Composite Sieving Techniques: Dihedral Action on Cluster Complexes

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Outline

* **Motivation**: cyclic and dihedral sieving
* **Our results**: dihedral sieving on cluster complexes
* **Future directions**

q- and q,t-analogues

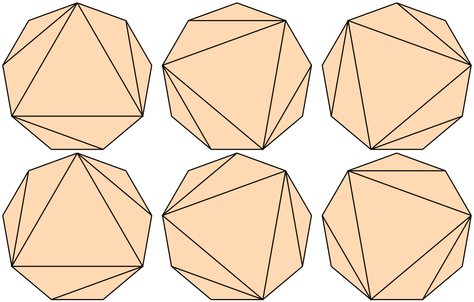
Definition

$$\{n\}_{q,t} := \sum_{i=0}^{n-1} q^i t^{n-1-i} \qquad [n]_q := \{n\}_{q,1} = \sum_{i=0}^{n-1} q^i$$

$$\{n\}!_{q,t} := \prod_{i=1}^{n} \{n\}_{q,t} \qquad [n]!_q := \{n\}!_{q,1} = \prod_{i=1}^{n} [n]_q$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,t} := \frac{\{n\}!_{q,t}}{\{k\}!_{q,t}\{n-k\}!_{q,t}} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,1} = \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

A strange behavior



$$\frac{1}{[8]_{\omega_9^3}} \begin{bmatrix} 14 \\ 7 \end{bmatrix}_{\omega_9^3} = 6 \quad \text{and} \qquad \frac{1}{[8]_{\omega_9^4}} \begin{bmatrix} 14 \\ 7 \end{bmatrix}_{\omega_9^4} = 0$$

What's going on?

Cyclic sieving

Definition (Reiner–Stanton–White '04)

If X is a finite set acted on by a cyclic group $C_n = \langle r \rangle$, and $X(q)$ is a polynomial in q, then the pair $(X \circlearrowleft C_n, X(q))$ has the **cyclic sieving phenomenon** (CSP) if for all $\ell \in [n]$,

$$\#\{x \in X : r^\ell x = x\} = X(\omega_n^\ell).$$

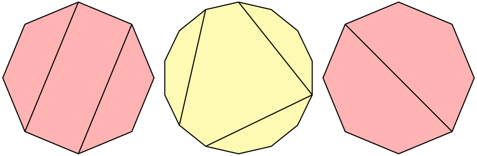
Examples of cyclic sieving

- Let X be the k-subsets of $[n]$. Then $\left(X \circlearrowleft C_n, \begin{bmatrix} n \\ k \end{bmatrix}_q\right)$ exhibits CSP. [Reiner–Stanton–White '04]

- Let X be the k-multisubsets of $[n]$. Then $\left(X \circlearrowleft C_n, \begin{bmatrix} n-k+1 \\ k \end{bmatrix}_q\right)$ exhibits CSP. [Reiner–Stanton–White '04]

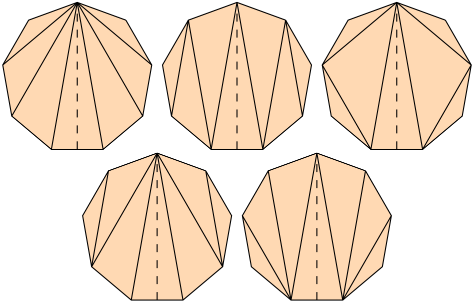
- Let X be the set of k-angulations of an n-gon. Then $\left(X \circlearrowleft C_n, \frac{1}{[m]_q} \begin{bmatrix} (k-1)m \\ m \end{bmatrix}_q\right)$ exhibits CSP, for $m := \frac{n-2}{k-2}$. [Eu–Fu '06]

k-angulations of an n-gon



It is easily verified that such a dissection exists iff $n \equiv 2 \bmod (k-2)$.

Another strange behavior?



$$\mathrm{Cat}_n(\omega_9^3, \omega_9^{-3}) = 6 \quad \text{and} \quad \mathrm{Cat}_n(\omega_9^4, \omega_9^{-4}) = 0 \quad \text{and} \quad \mathrm{Cat}_n(1, -1) = 5$$

What's this Cat_n? Is there something else going on?

Dihedral sieving

Definition (Rao–Suk '17)

If X is a finite set acted on by a dihedral group $I_2(n) = \langle r_1, r_2 \rangle$ for odd n, and $X(q,t)$ is a symmetric polynomial in q and t, then the pair $(X \circlearrowleft I_2(n), X(q,t))$ has the **dihedral sieving phenomenon** (DSP) if for all $g \in I_2(n)$ with $\{\lambda_1, \lambda_2\} = \begin{cases} \{\omega^k, \overline{\omega}^k\} & g \text{ a rotation} \\ \{1, -1\} & g \text{ a reflection} \end{cases}$,

$$\#\{x \in X : gx = x\} = X(\lambda_1, \lambda_2).$$

Examples of dihedral sieving

* Let X be the k-subsets of $[n]$. Then $\left(X \circlearrowleft I_2(n), \left\{ {n \atop k} \right\}(q,t)\right)$ exhibits DSP for odd n. [Rao–Suk '17]

* Let X be the k-multisubsets of $[n]$. Then $\left(X \circlearrowleft I_2(n), \left\{ {n-k+1 \atop k} \right\}(q,t)\right)$ exhibits DSP for odd n. [Rao–Suk '17]

* Let X be the set of **triangulations** of an n-gon. Then $\left(X \circlearrowleft I_2(n), \mathrm{Cat}_n(q,t)\right)$ exhibits DSP for odd n. [Rao–Suk '17]

Just triangulations?

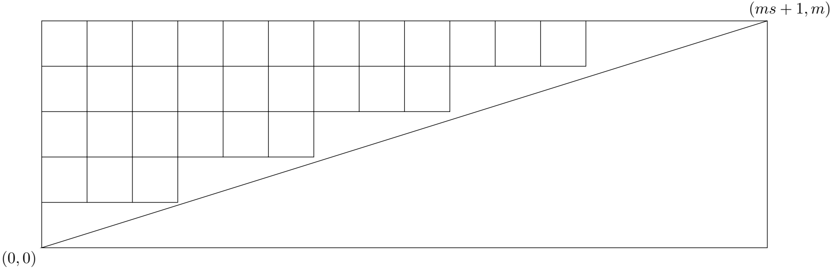
Question: Does anything stop us from obtaining DSP for k-angulations?

Answer: No.

Theorem (REU '19)

Let X be the set of k-angulations of an n-gon. Then $\left(X \circlearrowleft I_2(n), \mathrm{Cat}_n^k(q,t)\right)$ exhibits DSP for all odd n and k.

$\mathrm{Cat}_n^k(q,t)$

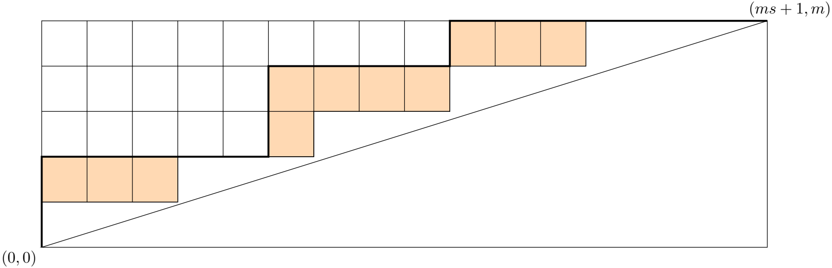


$$\mathrm{Cat}_n^k(q,t) := \sum_{\lambda} q^{\texttt{area}(\lambda)} t^{\texttt{area}(\texttt{sweep}(\lambda))}$$

$$\mathrm{Cat}_n^k(\omega,\overline{\omega}) = \frac{1}{[m]_\omega} \begin{bmatrix} (k-1)m \\ m-1 \end{bmatrix}_\omega$$

$$\mathrm{Cat}_n^k(-1,1) \equiv \#\text{ even area paths} - \#\text{ odd area paths}$$

$\mathrm{Cat}_n^k(q,t)$



$$\mathrm{Cat}_n^k(q,t) := \sum_{\lambda} q^{\texttt{area}(\lambda)} t^{\texttt{area}(\texttt{sweep}(\lambda))}$$

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$$\mathrm{Cat}_n^k(-1,1) \equiv \#\text{ even area paths} - \#\text{ odd area paths}$$

Raney numbers

$$R_{p,r}(m) := \frac{r}{pm+r}\binom{pm+r}{m}$$

$$R_{p,1}(m) = \sum_{i=0}^{m-1} R_{p,1}(i) R_{p,p-1}(m-1-i) \qquad \text{[Zhou–Yan '17]}$$

$$R_{p,r}(m) = \sum_{i=0}^{m} R_{p,r}(i) R_{p,r-1}(m-i) \qquad \text{[Zhou–Yan '17]}$$

DSP for k-angulations

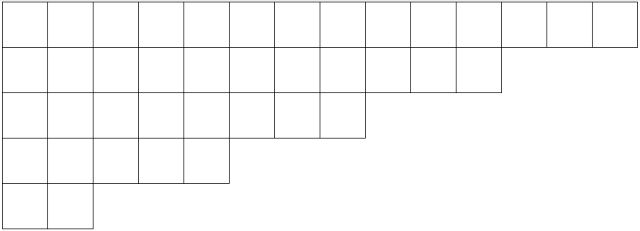
Theorem

For odd $k > 3$ and m,

$$\mathrm{Cat}_n^k(1,-1) = (-1)^{\frac{m-1}{2}} R_{s+1,\frac{s+1}{2}} \left(\frac{m-1}{2}\right).$$

Proof sketch.

We use recursion on Young diagrams of the shape shown before. We define $D_s(\ell, m) = \sum_{\lambda} (-1)^{\mathtt{area}(\lambda)}$.

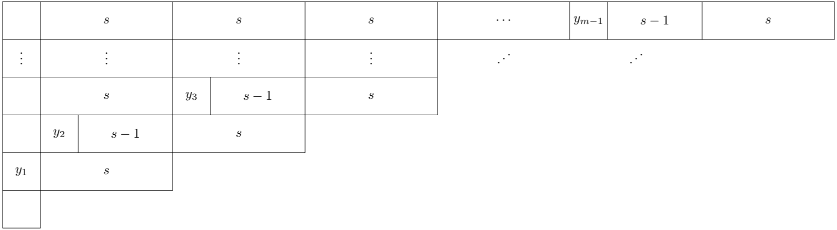


DSP for k-angulations

Proof sketch, cont.

$$D_s(1,m) = \sum_{y=0}^{m-2} (-1)^{y+1} D_s(1,y) D_s(2y-1, m-y-2)$$

follows by considering recursion on the SW-most marker contained above given a path.

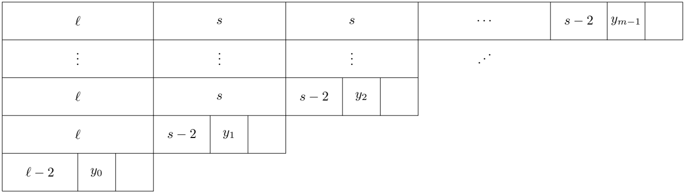


DSP for k-angulations

Proof sketch, cont.

$$D_s(\ell, m) = \sum_{y=0}^{m} (-1)^{(m+1)y} D_s(\ell - 2, y) D_s(1, m - y)$$

follows by considering recursion on the SW-most marker contained above given a path. (Note the different marker configuration.)



DSP for k-angulations

Proof sketch, cont.

$D_s(\ell, m) = 0$ for odd ℓ and m, so we can rewrite the recurrences for $D_s(\ell, m)$ to match those for $R_{s+1, \frac{\ell+1}{2}}\left(\frac{m}{2}\right)$ ■

The final case of is $k = 3$—triangulations, proved in in Rao–Suk '17.

Generalizing k-angulations?

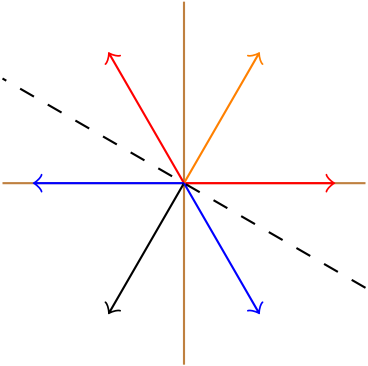
Question: Is there another layer of generality to look at?

Answer: Yes.

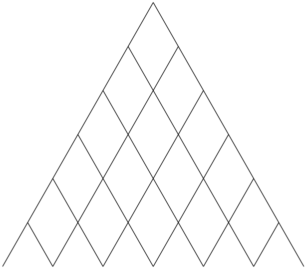
k-angulations arise as maximal clusters in $\Delta(\Phi(A_n))$. Let's go into what that means.

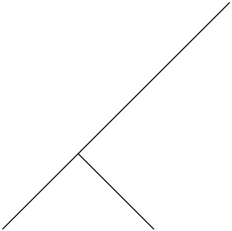
Root systems

Take a Coxeter group W with root system $\Phi = \Phi(W)$, simple roots Π, and positive roots Φ^+. Let $\Phi_{-1} := \Phi^+ \sqcup -\Pi$.



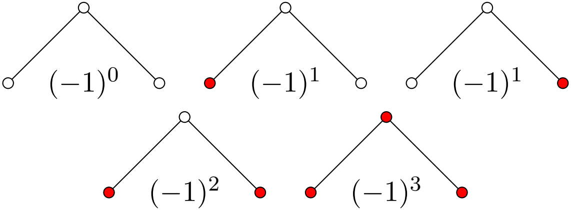
Positive root posets and $\mathrm{Cat}_W(q,t)$





$$\mathrm{Cat}_W\left(q, \frac{1}{q}\right) = q^{\blacksquare} \prod_i \frac{[h+d_i]_q}{[d_i]_q} \qquad \mathrm{Cat}_W(q,1) = q^{\blacksquare} \sum_{J \in I(P)} q^{|J|}$$

Example: $W = A_2$

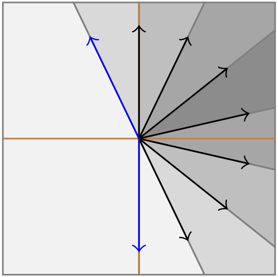


$$\sum_{J \in I(P)} (-1)^{|J|} = 1 - 1 - 1 + 1 - 1 = -1 = \pm 1 \quad \checkmark$$

Cluster complexes

Clusters are maximal sets of mutually-**compatible** roots, forming the **cluster complex** $\Delta(\Phi)$. Further, there is the action

$$\Phi_{-1} \circlearrowleft \langle \tau_-, \tau_+ \rangle \cong I_2(n).$$



$\Delta(\Phi)$ generalizes to $\Delta^{(s)}(\Phi)$, with a corresponding poset and $\mathrm{Cat}_W^{(s)}$.

Main results

In this language, we can reformulate the previous theorem:

Theorem (REU '19)

The pair $(\Delta^{(s)}(\Phi(A_{n-1})) \circlearrowleft I_2(n+2), \mathrm{Cat}^{(s)}_{A_{n-1}}(q,t))$ exhibits dihedral sieving for all odd n and s.

We also prove:

Theorem (REU '19)

The pair $(\Delta(\Phi) \circlearrowleft I_2(h+2), \mathrm{Cat}_W(q,t))$ exhibits dihedral sieving for any root system $\Phi = \Phi(W)$ when $h = \max\{d_i\}$ is odd.

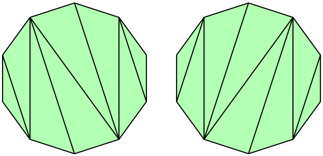
Type A

$$\Delta^{(s)}(\Phi(A_{n-1})) \longleftrightarrow \{k\text{-angulations of } (n+2)\text{-gon}\}$$

Since the action of $\langle \tau_-, \tau_+ \rangle$ is dihderal, for odd n—done!

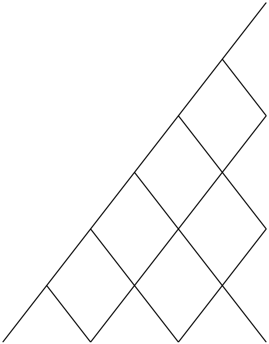
Type B cluster complexes

Clusters of $\Delta(\Phi(B_{n-1}))$ correspond to centrally symmetric k-angulations of a $2n$-gon with a diameter.



It is evident that no such k-angulation is fixed under a reflection.

Type B cluster complexes



Type B cluster complexes

The positive root poset of B_n is the trapezoid poset $T_{n,2n}$. Let the triangle poset be T_n.

Lemma

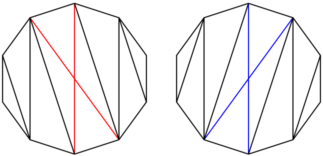
$$\sum_{J \in I(T_n)} (-1)^{|J|} = 0 \quad \text{and hence} \quad \sum_{J \in I(T_{n,2n})} (-1)^{|J|} = 0.$$

Proof sketch.

Induction on n and the number of minimal elements included in a given order ideal. ■

Type D cluster complexes

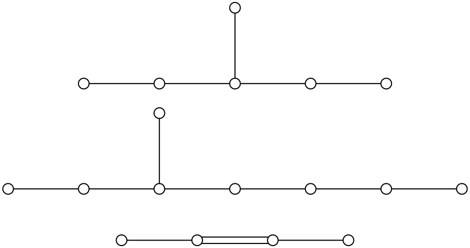
Clusters of $\Delta(\Phi(D_{n-1}))$ correspond to centrally symmetric k-angulations of a $2n$-gon with colored diameters that may intersect the same color. Reflection switches their color.



It is evident that no such k-angulation is fixed under a reflection. We can show that the desired polynomial vanishes in the same way as in Type B.

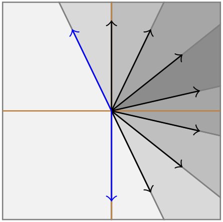
Type E, F cluster complexes

The exceptional cases of E_6, E_8, F_4 were verified though Sage.

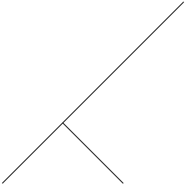


Type I cluster complexes

We showed that there is exactly 1 cluster fixed under reflection.



To show that the polynomial evaluates similarly, there is a counting argument on the relatively simple (retrofitted) poset.



Future directions in sieving

Odd DSP $\Delta^{(s)}$ for non-A_n. The main difficulty here is to actually conceptualize and work with the objects.

Even dihedral sieving. Recall that all of our results were for odd n. We hope to extend to even n, but only have partial results towards that end.

Symmetric sieving?! *In principle* nothing stops us from continuing to sieving with symmetric groups. We managed to show k-multisubset sieving of $[n]$ but have not succeeded beyond that.

Acknowledgements

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References

The following references were cited directly in this presentation. Please see our REU report for a full list of references.

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