $1 - x_i$ is invertible, the series $\frac{1}{1-x_i} = 1 + x_i + x_i^2 + \cdots$ being its inverse.

Example 1.11 The Hilbert series of S itself is the rational function

$$\begin{aligned} H(S;\mathbf{x}) \quad &= \quad \prod_{i=1}^{n} \frac{1}{1-x_i} \\ &= \quad \text{sum of all monomials in } S. \end{aligned}$$

Denote by $S(-\mathbf{a})$ the free module generated in degree $\mathbf{a}$, so $S(-\mathbf{a}) \cong \langle \mathbf{x}^\mathbf{a} \rangle$ as $\mathbb{N}^n$-graded modules. The Hilbert series

$$H(S(-\mathbf{a});\mathbf{x}) \quad = \quad \frac{\mathbf{x}^\mathbf{a}}{\prod_{i=1}^{n}(1-x_i)}$$

of such an $\mathbb{N}^n$*-graded translate* of S is just $\mathbf{x}^\mathbf{a} \cdot H(S;\mathbf{x})$. $\diamond$

In the rest of Part I, our primary examples of Hilbert series are

$$H(S/I;\mathbf{x}) \quad = \quad \text{sum of all monomials not in } I$$

for monomial ideals I. A running theme of Part I of this book is to analyze not so much the whole Hilbert series, but its numerator, as defined in Definition 1.12. (In fact, Parts II and III are frequently concerned with similar analyses of such numerators, for ideals in other gradings.)

Definition 1.12 If the Hilbert series of an $\mathbb{N}^n$-graded S-module M is expressed as a rational function $H(M;\mathbf{x}) = \mathcal{K}(M;\mathbf{x})/(1-x_1)\cdots(1-x_n)$, then its numerator $\mathcal{K}(M;\mathbf{x})$ is the ***K*-polynomial** of M.

We will eventually see in Corollary 4.20 (but see also Theorem 8.20) that the Hilbert series of every monomial quotient of S can in fact be expressed as a rational function as in Definition 1.12, and therefore every such quotient has a K-polynomial. That these K-polynomials are polynomials (as opposed to Laurent polynomials, say) is also proved in Corollary 4.20. Next we want to show that Stanley–Reisner rings S/I_Δ have K-polynomials by explicitly writing them down in terms of Δ.

Theorem 1.13 *The Stanley–Reisner ring S/I_Δ has the K-polynomial*

$$\mathcal{K}(S/I_\Delta;\mathbf{x}) \quad = \quad \sum_{\sigma\in\Delta}\Big(\prod_{i\in\sigma} x_i \cdot \prod_{j\notin\sigma}(1-x_j)\Big).$$

Proof. The definition of I_Δ says which *squarefree* monomials are not in I_Δ. However, because the generators of I_Δ are themselves squarefree, a monomial $\mathbf{x}^{\mathbf{a}}$ lies outside I_Δ precisely when the squarefree monomial $\mathbf{x}^{\operatorname{supp}(\mathbf{a})}$ lies outside I_Δ, where $\operatorname{supp}(\mathbf{a}) = \{i \in \{1,\ldots,n\} \mid a_i \neq 0\}$ is the *support* of $\mathbf{a}$. Therefore

$$\begin{aligned} H(S/I_\Delta;x_1,\ldots,x_n) &= \sum\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a}\in\mathbb{N}^n \text{ and } \operatorname{supp}(\mathbf{a})\in\Delta\} \\ &= \sum_{\sigma\in\Delta}\sum\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a}\in\mathbb{N}^n \text{ and } \operatorname{supp}(\mathbf{a})=\sigma\} \\ &= \sum_{\sigma\in\Delta}\prod_{i\in\sigma}\frac{x_i}{1-x_i}, \end{aligned}$$

and the result holds after multiplying the summand for σ by $\prod_{j\notin\sigma}\frac{1-x_j}{1-x_j}$ to bring the terms over a common denominator of $(1-x_1)\cdots(1-x_n)$. □

Example 1.14 Consider the simplicial complex Γ depicted in Fig. 1.1. (The reason for not calling it Δ is because we will compare Γ in Example 1.36 with the simplicial complex Δ of Examples 1.5 and 1.8.) The Stanley–Reisner ideal of Γ is

$$\begin{aligned} I_\Gamma &= \langle de, abe, ace, abcd\rangle \\ &= \langle a,d\rangle \cap \langle a,e\rangle \cap \langle b,c,d\rangle \cap \langle b,e\rangle \cap \langle c,e\rangle \cap \langle d,e\rangle, \end{aligned}$$

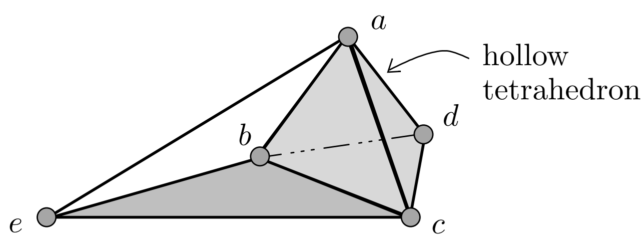


Figure 1.1: The simplicial complex Γ

and the Hilbert series of the quotient $\Bbbk[a,b,c,d,e]/I_\Gamma$ is

$$\begin{aligned} & 1+\tfrac{a}{1-a}+\tfrac{b}{1-b}+\tfrac{c}{1-c}+\tfrac{d}{1-d}+\tfrac{e}{1-e}+\tfrac{ab}{(1-a)(1-b)}+\tfrac{ac}{(1-a)(1-c)} \\ & \quad +\tfrac{ad}{(1-a)(1-d)}+\tfrac{ae}{(1-a)(1-e)}+\tfrac{bc}{(1-b)(1-c)}+\tfrac{bd}{(1-b)(1-d)}+\tfrac{be}{(1-b)(1-e)} \\ & \quad +\tfrac{cd}{(1-c)(1-d)}+\tfrac{ce}{(1-c)(1-e)}+\tfrac{abc}{(1-a)(1-b)(1-c)}+\tfrac{abd}{(1-a)(1-b)(1-d)} \\ & \quad +\tfrac{acd}{(1-a)(1-c)(1-d)}+\tfrac{bcd}{(1-b)(1-c)(1-d)}+\tfrac{bce}{(1-b)(1-c)(1-e)} \\ = & \ \frac{1-abcd-abe-ace-de+abce+abde+acde}{(1-a)(1-b)(1-c)(1-d)(1-e)}. \end{aligned}$$

See Example 1.25 for a hint at a quick way to get this series. $\diamond$

The formula for the Hilbert series of S/I_Δ perhaps becomes a little neater when we coarsen to the $\mathbb{N}$-grading.

Corollary 1.15 *Letting f_i be the number of i-faces of Δ, we get*

$$H(S/I_\Delta;t,\ldots,t) \quad = \quad \frac{1}{(1-t)^n}\sum_{i=0}^{d} f_{i-1}t^i(1-t)^{n-i},$$

where $d = \dim(\Delta)+1$.

Canceling $(1-t)^{n-d}$ from the sum and the denominator $(1-t)^n$ in Corollary 1.15, the numerator polynomial $h(t)$ on the right-hand side of

$$\frac{1}{(1-t)^d}\sum_{i=0}^{d} f_{i-1}\,t^i(1-t)^{d-i} \quad = \quad \frac{h_0+h_1t+h_2t^2+\cdots+h_dt^d}{(1-t)^d}$$

is called the *h-polynomial* of Δ. It and the *f-vector* $(f_{-1},f_0,\ldots,f_{d-1})$ are, to some approximation, the subjects of a whole chapter of Stanley's book [Sta96]; we refer the reader there for further discussion of these topics.

1.3 Simplicial complexes and homology

Much of combinatorial commutative algebra is concerned with analyzing various homological constructions and invariants, and in particular, the manner in which they are governed by combinatorial data. Often, the analysis reduces to related (and hopefully easier) homological constructions purely in the realm of simplicial topology. We review the basics here, referring the reader to [Hat02], [Rot88], or [Mun84] for a full treatment.

Let Δ be a simplicial complex on $\{1,\ldots,n\}$. For each integer i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ, and let $\mathbb{k}^{F_i(\Delta)}$ be a vector space over $\mathbb{k}$ whose basis elements e_σ correspond to i-faces $\sigma \in F_i(\Delta)$.

Definition 1.16 The (**augmented** or **reduced**) **chain complex** of Δ over $\mathbb{k}$ is the complex $\widetilde{\mathcal{C}}_\bullet(\Delta;\mathbb{k})$:

$$0 \longleftarrow \mathbb{k}^{F_{-1}(\Delta)} \xleftarrow{\partial_0} \cdots \longleftarrow \mathbb{k}^{F_{i-1}(\Delta)} \xleftarrow{\partial_i} \mathbb{k}^{F_i(\Delta)} \longleftarrow \cdots \xleftarrow{\partial_{n-1}} \mathbb{k}^{F_{n-1}(\Delta)} \longleftarrow 0.$$

The **boundary maps** ∂_i are defined by setting $\mathrm{sign}(j,\sigma) = (-1)^{r-1}$ if j is the r^{th} element of the set $\sigma \subseteq \{1,\ldots,n\}$, written in increasing order, and

$$\partial_i(e_\sigma) \quad = \quad \sum_{j\in\sigma} \mathrm{sign}(j,\sigma)\, e_{\sigma \smallsetminus j}.$$

If $i < -1$ or $i > n-1$, then $\mathbb{k}^{F_i(\Delta)} = 0$ and $\partial_i = 0$ by definition. The reader unfamiliar with simplicial complexes should make the routine check that $\partial_i \circ \partial_{i+1} = 0$. In other words, the image of the $(i+1)^{\text{st}}$ boundary map ∂_{i+1} lies inside the kernel of the i^{th} boundary map ∂_i.

Definition 1.17 For each integer i, the $\mathbb{k}$-vector space

$$\widetilde{H}_i(\Delta;\mathbb{k}) \quad = \quad \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$$

in **homological degree** i is the i^{th} **reduced homology** of Δ over $\mathbb{k}$.

In particular, $\widetilde{H}_{n-1}(\Delta;\mathbb{k}) = \ker(\partial_{n-1})$, and when Δ is not the irrelevant complex $\{\varnothing\}$, we get also $\widetilde{H}_i(\Delta;\mathbb{k}) = 0$ for $i < 0$ or $i > n-1$. The irrelevant complex $\Delta = \{\varnothing\}$ has homology only in homological degree -1, where $\widetilde{H}_{-1}(\Delta;\mathbb{k}) \cong \mathbb{k}$. The dimension of the zeroth reduced homology $\widetilde{H}_0(\Delta;\mathbb{k})$ as a $\mathbb{k}$-vector space is one less than the number of connected components of Δ. Elements of $\ker(\partial_i)$ are often called *i-cycles* and elements of $\mathrm{im}(\partial_{i+1})$ are often called *i-boundaries*.

Example 1.18 For Δ as in Example 1.5, we have

$$\begin{aligned}
F_2(\Delta) &= \{\{1,2,3\}\},\\
F_1(\Delta) &= \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\},\\
F_0(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\}\},\\
F_{-1}(\Delta) &= \{\varnothing\}.
\end{aligned}$$

Ordering the bases for $\Bbbk^{F_i(\Delta)}$ as suggested by the ordering of the faces listed above, the chain complex for Δ becomes

$$0 \longleftarrow \Bbbk \xleftarrow[\partial_0]{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}} \Bbbk^5 \xleftarrow[\partial_1]{\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \Bbbk^5 \xleftarrow[\partial_2]{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \Bbbk \longleftarrow 0,$$

where vectors in $\Bbbk^{F_i(\Delta)}$ are viewed as columns of length $f_i = |F_i(\Delta)|$. For example, $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} - e_{\{1,3\}} + e_{\{1,2\}}$, which we identify with the vector $(1, -1, 1, 0, 0)$. The homomorphisms ∂_2 and ∂_0 both have rank 1 (that is, they are injective and surjective, respectively). Since the matrix ∂_1 has rank 3, we conclude that $\widetilde{H}_0(\Delta; \Bbbk) \cong \widetilde{H}_1(\Delta; \Bbbk) \cong \Bbbk$, and the other homology groups are 0. Geometrically, $\widetilde{H}_0(\Delta; \Bbbk)$ is nontrivial because Δ is disconnected, and $\widetilde{H}_1(\Delta; \Bbbk)$ is nontrivial because Δ contains a triangle that does not bound a face of Δ. $\diamond$

Remark 1.19 We would avoid making such a big deal about the difference between the irrelevant complex $\{\varnothing\}$ and the void complex $\{\}$ if it did not come up so much. Many of the formulas for Betti numbers, dimensions of local cohomology, and so on depend on the fact that $\widetilde{H}_i(\{\varnothing\}; \Bbbk)$ is nonzero for $i = -1$, whereas $\widetilde{H}_i(\{\}; \Bbbk) = 0$ for all i.

In some situations, the notion dual to homology arises more naturally. In what follows, we write $(\_)^*$ for vector space duality $\mathrm{Hom}_\Bbbk(\_, \Bbbk)$.

Definition 1.20 The (**reduced**) **cochain complex** of Δ over $\Bbbk$ is the vector space dual $\widetilde{\mathcal{C}}^\bullet(\Delta; \Bbbk) = (\widetilde{\mathcal{C}}_\bullet(\Delta; \Bbbk))^*$ of the chain complex, with **coboundary maps** $\partial^i = \partial_i^*$. For $i \in \mathbb{Z}$, the $\Bbbk$-vector space

$$\widetilde{H}^i(\Delta; \Bbbk) \quad = \quad \ker(\partial^{i+1})/\mathrm{im}(\partial^i)$$

is the i^{th} **reduced cohomology** of Δ over $\Bbbk$.

Explicitly, let $\Bbbk^{F_i^*(\Delta)} = (\Bbbk^{F_i(\Delta)})^*$ have basis $F_i^*(\Delta) = \{e_\sigma^* \mid \sigma \in F_i(\Delta)\}$ dual to the basis of $\Bbbk^{F_i(\Delta)}$. Then

$$0 \longrightarrow \Bbbk^{F_{-1}^*(\Delta)} \xrightarrow{\partial^0} \cdots \longrightarrow \Bbbk^{F_{i-1}^*(\Delta)} \xrightarrow{\partial^i} \Bbbk^{F_i^*(\Delta)} \longrightarrow \cdots \xrightarrow{\partial^{n-1}} \Bbbk^{F_{n-1}^*(\Delta)} \longrightarrow 0$$

is the cochain complex $\widetilde{\mathcal{C}}^\bullet(\Delta; \Bbbk)$ of Δ, where for an $(i-1)$-face σ,

$$\partial^i(e_\sigma^*) \quad = \quad \sum_{\substack{j \notin \sigma \\ j \cup \sigma \in \Delta}} \mathrm{sign}(j, \sigma \cup j)\, e_{\sigma \cup j}^*$$

is the transpose of ∂_i.

Since $\mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$ takes exact sequences to exact sequences, there is a canonical isomorphism $\widetilde{H}^i(\Delta; \Bbbk) = \widetilde{H}_i(\Delta; \Bbbk)^*$. Elements of $\ker(\partial^{i+1})$ are called *i-cocycles* and elements of $\mathrm{im}(\partial^i)$ are called *i-coboundaries*.

Example 1.21 The cochain complex for Δ as in Example 1.18 is exactly the same as the chain complex there, except that the arrows should be reversed and the elements of the vector spaces should be considered as row vectors, with the matrices acting by multiplication on the right. The nonzero reduced cohomology of Δ is $\widetilde{H}^0(\Delta; \Bbbk) \cong \widetilde{H}^1(\Delta; \Bbbk) \cong \Bbbk$. $\diamond$

1.4 Monomial matrices

The central homological objects in Part I of this book, as well as in Chapter 9, are free resolutions. To begin, a *free S-module* of finite rank is a direct sum $F \cong S^r$ of copies of S, for some nonnegative integer r. In our combinatorial context, F will usually be $\mathbb{N}^n$-graded, which means that $F \cong S(-\mathbf{a}_1) \oplus \cdots \oplus S(-\mathbf{a}_r)$ for some vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathbb{N}^n$. A sequence

$$\mathcal{F}_{\bullet}: \quad 0 \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{\ell-1} \xleftarrow{\phi_\ell} F_\ell \longleftarrow 0 \tag{1.1}$$

of maps of free S-modules is a *complex* if $\phi_i \circ \phi_{i+1} = 0$ for all i. The complex is *exact* in homological degree i if $\ker(\phi_i) = \mathrm{im}(\phi_{i+1})$. When the free modules F_i are $\mathbb{N}^n$-graded, we require that each homomorphism ϕ_i be degree-preserving (or $\mathbb{N}^n$*-graded of degree* $\mathbf{0}$), so that it takes elements in F_i of degree $\mathbf{a} \in \mathbb{N}^n$ to degree $\mathbf{a}$ elements in F_{i-1}.

Definition 1.22 A complex $\mathcal{F}_{\bullet}$ as in (1.1) is a **free resolution** of a module M over $S = \Bbbk[x_1, \ldots, x_n]$ if $\mathcal{F}_{\bullet}$ is exact everywhere except in homological degree 0, where $M = F_0/\mathrm{im}(\phi_1)$. The image in F_i of the homomorphism ϕ_{i+1} is the i^{th} **syzygy module** of M. The **length** of the resolution is the greatest homological degree of a nonzero module in the resolution; this equals ℓ in (1.1), assuming $F_\ell \neq 0$.

Often we *augment* the free resolution $\mathcal{F}_{\bullet}$ by placing $0 \leftarrow M \xleftarrow{\phi_0} F_0$ at its left end instead, to make the complex exact everywhere.

The Hilbert Syzygy Theorem says that every module M over the polynomial ring S has a free resolution with length at most n. In cases that interest us here, $M = S/I$ is $\mathbb{N}^n$-graded, so it has an $\mathbb{N}^n$-graded free resolution. Indeed, the kernel of an $\mathbb{N}^n$-graded module map is $\mathbb{N}^n$-graded, so the syzygy modules—and hence the whole free resolution—of S/I are automatically $\mathbb{N}^n$-graded. Before giving examples, it would help to be able to write down maps between $\mathbb{N}^n$-graded free modules efficiently. To do this, we offer the following definition, in which the "$\succeq$" symbol is used to denote the partial order on $\mathbb{N}^n$ in which $\mathbf{a} \succeq \mathbf{b}$ if $a_i \geq b_i$ for all $i \in \{1, \ldots, n\}$.

Definition 1.23 A **monomial matrix** is an array of **scalar entries** λ_{qp} whose columns are labeled by **source degrees** $\mathbf{a}_p$, whose rows are labeled by **target degrees** $\mathbf{a}_q$, and whose entry $\lambda_{qp} \in \mathbb{k}$ is zero unless $\mathbf{a}_p \succeq \mathbf{a}_q$.

The general monomial matrix represents a map that looks like

$$\bigoplus_q S(-\mathbf{a}_q) \xleftarrow{\begin{array}{c|c} & \cdots\ \mathbf{a}_p\ \cdots \\ \hline \vdots & \\ \mathbf{a}_q & \left[\quad \lambda_{qp} \quad\right] \\ \vdots & \end{array}} \bigoplus_p S(-\mathbf{a}_p).$$

Sometimes we label the rows and columns with monomials $\mathbf{x}^{\mathbf{a}}$ instead of vectors $\mathbf{a}$. The scalar entry λ_{qp} indicates that the basis vector of $S(-\mathbf{a}_p)$ should map to an element that has coefficient λ_{qp} on the monomial that is $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}$ times the basis vector of $S(-\mathbf{a}_q)$. Observe that this monomial sits in degree $\mathbf{a}_p$, just like the basis vector of $S(-\mathbf{a}_p)$. The requirement $\mathbf{a}_p \succeq \mathbf{a}_q$ precisely guarantees that $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}$ has nonnegative exponents.

When the maps in a free resolution are written using monomial matrices, the top border row (source degrees $\mathbf{a}_p$) on a monomial matrix for ϕ_i equals the left border column (target degrees $\mathbf{a}_q$) on a monomial matrix for ϕ_{i+1}.

Each $\mathbb{N}^n$-graded free module can also be regarded as an ungraded free module, and most readers will have seen already matrices used for maps of (ungraded) free modules over arbitrary rings. In order to recover the more usual notation, simply replace each matrix entry λ_{qp} by $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}\lambda_{qp}$, and then forget the border row and column. Because of the conditions defining monomial matrices, $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}\lambda_{qp} \in S$ for all p and q.

Definition 1.24 A monomial matrix is **minimal** if $\lambda_{qp} = 0$ when $\mathbf{a}_p = \mathbf{a}_q$. A homomorphism of free modules, or a complex of such, is **minimal** if it can be written down with minimal monomial matrices.

Given that $\mathbb{N}^n$-graded free resolutions exist, it is not hard to show (by "pruning" the nonzero entries λ_{qp} for which $\mathbf{a}_p = \mathbf{a}_q$) that every finitely generated graded module possesses a minimal free resolution. In fact, minimal free resolutions are unique up to isomorphism. For more details on these issues, see Exercises 1.10 and 1.11; for a full treatment, see [Eis95, Theorem 20.2 and Exercise 20.1].

Minimal free resolutions are characterized by having scalar entry $\lambda_{qp} = 0$ whenever $\mathbf{a}_p = \mathbf{a}_q$ in any of their monomial matrices. If the monomial matrices are made ungraded as above, this simply means that the nonzero entries in the matrices are nonconstant monomials (with coefficients), so it agrees with the usual notion of minimality for $\mathbb{N}$-graded resolutions.

Example 1.25 Let Γ be the simplicial complex from Example 1.14. The Stanley–Reisner ring S/I_Γ has minimal free resolution

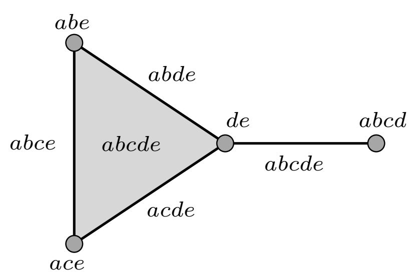
$$0 \leftarrow \underset{00000}{S} \xleftarrow{\begin{array}{cc} & \begin{matrix} de & abe & ace & abcd \end{matrix} \\ 1 & \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \end{array}} \underset{\substack{00011\\11001\\10101\\11110}}{S^4} \xleftarrow{\begin{array}{cc} & \begin{matrix} abce & abde & acde & abcde \end{matrix} \\ \begin{matrix} de \\ abe \\ ace \\ abcd \end{matrix} & \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}} \underset{\substack{11101\\11011\\10111\\11111}}{S^4} \xleftarrow{\begin{array}{cc} & abcde \\ \begin{matrix} abce \\ abde \\ acde \\ abcde \end{matrix} & \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \end{array}} \underset{11111}{S} \leftarrow 0$$

in which the maps are denoted by monomial matrices. We have used the more succinct monomial labels $\mathbf{x}^{\mathbf{a}_p}$ and $\mathbf{x}^{\mathbf{a}_q}$ instead of the vector labels $\mathbf{a}_p$ and $\mathbf{a}_q$. Below each free module is a list of the degrees in $\mathbb{N}^5$ of its generators. For an example of how to recover the usual matrix notation for maps of free S-modules, this free resolution can be written as

$$0 \leftarrow S \xleftarrow{\begin{bmatrix} de & abe & ace & abcd \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} 0 & -ab & -ac & -abc \\ c & d & 0 & 0 \\ -b & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} -d \\ c \\ -b \\ 0 \end{bmatrix}} S \leftarrow 0,$$

without the border entries and forgetting the grading.

As a preview to Chapter 4, the reader is invited to figure out how the labeled simplicial complex below corresponds to the above free resolution.



Hint: Compare the free resolution and the labeled simplicial complex with the numerator of the Hilbert series in Example 1.14. ◇

Recall that in reduced chain complexes of simplicial complexes, the basis vectors are called e_σ for subsets $\sigma \subseteq \{1, \ldots, n\}$.

Definition 1.26 The **Koszul complex** is the complex $\mathbb{K}_{\bullet}$ of free modules given by monomial matrices as follows: in the reduced chain complex of the simplex consisting of all subsets of $\{1, \ldots, n\}$, label the column and the row corresponding to e_σ by σ itself (or $\mathbf{x}^\sigma$), and renumber the homological degrees so that the empty set $\varnothing$ sits in homological degree 0.

Example 1.27 The Koszul complex for $n = 3$ is

$$\mathbb{K}_\bullet : \quad 0 \longleftarrow S \xleftarrow{\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ 1 & \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array}} S^3 \xleftarrow{\begin{array}{c c} & \begin{array}{ccc} yz & xz & xy \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \end{array}} S^3 \xleftarrow{\begin{array}{c c} & xyz \\ \begin{array}{c} yz \\ xz \\ xy \end{array} & \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{array}} S \longleftarrow 0$$

after replacing the variables $\{x_1, x_2, x_3\}$ by $\{x, y, z\}$. $\diamond$

The method of proof for many statements about resolutions of monomial ideals is to determine what happens in each $\mathbb{N}^n$-graded degree of a complex of S-modules. To illustrate, we do this now for $\mathbb{K}_\bullet$ in some detail.

Proposition 1.28 *The Koszul complex* $\mathbb{K}_\bullet$ *is a minimal free resolution of* $\Bbbk = S/\mathfrak{m}$ *for the maximal ideal* $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$*.*

Proof. The essential observation is that a free module generated by 1_τ in squarefree degree τ is nonzero in squarefree degree σ precisely when $\tau \subseteq \sigma$ (equivalently, when $\mathbf{x}^\tau$ divides $\mathbf{x}^\sigma$). The only contribution to the degree $\mathbf{0}$ part of $\mathbb{K}_\bullet$, for example, comes from the free module corresponding to $\varnothing$, whose basis vector $1_\varnothing$ sits in degree $\mathbf{0}$.

More generally, for $\mathbf{b} \in \mathbb{N}^n$ with support σ, the degree $\mathbf{b}$ part $(\mathbb{K}_\bullet)_\mathbf{b}$ of the complex $\mathbb{K}_\bullet$ comes from those rows and columns labeled by faces of σ. In other words, we restrict $\mathbb{K}_\bullet$ to its degree $\mathbf{b}$ part by ignoring summands $S \cdot 1_\tau$ for which τ is not a face of σ. Therefore, $(\mathbb{K}_\bullet)_\mathbf{b}$ is, as a complex of $\Bbbk$-vector spaces, precisely equal to the reduced chain complex of the simplex σ! This explains why the homology of $\mathbb{K}_\bullet$ is just $\Bbbk$ in degree $\mathbf{0}$ and zero elsewhere: a simplex σ is contractible, so it has no reduced homology—that is, unless $\sigma = \{\varnothing\}$ is the irrelevant complex (see Remark 1.19). $\square$

1.5 Betti numbers

Since every free resolution of an $\mathbb{N}^n$-graded module M contains a minimal resolution as a subcomplex (Exercise 1.11), minimal resolutions of M are characterized by having the ranks of their free modules F_i all simultaneously minimized, among free resolutions (1.1) of M.

Definition 1.29 If the complex $\mathcal{F}_\bullet$ in (1.1) is a minimal free resolution of a finitely generated $\mathbb{N}^n$-graded module M and $F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$, then the i^{th} **Betti number** of M in degree $\mathbf{a}$ is the invariant $\beta_{i,\mathbf{a}} = \beta_{i,\mathbf{a}}(M)$.

There are other equivalent ways to describe the $\mathbb{N}^n$-graded Betti number $\beta_{i,\mathbf{a}}(M)$. For example, it measures the minimal number of generators required in degree $\mathbf{a}$ for any i^{th} syzygy module of M. A more natural (by which we mean functorial) characterization of Betti numbers uses tensor products and Tor, which we now review in some detail.

If M and N are $\mathbb{N}^n$-graded modules, then their tensor product $N \otimes_S M$ is $\mathbb{N}^n$-graded, with degree $\mathbf{c}$ component $(N \otimes_S M)_\mathbf{c}$ generated by all elements $f_\mathbf{a} \otimes g_\mathbf{b}$ such that $f_\mathbf{a} \in N_\mathbf{a}$ and $g_\mathbf{b} \in M_\mathbf{b}$ satisfy $\mathbf{a} + \mathbf{b} = \mathbf{c}$. For example, $S(-\mathbf{a}) \otimes_S M$ is a module denoted by $M(-\mathbf{a})$ and called the $\mathbb{N}^n$-*graded translate* of M by $\mathbf{a}$. Its degree $\mathbf{b}$ component is $M(-\mathbf{a})_\mathbf{b} = 1_\mathbf{a} \otimes M_{\mathbf{b}-\mathbf{a}}$, where $1_\mathbf{a}$ is a basis vector for $S(-\mathbf{a})$, so that $S \cdot 1_\mathbf{a} = S(-\mathbf{a})$. In particular, $S(-\mathbf{a}) \otimes_S \Bbbk$ is a copy $\Bbbk(-\mathbf{a})$ of the vector space $\Bbbk$ in degree $\mathbf{a} \in \mathbb{N}^n$.

Example 1.30 Tensoring the minimal free resolution in Example 1.25 with $\Bbbk = S/\mathfrak{m}$ yields a complex

$$
\begin{array}{ccccccccc}
0 \longleftarrow & \Bbbk & \longleftarrow & \Bbbk^4 & \longleftarrow & \Bbbk^4 & \longleftarrow & \Bbbk & \longleftarrow 0 \\
 & 00000 & & 00011 & & 11101 & & 11111 & \\
 & & & 11001 & & 11011 & & & \\
 & & & 10101 & & 10111 & & & \\
 & & & 11110 & & 11111 & & &
\end{array}
$$

of S-modules, each of which is a direct sum of translates of $\Bbbk$, and where all the maps are zero. The translation vectors, which are listed below each direct sum, are identified with the row labels to the right of the corresponding free module in Example 1.25, or the column labels to the left. $\diamond$

The modules $\operatorname{Tor}_i^S(M, N)$ are by definition calculated by applying $\_ \otimes N$ to a free resolution of M and taking homology [Wei94, Definition 2.6.4]. However, it is a general theorem from homological algebra (see [Wei94, Application 5.6.3] or do Exercise 1.12) that $\operatorname{Tor}_i^S(M, N)$ can also be calculated by applying $M \otimes \_$ to a free resolution of N and taking homology. When both M and N are $\mathbb{N}^n$-graded, we can choose the free resolutions to be $\mathbb{N}^n$-graded, so the Tor modules are also $\mathbb{N}^n$-graded.

Example 1.31 The homology of the complex in Example 1.30 is the complex itself, considered as a homologically and $\mathbb{N}^n$-graded module. By definition, this module is $\operatorname{Tor}_\bullet^S(S/I_\Gamma, \Bbbk)$. It agrees with the result of tensoring the Koszul complex with S/I_Γ, where again Γ is the simplicial complex from Examples 1.25 and 1.14. The reader is encouraged to check this explicitly, but we shall make this calculation abstractly in the proof of Corollary 5.12. $\diamond$

Now we can see that Betti numbers tell us the vector space dimensions of certain Tor modules.

Lemma 1.32 *The i^{th} Betti number of an $\mathbb{N}^n$-graded module M in degree $\mathbf{a}$ equals the vector space dimension* $\dim_\Bbbk \operatorname{Tor}_i^S(\Bbbk, M)_\mathbf{a}$.

Proof. Tensoring a minimal free resolution of M with $\Bbbk = S/\mathfrak{m}$ turns all of the differentials ϕ_i into zero maps. $\square$

There is no general formula for the maps in a minimal free resolution of an arbitrary squarefree monomial ideal I_Δ. However, we can figure out what its Betti numbers are in terms of simplicial topology. More generally, we can get simplicial formulas for Betti numbers of quotients by arbitrary monomial ideals.

Definition 1.33 For a monomial ideal I and a degree $\mathbf{b} \in \mathbb{N}^n$, define

$$K^{\mathbf{b}}(I) \quad = \quad \{\text{squarefree vectors } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$$

to be the **(upper) Koszul simplicial complex** of I in degree $\mathbf{b}$.

Theorem 1.34 *Given a vector* $\mathbf{b} \in \mathbb{N}^n$*, the Betti numbers of* I *and* S/I *in degree* $\mathbf{b}$ *can be expressed as*

$$\beta_{i,\mathbf{b}}(I) \quad = \quad \beta_{i+1,\mathbf{b}}(S/I) \quad = \quad \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{b}}(I); \Bbbk).$$

Proof. For the first equality, use a minimal free resolution of I achieved by snipping off the copy of S occurring in homological degree 0 of a minimal free resolution of S/I. To equate $\beta_{i,\mathbf{b}}(I)$ with the dimension of the indicated homology, use Lemma 1.32 and Proposition 1.28 to write $\beta_{i,\mathbf{b}}(I)$ as the vector space dimension of the i^{th} homology of the complex $\mathbb{K}_\bullet \otimes I$ in $\mathbb{N}^n$-graded degree $\mathbf{b}$. Then calculate this homology as follows.

Since I is a submodule of S, the complex in degree $\mathbf{b}$ of $\mathbb{K}_\bullet \otimes_S I$ is naturally a subcomplex of $(\mathbb{K}_\bullet)_{\mathbf{b}}$, which we saw in the proof of Proposition 1.28 is the reduced chain complex of the simplex with facet $\sigma = \operatorname{supp}(\mathbf{b})$. It suffices to identify which faces of σ contribute $\Bbbk$-basis vectors to $(\mathbb{K}_\bullet)_{\mathbf{b}}$.

The summand of $\mathbb{K}_\bullet$ corresponding to a squarefree vector τ is a free S-module of rank 1 generated in degree τ. Tensoring this summand with I yields $I(-\tau)$, which contributes a nonzero vector space to degree $\mathbf{b}$ if and only if I is nonzero in degree $\mathbf{b} - \tau$, which is equivalent to $\mathbf{x}^{\mathbf{b}-\tau} \in I$. □

In the special case of squarefree ideals, the Koszul simplicial complexes have natural interpretations in terms of a simplicial complex closely related to Δ. In fact, the simplicial complex we are about to introduce is determined just as naturally from the data defining Δ as is Δ itself.

Definition 1.35 The squarefree **Alexander dual** of $I = \langle \mathbf{x}^{\sigma_1}, \ldots, \mathbf{x}^{\sigma_r} \rangle$ is

$$I^\star \quad = \quad \mathfrak{m}^{\sigma_1} \cap \cdots \cap \mathfrak{m}^{\sigma_r}.$$

If Δ is a simplicial complex and $I = I_\Delta$ its Stanley–Reisner ideal, then the simplicial complex $\Delta^\star$ **Alexander dual** to Δ is defined by $I_{\Delta^\star} = I_\Delta^\star$.

Example 1.36 The Stanley–Reisner ideals I_Δ and I_Γ from Examples 1.8 and 1.14 are Alexander dual; their generators and irreducible components are arranged to make this clear. ◇

The following is a direct description of the Alexander dual simplicial complex. Recall that $\overline{\sigma} = \{1, \ldots, n\} \smallsetminus \sigma$ is the complement of σ in the vertex set.

Proposition 1.37 *If Δ is a simplicial complex, then its Alexander dual is $\Delta^\star = \{\overline{\tau} \mid \tau \notin \Delta\}$, consisting of the complements of the nonfaces of Δ.*

Proof. By Definition 1.6, $I_\Delta = \langle \mathbf{x}^\tau \mid \tau \notin \Delta \rangle$, so $I_{\Delta^\star} = \bigcap_{\tau \notin \Delta} \mathfrak{m}^\tau$ by Definition 1.35. However, this intersection equals $\bigcap_{\overline{\tau} \in \Delta^\star} \mathfrak{m}^\tau$ by Theorem 1.7, so we conclude that $\tau \notin \Delta$ if and only if $\overline{\tau} \in \Delta^\star$, as desired. □

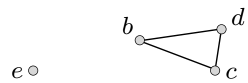
Specializing Theorem 1.34 to squarefree ideals requires one more notion.

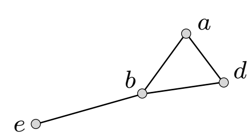
Definition 1.38 The **link** of σ inside the simplicial complex Δ is

$$\mathrm{link}_\Delta(\sigma) \quad = \quad \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\},$$

the set of faces that are disjoint from σ but whose unions with σ lie in Δ.

Example 1.39 Consider the simplicial complex Γ from Examples 1.14 and 1.25, depicted in Fig. 1.1. The link of the vertex a in Γ consists of the vertex e along with all proper faces of the triangle $\{b, c, d\}$. The link of the vertex c in Γ is pure of dimension 1, its four facets being the three edges of the triangle $\{a, b, d\}$ plus the extra edge $\{b, e\}$ sticking out.

$\mathrm{link}_\Gamma(a) \;=\;$ 

$\mathrm{link}_\Gamma(c) \;=\;$ 

The simplicial complex $\mathrm{link}_\Gamma(e)$ consists of the vertex a along with the edge $\{b, c\}$ and its subsets. The link of the edge $\{b, c\}$ in Γ consists of the three remaining vertices: $\mathrm{link}_\Gamma(\{b, c\}) = \{\varnothing, a, d, e\}$. The link in Γ of the edge through a and e is the irrelevant complex: $\mathrm{link}_\Gamma(\{a, e\}) = \{\varnothing\}$. ◇

The next result is called the "dual version" of Hochster's formula because it gives Betti numbers of I_Δ by working with the Alexander dual complex $\Delta^\star$, and because it is dual to Hochster's original formulation, which we will see in Corollary 5.12.

Corollary 1.40 (Hochster's formula, dual version) *All nonzero Betti numbers of I_Δ and S/I_Δ lie in squarefree degrees σ, where*

$$\beta_{i,\sigma}(I_\Delta) \quad = \quad \beta_{i+1,\sigma}(S/I_\Delta) \quad = \quad \dim_{\Bbbk} \widetilde{H}_{i-1}(\mathrm{link}_{\Delta^\star}(\overline{\sigma}); \Bbbk).$$

Proof. For squarefree degrees, apply Theorem 1.34 by first checking that $K^\sigma(I_\Delta) = \mathrm{link}_{K^{\mathbf{1}}(I_\Delta)}(\overline{\sigma})$ and then verifying that $K^{\mathbf{1}}(I_\Delta) = \Delta^\star$. Both of these claims are straightforward from the definitions and hence omitted. For degrees $\mathbf{b}$ with $b_i \geq 2$, the monomial $\mathbf{x}^{\mathbf{b}-(\tau \cup i)}$ lies in I_Δ if and only if $\mathbf{x}^{\mathbf{b}-\tau}$ does. This means that $K^{\mathbf{b}}(I_\Delta)$ is a cone with vertex i. Cones, being contractible, have zero homology (see [Wei94, Section 1.5], for example). □

We will have a lot more to say about Alexander duality in Chapter 5. The interested reader may even wish to skip directly to Sections 5.1, 5.2, and 5.5 (except the end), as these require no additional prerequisites.

Remark 1.41 Since we are working over a field $\Bbbk$, one may substitute reduced homology for reduced cohomology when calculating Betti numbers, since these have the same dimension.

Exercises

1.1 Let $n = 6$ and let Δ be the boundary of an octahedron.

(a) Determine I_Δ and $I_\Delta^\star$.
(b) Compute their respective Hilbert series.
(c) Compute their minimal free resolutions.
(d) Interpret the Betti numbers in part (c) in terms of simplicial homology.

1.2 Suppose that $\mathbf{x}^\mathbf{b}$ is not the least common multiple of some subset of the minimal monomial generators of I. Explain why $K^\mathbf{b}(I)$ is the cone over some subcomplex. Conclude that all nonzero Betti numbers of I occur in $\mathbb{N}^n$-graded degrees $\mathbf{b}$ for which $\mathbf{x}^\mathbf{b}$ equals a least common multiple of some minimal generators.

1.3 Fix a simplicial complex Δ. Exhibit a monomial ideal I and a degree $\mathbf{b}$ in $\mathbb{N}^n$ such that $\Delta = K^\mathbf{b}(I)$ is a Koszul simplicial complex. Is your ideal I squarefree?

1.4 Fix a set of monomials in $x_1, \ldots, x_n$, and let $I(\Bbbk)$ be the ideal they generate in $S = \Bbbk[x_1, \ldots, x_n]$, for varying fields $\Bbbk$.

(a) Can the $\mathbb{N}^n$-graded Hilbert series of $I(\Bbbk)$ depend on the characteristic of $\Bbbk$?
(b) Is the same true for Betti numbers instead of Hilbert series?
(c) Show that the Betti numbers of $S/I(\Bbbk)$ in homological degrees 0, 1, 2, and n are independent of $\Bbbk$.
(d) Prove that all Betti numbers of $S/I(\Bbbk)$ in homological degrees 0, 1, and n lie in distinct $\mathbb{N}^n$-graded degrees. Why is 2 not on this list? Give an example.

1.5 Let $\Bbbk = \mathbb{C}$ be the field of complex numbers. For each monomial $\mathbf{x}^\mathbf{a} \in \mathbb{C}[\mathbf{x}]$, the exponent vector $\mathbf{a}$ can be considered as a vector in $\mathbb{C}^n$. Show that $\mathbf{a}$ lies in the zero set of a Stanley–Reisner ideal I_Δ if and only if $\mathbf{x}^\mathbf{a}$ is nonzero in $\mathbb{C}[\mathbf{x}]/I_\Delta$.

1.6 For a monomial ideal $I = \langle m_1, \ldots, m_r \rangle$ and integers $t \geq 1$, the **Frobenius powers** of I are the ideals $I^{[t]} = \langle m_1^t, \ldots, m_r^t \rangle$. Given a simplicial complex Δ, write an expression for the K-polynomial of $S/I_\Delta^{[2]}$. What about $S/I_\Delta^{[3]}$? $S/I_\Delta^{[t]}$?

1.7 Is there a way to construct monomial matrices for a (minimal) free resolution of $I^{[t]}$ starting with monomial matrices for a (minimal) free resolution of I?

1.8 Let Δ be as in Examples 1.5 and 1.8. Use the links in Example 1.39 to compute as many nonzero Betti numbers of I_Δ as possible.

1.9 Which links in the simplicial complex Δ from Example 1.5 have nonzero homology? Verify your answer using Hochster's formula by comparing it to the Betti numbers of S/I_Γ that appear in Examples 1.25 and 1.30.

1.10 Suppose that ϕ is a nonminimal $\mathbb{N}^n$-graded homomorphism of free modules. Show that ϕ can be represented by a block diagonal monomial matrix Λ in which one of its blocks is a nonzero 1×1 matrix with equal row and column labels.

1.11 Using the fact that every $\mathbb{N}^n$-graded module M has a finite $\mathbb{N}^n$-graded free resolution, deduce from Exercise 1.10 that every $\mathbb{N}^n$-graded free resolution of M is the direct sum of a minimal free resolution of M and a free resolution of zero.

1.12 This exercise provides a direct proof that $\mathrm{Tor}_i^S(M,N) \cong \mathrm{Tor}_i^S(N,M)$. Let $\mathcal{F}$ and $\mathcal{G}$ be free resolutions of M and N, respectively, with differentials ϕ and ψ. Denote by $\mathcal{F} \otimes \mathcal{G}$ the free module $\bigoplus_{i,j} F_i \otimes G_j$, and think of the summands as lying in a rectangular array, with $F_i \otimes G_j$ in row i and column j.

(a) Explain why the *horizontal differential* $(-1)^i \otimes \psi$ on row i of $\mathcal{F} \otimes \mathcal{G}$, induced by ψ on $\mathcal{G}$ and multiplication by ± 1 on F_i, makes $F_i \otimes \mathcal{G}$ into a free resolution of $F_i \otimes N$. (The sign $(-1)^i$ is innocuous, but is needed for ∂, defined next.)

(b) Define a *total differential* ∂ on $\mathcal{F} \otimes \mathcal{G}$ by requiring that

$$\partial(f \otimes g) \quad = \quad \phi_i(f) \otimes g \ + \ (-1)^i f \otimes \psi_j(g)$$

for $f \in F_i$ and $g \in G_j$. Show that $\partial^2 = 0$, so we get a *total complex* $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G})$ by setting $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G})_k = \bigoplus_{i+j=k} F_i \otimes G_j$ in homological degree k.

(c) Prove that the map $\mathcal{F} \otimes \mathcal{G} \to \mathcal{F} \otimes N$ that kills $F_i \otimes G_j$ for $j > 0$ and maps $F_i \otimes G_0 \twoheadrightarrow F_i \otimes N$ induces a morphism $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{F} \otimes N$ of *complexes*, where the i^{th} differential on $\mathcal{F} \otimes N$ is the map $\phi_i \otimes 1$ induced by ϕ.

(d) Using the exactness of the horizontal differential, verify that the morphism $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{F} \otimes N$ induces an isomorphism on homology. (The arguments for injectivity and surjectivity are each a diagram chase.)

(e) Deduce that the i^{th} homology of $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G})$ is isomorphic to $\mathrm{Tor}_i^S(M,N)$.

(f) Transpose the above argument, leaving the definition of $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G})$ unchanged but replacing $(-1)^i \otimes \psi$ with the *vertical differential* $\phi \otimes 1$ on the j^{th} column of $\mathcal{F} \otimes \mathcal{G}$, to deduce that $\mathrm{tot}(\mathcal{F} \otimes \mathcal{G})$ has j^{th} homology $\mathrm{Tor}_j^S(N,M)$.

(g) Conclude that $\mathrm{Tor}_i^S(M,N) \cong H_i\big(\mathrm{tot}(\mathcal{F} \otimes \mathcal{G})\big) \cong \mathrm{Tor}_i^S(N,M)$.

1.13 Let $m \leq n$ be positive integers, and $S = \Bbbk[x_1, \ldots, x_{m+n}]$. Setting $M = S/\langle x_{m+1}, \ldots, x_{m+n}\rangle$ and $N = S/\langle x_{n+1}, \ldots, x_{m+n}\rangle$, find the Hilbert series of the isomorphic modules $\mathrm{Tor}_i^S(M,N)$ and $\mathrm{Tor}_i^S(N,M)$. Which is easier to calculate? Write a succinct expression for the result of setting $x_i = q^i$ for all i in this series.

Notes

Stanley–Reisner rings and Stanley–Reisner ideals are sometimes called *face rings* and *face ideals*. Their importance in combinatorial commutative algebra cannot be overstated. Stanley's green book [Sta96] contains a wealth of information about them, including a number of important applications, such as Stanley's proof of the Upper Bound Theorem for face numbers of convex polytopes. We also recommend Chapter 5 of the book of Bruns and Herzog [BH98] and Hibi's book [Hib92] for more background on squarefree monomial ideals. The first two of these references contain versions of Hochster's formula, whose original form appeared in [Hoc77]; the form taken by Theorem 1.34 is that of [BCP99].

We have only presented the barest prerequisites in simplicial topology. The reader wishing a full introduction should consult [Hat02], [Mun84], or [Rot88].

Monomial matrices were introduced in [Mil00a] for the purpose of working efficiently with resolutions and Alexander duality. Monomial matrices will be convenient for the purpose of cellular resolutions in Chapters 4, 5, and 6. Other applications and generalizations will appear in the context of injective resolutions (Section 11.3) and local cohomology (Chapter 13).

The reader is encouraged to do explicit computations with the objects in this chapter, and indeed, in all of the chapters to come. Those who desire to compute numerous or complicated examples should employ a computer algebra system such as CoCoA, Macaulay2, or Singular [CoC, GS04, GPS01].

We included Exercise 1.12 because there seems to be no accessible proof of the symmetry of Tor in the literature. The proof outlined here shows that the natural map from the total complex of any bicomplex to its horizontal homology complex is an isomorphism on homology when the rows are resolutions (so their homology lies only in homological degree zero). This statement forms the crux of a great number of arguments producing isomorphisms arising in local cohomology and other parts of homological algebra. The argument given in Exercise 1.12 is the essence behind the spectral sequence method of deriving the same result. Those who desire to brush up on their abstract homological algebra should employ a textbook such as Mac Lane's classic [MacL95] or Weibel's book [Wei94].