A Partial Characterization of Virtually Cohen-Macaulay Simplicial Complexes

Nathan Kenshur, **Feiyang Lin**, Sean McNally, **Zixuan Xu**, Teresa Yu

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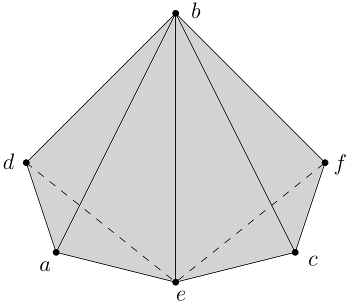
Outline

Definition

An **abstract simplicial complex** Δ on vertex set X is a collection of subsets of X such that $A \in \Delta$ whenever $A \subseteq B \in \Delta$.

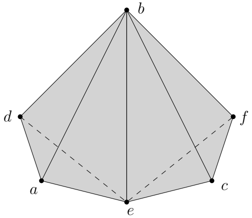


$X = \{a, b, c, d, e, f\}$
$\Delta = 2^{\{a,b,d,e\}} \cup 2^{\{b,c,e,f\}}$
facets: $\{a, b, d, e\}, \{b, c, e, f\}$
dimension: 3
pure? yes
gallery-connected? no

Stanley-Reisner Theory

Given a simplicial complex Δ on X, the **Stanley-Reisner ideal** of Δ is the following ideal in $\Bbbk[X]$:

$$I_\Delta = \bigcap_{A \in \Delta} (x_i : x_i \notin A) = (m_A : A \notin \Delta).$$

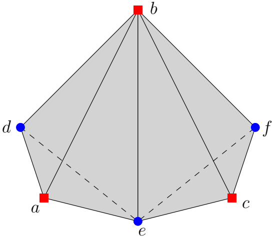


$$I_\Delta = \langle c, f \rangle \cap \langle a, d \rangle$$
$$= \langle ac, af, cd, df \rangle.$$

Simplicial Complex in $\mathbb{P}^{\vec{n}}$

From now on we will be working in the product projective space $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and we use the following notation.

- $S := \mathbb{k}[x_{i,j} : 1 \leq i \leq r, 0 \leq j \leq n_i]$
- $B := \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$ is the **irrelevant ideal** of S. Note that $V(B) = \emptyset$.
- A **simplicial complex in** $\mathbb{P}^{\vec{n}}$ is a simplicial complex on the vertex set $X_{\vec{n}} = \bigcup_{i=1}^{r} \{x_{i,j} : 0 \leq j \leq n_i\}$.
- The **Stanley-Reisner ring** of Δ is the quotient ring $\mathbb{k}[\Delta] := S/I_\Delta$.



Free Resolutions & Virtual Resolutions

Definition

A complex of free S-modules,

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_n} F_n,$$

is a **free resolution** of S/I if

1. $\widetilde{H}_i(\mathcal{F}_\bullet) = 0$ for $i \geq 1$
2. $\widetilde{H}_0(\mathcal{F}_\bullet) = F_0 / \operatorname{im} \phi_1 = S/I$

It is a **virtual resolution of** S/I if

1. $\operatorname{rad} \operatorname{ann} H_i \mathcal{F}_\bullet \supseteq B$ for all $i > 0$
2. $\operatorname{ann} H_0 \mathcal{F}_\bullet : B^\infty = I : B^\infty$

Cohen-Macaulay & Virtually Cohen-Macaulay

Definition (Cohen-Macaulay)

A simplicial complex Δ on X is **Cohen-Macaulay** if there exists a free resolution of $\mathbb{k}[\Delta]$ of length codim I_Δ.

Definition (Virtually Cohen-Macaulay)

A simplicial complex Δ on $X_{\vec{n}}$ is **virtually Cohen-Macaulay** if there exists a virtual resolution of $\mathbb{k}[\Delta]$ of length codim I_Δ.

Resolutions of Ideals with Same Variety

Lemma

For two ideals $I, J \subset S$ with $V(I) = V(J)$, then any free resolution r of S/J is a virtual resolution of S/I.

Recall that $B = \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$. Let $B^{\vec{u}}$ be $\bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle^{u_i}$. Since $V(I \cap B^{\vec{u}}) = V(I) \cup V(B^{\vec{u}}) = V(I)$, a free resolution of $S/(I \cap B^{\vec{u}})$ is a virtual resolution of S/I.

Irrelevant & Relevant Faces

Since $I_\Delta = \bigcap_{A \in \Delta}(x_i : x_i \notin A)$, adding a face F to Δ is equivalent to intersecting I_Δ with the ideal $I = (x : x \notin F)$.

Definition

A face F of a simplicial complex Δ is **relevant** if it contains at least one vertex from every color; otherwise it is **irrelevant**.

$V(I) = \varnothing$ if and only if F is irrelevant.

Virtually Equivalent Simplical Complexes

From the previous observation, we have the following important lemma.

Lemma

Let Δ, Δ' be two simplicial complexes in $\mathbb{P}^{\vec{n}}$ such that $\Delta \setminus \Delta'$ and $\Delta' \setminus \Delta$ contain only irrelevant faces. Then the free resolution of $I_{\Delta'}$ is a virtual resolution of I_{Δ}.

We call such Δ and Δ' **virtually equivalent**.

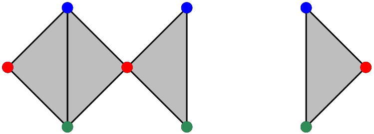


Figure 1: Δ, in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$

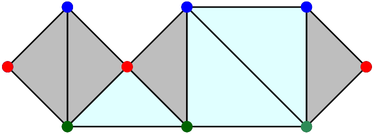


Figure 2: $\Delta' = \Delta \cup \{\text{Irrelevant Facets}\}$

The Intersection Method

Theorem (Herzog-Takayama-Terai)

Let I be a monomial ideal, then if I is Cohen-Macaulay, $\text{rad}(I)$ is also Cohen-Macaulay.

Lemma

If there exists $\vec{u} \in \{0,1\}^r$ such that $I' = I \cap B^{\vec{u}}$ is Cohen-Macaulay, then I is virtually Cohen-Macaulay.

Then we obtain the following:

Proposition

Let Δ be a simplicial complex on the product projective space $\mathbb{P}^{\vec{n}}$. If there exists J a monomial ideal with $V(J) = \varnothing$ such that $I_\Delta \cap J$ is Cohen-Macaulay, then there exists Δ' containing only irrelevant facets such that $\text{rad}(J) = I_{\Delta'}$ and $I_\Delta \cap I_{\Delta'}$ is Cohen-Macaulay. In particular, this implies $\Delta \cup \Delta'$ is Cohen-Macaulay and Δ is virtually Cohen-Macaulay.

The Intersection Method

Fact

Cohen-Macaulay complexes are pure and gallery-connected.

Corollary

For a simplicial complex Δ, if there exists $\vec{u} \in \mathbb{Z}^r$ such that $I_\Delta \cap B^{\vec{u}}$ is Cohen-Macaulay:

- Consider $\operatorname{supp} \vec{u} \in \{0,1\}^r$, then $(\operatorname{supp} \vec{u})_i$ can be 1 only if $\dim \mathbb{P}^{n_i} = \dim \Delta$.
- Δ is pure and gallery-connected up to adding irrelevant facets.

Balanced Complexes

Definition

Let Δ be a pure simplicial complex on the product of projective spaces $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. We say that a facet $F \in \Delta$ is **balanced** if it contains exactly one vertex of every component. We say that a simplicial complex is **balanced** if all of its facets are balanced.

Theorem

The Stanley-Reisner ring of a pure shellable simplicial complex is Cohen-Macaulay.

Strategy: Add all possible irrelevant facets of same dimension and show the new complex is shellable.

Balanced Complex

Definition (Shellability)

A *shelling* of Δ is an ordered list $F_1, F_2, \ldots, F_m$ of its facets such that for all $i = 2, \ldots, m$, $(\bigcup_{k=1}^{i-1} F_k) \cap F_i$ is pure of codimension 1. If a simplicial complex is pure and has a shelling, then it is *shellable*.

Definition

Given a vertex set V on the product projective space $\mathbb{P}^{\vec{n}}$. Then the **irrelevant complex supported on** V is defined to be

$$\Delta_{irr} := \{\sigma \in 2^V \mid |\sigma| = n, |\mathrm{col}(\sigma)| < n\}.$$

Strategy: show that any balanced complex with all the irrelevant facets added in yields a shellable complex.

Balanced Complex

Proposition

Let Δ_{irr} be the irrelevant complex supported on V in the product projective $\mathbb{P}^n$. Then there exists a balanced facet R such that $\Delta = \Delta_{irr} \cup \{R\}$ is shellable.

Observation: Adding more balanced facet still maintains a shelling.

Balanced Complex

Theorem

If Δ is a pure and balanced in the product projective space $\mathbb{P}^{\vec{n}}$, then Δ is virtually Cohen-Macaulay.

Future work

- Analogue for Reisner's criterion for virtual Cohen-Macaulayness?

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References

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Questions?

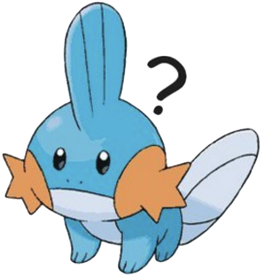

Figure 3: confused mudkip.